Economics Bulletin

Volume 34, Issue 2

Multiplicity of Equilibrium Payoffs in Three-Player Baron-Ferejohn Model

Duozhe Li
Chinese University of Hong Kong

Abstract

This paper studies the three-player sequential bargaining game with a simple majority voting rule due to Baron and Ferejohn (1989). We show that there is a vast multiplicity of equilibrium payoffs, and that as the discount factor tends to one, the set of equilibrium payoffs evolves monotonically towards the entire feasible set. The multiplicity result can be easily extended to an interesting variant of the bargaining game, in which the responders only observe their own offers.

I would like to thank the editor and two anonymous referees for their helpful comments on a previous version of this paper.
Citation: Duozhe Li, (2014) "Multiplicity of Equilibrium Payoffs in Three-Player Baron-Ferejohn Model", *Economics Bulletin*, Vol. 34 No. 2 pp. 1122-1132.
Contact: Duozhe Li - duozheli@cuhk.edu.hk.
Submitted: November 19, 2013. **Published:** May 25, 2014.

1. Introduction

In their seminal paper, Baron and Ferejohn (1989) study a simple model of multilateral bargaining under majority voting rule, which, in the past two decades, has become a basic framework for understanding legislative bargaining. They establish the uniqueness of stationary subgame perfect equilibrium under a symmetric restriction on the recognition process that determines who makes the proposal. The uniqueness result is nicely generalized to the asymmetric cases by Eraslan (2002).[1]

A well-known issue in multilateral bargaining is the multiplicity of equilibrium outcomes.[2] Although the stationary equilibrium notion is appealing for its simplicity (see Baron and Kalai 1993), the restriction to stationary strategies is sometimes considered problematic. Indeed, Baron and Ferejohn (1989) show that when their bargaining game involves at least five parties, any division of the surplus can be sustained in a subgame perfect equilibrium. What happens when there are only three negotiating parties? Is there a unique equilibrium outcome? If not, what is the set of equilibrium outcomes? This needs to be answered to complete the analysis of the Baron-Ferejohn model. Meanwhile, as one can easily think of many real-life negotiations that involve exactly three parties, the answer is also important from a practical point of view.

In this paper we show that in the three-player Baron-Ferejohn model there is a vast multiplicity of equilibria as well. Each player's equilibrium payoff is bounded below by a positive value, and by the same token, no one can obtain the entire surplus in equilibrium. As the discount factor tends to 1, the set of the equilibrium payoffs evolves monotonically towards the entire feasible set. We focus on the equilibria with minimal winning coalition, that is, the proposer is required to offer one of the responders a share that does not fall below a certain acceptance threshold value, while the other responder receives nothing. Any deviation will be punished by switching to a continuation equilibrium in which the deviator's payoff is held down to the minimum. The key step of the equilibrium construction is to ensure that both responders find it optimal to reject an off-equilibrium proposal.

Baliga and Serrano (1995) introduce a multilateral bargaining game with imperfect information, in which players take turns to make proposal and the offer to each responder is made in a sealed envelope. In other words, each responder observes only the share proposed for him. Responses are public and occur sequentially, and an agreement requires unanimous acceptance. They also obtain a large multiplicity of equilibria. This particular extension of the multilateral bargaining model is interesting and important because many real-life negotiations do involve under-the-table dealing.

[1] Banks and Duggan (2000) consider majoritarian bargaining with multidimensional alternative set, and they show by example that stationary equilibrium outcome is not necessarily unique. Eraslan and Merlo (2002) extend the Baron-Ferejohn model to a stochastic environment, in which stationary equilibrium payoffs need not be unique and efficient.

[2] Under unanimity rule, it is shown that any feasible agreement can be supported by a subgame perfect equilibrium for sufficiently large discount factors. See Herrero (1985), Sutton (1986), and Osborne and Rubinstein (1990) for details on these results.

Following Baliga and Serrano (1995), we analyze the majoritarian bargaining game with sealed offers. The multiplicity result is obtained as well. The equilibria constructed for the original three-player Baron-Ferejohn bargaining game can be easily modified into equilibria of the game with sealed offers. A particular feature of the equilibria is that each player, when in the role of a responder, adopts a monotone acceptance rule, i.e., accepting an offer if and only if it does not fall below a threshold value.

In a recent independent work, Herings, Meshalkin and Predtetchinski (2013) also study a three-player majoritarian bargaining model. In the current paper, we assume that players respond to a proposal simultaneously; in contrast, they consider sequential responses. While we focus on the equilibria with minimal winning coalition and identify the set of efficient equilibrium payoffs, they aim to identify the set of all possible equilibrium divisions. These two sets turn out to be identical. Finally, by allowing mixed strategies, our equilibrium characterization appears to be simpler than theirs, and moreover, our equilibria can be easily modified to fit in the model with sealed offers.

Section 2 establishes the multiplicity of the equilibrium payoffs in the three-player Baron-Ferejohn bargaining game. Section 3 studies the majoritarian bargaining game with sealed offers. All proofs are relegated to the Appendix.

2. Baron-Ferejohn Bargaining Game with Three Players

We consider the Baron-Ferejohn bargaining game with three players. The index i is used to refer to each player, and the indices j and k, when they appear, refer to the players other than i. The bargaining protocol is described as follows. In each period $t = 0, 1, 2...$, player i is recognized with probability $1/3$ to make a proposal $\mathbf{x} \in \mathbf{X} \equiv \left\{ \mathbf{x} \in \mathbb{R}^3_+ : \sum_{i=1}^{3} x_i = 1 \right\}$, where x_i denotes player i's share of a fixed surplus of size one. Players j and k simultaneously respond by either accepting or rejecting the proposal. If at least one responder accepts, the proposal is passed and the game ends; otherwise, the game proceeds to the next period. If a proposal $\mathbf{x}$ is passed in period t, player i's payoff is $\delta^t x_i$, where $\delta \in (0, 1)$ is the common discount factor.[3] The bargaining game is denoted by $G(\delta)$.

We adopt subgame perfect equilibrium (henceforth equilibrium) as the solution concept. As responses are made simultaneously, there is a trivial equilibrium in which every player accepts all proposals. To rule out this trivial equilibrium, we further impose on the equilibria a restriction that no one uses a weakly dominated strategy. It is weakly dominated to accept any proposal that gives a player less than his discounted payoff in the continuation play.

An equilibrium outcome is efficient if a proposal is passed in period 0. We characterize the entire set of efficient equilibrium payoffs of $G(\delta)$. Lemma 1 establishes the upper and the lower bounds of each player's equilibrium payoffs.

[3] For simplicity, we assume that all players have the same linear utility function and a common discount factor, and that the probability with which each player is recognized as the proposer is also identical. To include asymmetry into the analysis would add algebraic complexity without causing any qualitative changes in our results.

Lemma 1 *Let $M(\delta)$ and $m(\delta)$ be the supremum and the infimum of a player's equilibrium payoffs in $G(\delta)$, then, $M(\delta) \leq (3-\delta)/\left(9-6\delta-\delta^2\right)$ and $m(\delta) \geq (3-3\delta)/\left(9-6\delta-\delta^2\right)$.*

Recall that when there are at least five players, Baron and Ferejohn (1989) show that for all δ no less than some δ^*, any division of the surplus can be sustained in equilibrium, i.e., $M(\delta)=1$ and $m(\delta)=0$ for all $G(\delta)$ with $\delta \geq \delta^*$. Here, for any $\delta<1$, we have $M(\delta)<1$ and $m(\delta)>0$, and the range of equilibrium payoffs depends on δ. Hence, it is already clear that the three-player case is different from the cases with five or more players. With a slight abuse of notation, from now on, we use M and m to denote the boundary values of the equilibrium payoffs as characterized in Lemma 1.

In the rest of this section, we focus on a specific type of efficient equilibria, namely, the equilibria with minimal winning coalition. In such an equilibrium, the proposer offers one of the responders a positive share, and the offer is accepted by that responder, while the player being excluded from the winning coalition receives nothing.[4]

Denote an equilibrium with minimal winning coalition by $e(\mathbf{z},\mathbf{p})$, where $\mathbf{z}=(z_1,z_2,z_3)$ and $\mathbf{p}=(\mathbf{p}^1,\mathbf{p}^2,\mathbf{p}^3)$ are referred to as the equilibrium configuration. More specifically, it says that, player i, when serving as the proposer, offers a share of $z_i \in [\delta m, \delta M]$ to player j (or k resp.) with probability $p^i_j \in [0,1]$ ($p^i_k = 1-p^i_j$ resp.).[5] The proposal will be accepted by the responder receiving the offer of z_i. Thus, the equilibrium $e(\mathbf{z},\mathbf{p})$ yields a payoff vector $\mathbf{u}(e(\mathbf{z},\mathbf{p})) \in \mathbb{R}^3_+$, where

$$u_i(e(\mathbf{z},\mathbf{p})) = (1-z_i)/3 + \left(p^j_i z_j + p^k_i z_k\right)/3.$$

Note that the equilibrium configuration specifies only the actions on the equilibrium path, not the complete strategy profile.[6] In order to completely characterize an equilibrium, we need to specify a continuation equilibrium configuration for every subgame.

First, we show that the upper and lower bounds of the equilibrium payoffs identified in Lemma 1 can be achieved in equilibrium. Denote as E^m_i the set of configurations, in which player i's payoff is minimized at m. One arbitrary configuration in E^m_i is denoted as $e^m_i(\mathbf{z},\mathbf{p})$, where $z_i=\delta M$, $p^j_i=p^k_i=0$, and z_j, z_k and $p^i_{j/k}$ can take any value within the aforementioned range. By the configuration $e^m_i(\mathbf{z},\mathbf{p})$, player i offers δM to either j or k with probability p^i_j and $p^i_k=1-p^i_j$, and player j (k resp.) offers z_j (z_k resp.) to player k (j resp.) with probability 1. The acceptance rule is specified as follows: when player i makes proposal, player j (k resp.) accepts if his offer is at least δm and the sum of the two offers is at least δM; when player j (k resp.) makes proposal, players k (j resp.) and i accept if one's own offer is at least δm and the sum of the two offers is at least z_j (z_k resp.).

With this configuration, player i's payoff is $m=(1-\delta M)/3$, and the total payoff of players j and k is $1-m$. Depending on the specific values of z_j, z_k and $p^i_{j/k}$, the payoff of player j (k resp.) in $e^m_i(\mathbf{z},\mathbf{p})$ can take any value between $1-M-m=(1-\delta M)/3+\delta m/3$

[4] The unique stationary equilibrium in Baron and Ferejohn (1989) has the same feature.

[5] Formally, player 1 randomizes between two proposals $(1-z_1, z_1, 0)$ and $(1-z_1, 0, z_1)$ with probabilities p^1_2 and p^1_3, and similar for the other two players.

[6] One exception is the stationary equilibrium, in which players' actions are history independent.

and $M = (1-\delta m)/3 + 2\delta M/3$. For example, let $z_j = \delta m$, $z_k = \delta M$ and $p_j^i = 1$, then player j's payoff is M and k's payoff is $(1-M-m)$.

Now, we specify the continuation play following any possible deviation. To guarantee that player i cannot benefit from making a different proposal, in the continuation play each responder's discounted payoff cannot be lower than that from accepting the proposal. More precisely, if player i deviates by proposing $\mathbf{x}$ with $x_j + x_k < \delta M$, it will be rejected by both j and k, and the continuation play is $e_i^m(\hat{\mathbf{z}}, \hat{\mathbf{p}}) \in E_i^m$, where

$$\begin{aligned} \hat{p}_j^i &= 1 - \hat{p}_k^i = \frac{x_j}{x_j + x_k} \in [0,1] \\ \hat{z}_j &= \min\left\{\delta M, \max\left\{\delta m, \frac{x_k}{x_j + x_k}\right\}\right\} \in [\delta m, \delta M] \\ \hat{z}_k &= \min\left\{\delta M, \max\left\{\delta m, \frac{x_j}{x_j + x_k}\right\}\right\} \in [\delta m, \delta M]. \end{aligned}$$

Denote as $\widehat{\pi}_i$ ($\widehat{\pi}_j$ and $\widehat{\pi}_k$ resp.) the payoff of player i (j and k resp.) in the continuation play. Then,

$$\begin{aligned} \widehat{\pi}_i &= m \\ \widehat{\pi}_j &= \frac{1}{3}\hat{p}_j^i \delta M + \frac{1}{3}\hat{z}_k + \frac{1}{3}(1-\hat{z}_j) \in [1-M-m, M] \\ \widehat{\pi}_k &= \frac{1}{3}\hat{p}_k^i \delta M + \frac{1}{3}\hat{z}_j + \frac{1}{3}(1-\hat{z}_k) \in [1-M-m, M]. \end{aligned}$$

Since player i's discounted payoff from the continuation play is $\delta\widehat{\pi}_i = \delta m < 1-\delta M$, deviation is unprofitable for him. The following lemma shows that both j and k will find it optimal to reject any proposal $\mathbf{x}$ from i with $x_j + x_k < \delta M$.

Lemma 2 *For any $\mathbf{x}$ with $x_j + x_k < \delta M$, $\delta\widehat{\pi}_j > x_j$ and $\delta\widehat{\pi}_k > x_k$.*

Similarly, if player j (k resp.) deviates by proposing $\mathbf{x}$ with $x_i + x_k < z_j$ ($x_i + x_j < z_k$ resp.), it will be rejected by both responders and the continuation play is $e_j^m(\hat{\mathbf{z}}, \hat{\mathbf{p}})$ ($e_k^m(\hat{\mathbf{z}}, \hat{\mathbf{p}})$ resp.), in which $(\hat{\mathbf{z}}, \hat{\mathbf{p}})$ is specified similarly as above.

A responder's deviation, i.e., rejecting an acceptable offer, is said to be effective if the rejection leads the game to the next period. Following an effective deviation of player j as a responder, the continuation play becomes $e_j^m(\mathbf{z}, \mathbf{p})$, which holds j's payoff down to m while giving the other responder the highest payoff of M. Clearly, this is sufficient to prevent any possible deviation of a responder.[7]

The last step of the construction is to note that every continuation play specified above is a configuration in the same form as $e_i^m(\mathbf{z}, \mathbf{p})$, and thus, further deviation from the continuation play can be prevented in the same way as above.

[7] As in the folk theorem of the repeated games, we ignore simultaneous deviation in which both responders reject acceptable offers.

To sum up, we obtain, at the same time, three sets of extremal equilibria E_i^m $(i = 1, 2, 3)$, each of which yields the lowest payoff m to one of the players, and the main force holding together this construction is the proper transition from one extremal equilibrium to another following any deviation: if player i deviates in the role of proposer, the continuation play is another $e_i^m(\hat{\mathbf{z}}, \hat{\mathbf{p}}) \in E_i^m$, where the specification of $\hat{\mathbf{z}}$ and $\hat{\mathbf{p}}$ depends on the rejected proposal; if player j (k resp.) deviates effectively as a responder, the continuation play becomes $e_j^m(\mathbf{z}, \mathbf{p})$ ($e_k^m(\mathbf{z}, \mathbf{p})$ resp.). Since m is the lower bound of the equilibrium payoffs, no one has a profitable one-step deviation.

Project the equilibrium set E_1^m onto the unit simplex of feasible payoff vectors, we obtain the interval between points $A = (m, M, 1 - M - m)$ and $A' = (m, 1 - M - m, M)$. Similarly, E_2^m corresponds to the interval between $B = (1 - M - m, m, M)$ and $B' = (M, m, 1 - M - m)$, and E_3^m corresponds to the interval between $C = (M, 1 - M - m, m)$ and $C' = (1 - M - m, M, m)$. Let $\mathcal{U}$ be the convex hull of the points A, A', B, B', C and C' (see Figure 1). Our main result is that any payoff vector in $\mathcal{U}$ can be supported in an equilibrium.

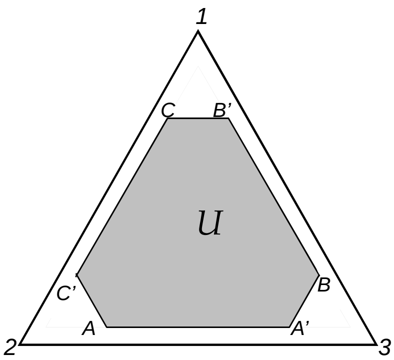

Figure 1. The Set of Equilibrium Payoffs

Proposition 1 *For any payoff vector* $\mathbf{u} \in \mathcal{U}$*, there is an equilibrium in the three-player Baron-Ferejohn bargaining game, in which player i's payoff is* u_i*.*

The key step of the proof is to show that any $\mathbf{u} \in \mathcal{U}$ can be achieved in a well-defined equilibrium configuration. Then, the specification of the strategy profile will be similar to that of $e_i^m(\mathbf{z}, \mathbf{p})$. Players follow the equilibrium configuration and bargaining is concluded immediately. Deviation is deterred by the fear of switching to an extremal equilibrium in which the deviator's payoff is minimized.

Remark 1 *The equilibrium configuration involves randomization of the proposer in choosing with whom to form the winning coalition. The randomization process does not need to be observable as the construction of continuation equilibria depends only on the proposal.*

Remark 2 *In the case with five or more players, the set of the equilibrium payoffs coincides with the feasible set. Here,* $\mathcal{U}$ *is a proper subset of the feasible set for any* $\delta < 1$*. Nevertheless, as* δ *tends to* 1*,* $\mathcal{U}$ *evolves monotonically towards the entire feasible set.*

Remark 3 *More generally, if we consider bargaining among $N \geq 3$ players with a q-quota $(2 \leq q \leq N)$ voting rule, the multiple equilibria constructed in Baron and Ferejohn (1989) extend to all cases with $N \geq q \geq 3$, whereas our equilibria extend to cases with $N > q = 2$.*

3. Majoritarian Bargaining with Sealed Offers

This section considers the majoritarian bargaining game with sealed offers, which combines the main features of the Baron-Ferejohn model and the Baliga-Serrano model. For expositional ease we still focus on the three-player case, and the extension to the N-player case is briefly discussed at the end of this section. In each period, the randomly recognized proposer makes offers in sealed envelopes, that is, each responder observes only the share proposed for him. Responses to offers are made simultaneously, and the proposal passes if at least one responder accepts the offer. If both responders reject, the game proceeds to the next period with previous offers being revealed.

The key feature of the bargaining game with sealed offers is that each responder's information set is never singleton, and his response can only depend on his own offer, not the entire proposal as in the original Baron-Ferejohn model. Upon receiving his own offer, a responder forms a belief about another responder's offer and response, and chooses his own optimal response, taking into consideration the continuation equilibrium. Therefore, we shall work with perfect Bayesian equilibrium (henceforth PBE). This equilibrium notion consists of two elements: (i) players' actions are sequentially rational given their beliefs, and (ii) players update their beliefs using Bayes' rule whenever possible.

In Section 2, we focus on the equilibria in which only one responder is offered a positive share, but it is easy to see that there are also equilibria in which all parties receive positive shares. With sealed offers, an immediate observation is that at most two players (the proposer and one responder) receive positive shares in any equilibrium outcome. In other words, only the minimal winning coalition can form in equilibrium.

Next we show that after simple modification, the equilibria constructed in Section 2 will fit in the current model. Again, denote a PBE by $e(\mathbf{z}, \mathbf{p})$. In the equilibrium configuration, player i makes an offer of z_i to either player j or k with probability p_j^i (p_k^i resp.), and makes an empty offer to the other player. Since a player's response can only depend on his own offer, not the entire proposal, we need to modify the acceptance rule as follows: player j (k resp.) accepts his offer if and only if it is no less than z_i.

Given the acceptance rule, it is sequentially rational for the proposer to follow the equilibrium configuration. To see that the acceptance rule is also sequentially rational, we now specify the responder's beliefs: (i) when player j receives the equilibrium offer z_i, his information set is on the equilibrium path, in which case Bayesian updating leads him to believe that the other responder has been offered nothing; (ii) when player j receives an empty offer, his belief does not matter as it is weakly dominant to reject; (iii) when player j receives a positive offer different from z_i, the play is clearly off the equilibrium path, in which case Bayes' rule puts no restriction on his belief and player j again believes that the other responder has been offered nothing. The simple belief specified here is consistent with Bayes' rule. Based on this belief, player j always views his own response as being crucial, thus it is sequentially rational for him to accept a proposal as long as his payoff is no less than that from the continuation equilibrium, which is constructed in the same way as in Section 2.[8]

Hence, we have the following proposition.

Proposition 2 *In the three-player majoritarian bargaining game with sealed offers, any payoff vector* $\mathbf{u} \in \mathcal{U}$ *can be supported in a PBE.*

With five or more players, Baron and Ferejohn's multiple equilibria do not extend to the model with sealed offers. It is because players' acceptance rules in their equilibria are not monotone, that is, a responder may reject an off-equilibrium proposal that gives him a share greater than that specified in the equilibrium proposal. With sealed offers, this cannot happen in any equilibrium.

The equilibria that we construct for the three-player case can be extended to the general N-player case in a straightforward way. As in the three-player case, only the minimal winning coalition forms in equilibrium, and those excluded from the winning coalition receive empty offers. The equilibrium strategy profile specifies the winning coalition that should be formed conditional on who is recognized as the proposer, and what offers should be made to the players in the winning coalition. The responders adopt monotone acceptance rule. Specifically, for every player in the winning coalition, the acceptance threshold is simply his equilibrium offer; for every player excluded from the winning coalition, let his threshold be the highest offer that should be made to the players in the winning coalition. It is easy to see that the upper bound of each player's equilibrium payoffs must be less than 1, but the lower bound can be 0 when the discount factor is sufficiently large.

Appendix

Proof of Lemma 1.

The following three inequality conditions should be satisfied by M and m:

$$\begin{aligned}
&\text{(i)} \quad 0 &\leq\; & m(\delta) \leq M(\delta) \leq 1 \\
&\text{(ii)}\; M(\delta) &\leq\; & \frac{1}{3}\left[1-\delta m(\delta)\right]+\frac{2}{3}\delta M(\delta) \\
&\text{(iii)}\; m(\delta) &\geq\; & \frac{1}{3}\left[1-\delta M(\delta)\right].
\end{aligned}$$

Condition (i) is self-explanatory. In any equilibrium, a player will always reject a proposal that gives him less than $\delta m(\delta)$ and accept one that gives him more than $\delta M(\delta)$. Hence, the best outcome that a player can receive is: (a) when he is recognized as the proposer (with probability 1/3), at least one of the other two players is willing to accept $\delta m(\delta)$, and (b) when another player proposes, he will be offered $\delta M(\delta)$ with probability 1. This establishes

[8] A crucial feature of the sealed-offer model is that after each period the offers are revealed to all players, and thus, the construction of the continuation equilibrium can depend on the entire rejected proposal.

Condition (ii). Finally, the worst outcome that a player can receive is that he has to offer $\delta M(\delta)$ to one of the responders when in the role of proposer, and he will not receive a positive offer from other players. Thus, Condition (iii) should also be satisfied. The three conditions together lead to

$$M(\delta) \leq \frac{3-\delta}{9-6\delta-\delta^2} \quad \text{and} \quad m(\delta) \geq \frac{3-3\delta}{9-6\delta-\delta^2}.$$

Proof of Lemma 2.

Let $\alpha_j = x_j/(x_j+x_k)$ and $\alpha_k = x_k/(x_j+x_k)$. Observe that by construction, we have

$$\widehat{\pi}_j + \widehat{\pi}_k = \frac{\delta M}{3} + \frac{2}{3} = 1 - m > M = \frac{2\delta M}{3} + \frac{1-\delta m}{3}$$

and $x_j < \alpha_j \delta M$. Below we show that $\delta\widehat{\pi}_j > x_j$ ($\delta\widehat{\pi}_k > x_k$ similarly) in all possible cases:

Case 1. $\alpha_j \leq \delta m$. It must be that $\alpha_k = 1 - \alpha_j > \delta M$ because $M + m < 1$. Then,

$$\widehat{\pi}_j = \frac{\alpha_j \delta M}{3} + \frac{\delta m}{3} + \frac{1-\delta M}{3} > \alpha_j\left(\frac{\delta M}{3} + \frac{2}{3}\right) = \alpha_j(\widehat{\pi}_j + \widehat{\pi}_k) > \alpha_j M > x_j/\delta.$$

Case 2. $\delta m \leq \alpha_j \leq \delta M$. It must be that $\alpha_k > \delta m$.

(2a) If $\alpha_k \leq \delta M$, then

$$\widehat{\pi}_j = \frac{\alpha_j \delta M}{3} + \frac{\alpha_j}{3} + \frac{1-\alpha_k}{3} = \alpha_j(\widehat{\pi}_j + \widehat{\pi}_k) > x_j/\delta.$$

(2b) If $\alpha_k > \delta M$, then

$$\widehat{\pi}_j = \frac{\alpha_j \delta M}{3} + \frac{\alpha_j}{3} + \frac{1-\delta M}{3} > \alpha_j\left(\frac{\delta M}{3} + \frac{2}{3}\right) > x_j/\delta.$$

Case 3. $\alpha_j > \delta M$.

(3a) If $\alpha_k < \delta m$, then

$$\widehat{\pi}_j = \frac{\alpha_j \delta M}{3} + \frac{\delta M}{3} + \frac{1-\delta m}{3} > \alpha_j\left(\frac{2\delta M}{3} + \frac{1-\delta m}{3}\right) = \alpha_j M > x_j/\delta.$$

(3b) If $\delta m \leq \alpha_k \leq \delta M$, then

$$\widehat{\pi}_j = \frac{\alpha_j \delta M}{3} + \frac{\delta M}{3} + \frac{1-\alpha_k}{3} > \alpha_j\left(\frac{2\delta M}{3} + \frac{1}{3}\right) > \alpha_j M > x_j/\delta.$$

(3c) If $\alpha_k > \delta M$ (possible when δ is sufficiently small), then $1 - \delta M > \alpha_j$ and

$$\widehat{\pi}_j = \frac{\alpha_j \delta M}{3} + \frac{\delta M}{3} + \frac{1-\delta M}{3} > \alpha_j\left(\frac{2\delta M}{3} + \frac{1}{3}\right) > \alpha_j M > x_j/\delta.$$

Proof of Proposition 1.
Note that we have constructed the extremal equilibrium corresponding to each vertex of $\mathcal{U}$. It suffices to show that for any two equilibrium configurations $e(\mathbf{x},\mathbf{p})$ and $e(\mathbf{y},\mathbf{q})$, there is another equilibrium configuration $e(\mathbf{z},\mathbf{s})$ such that for any $\lambda \in [0,1]$,

$$u_i(e(\mathbf{z},\mathbf{s})) = \lambda u_i(e(\mathbf{x},\mathbf{p})) + (1-\lambda) u_i(e(\mathbf{y},\mathbf{q})).$$

Denote as $c_j^i(e(\mathbf{x},\mathbf{p})) \equiv p_j^i x_i/3$ the contribution of player i to player j's payoff in $e(\mathbf{x},\mathbf{p})$. Player i's payoff in $e(\mathbf{x},\mathbf{p})$ can be rewritten as:

$$u_i(e(\mathbf{x},\mathbf{p})) = \left[1 - c_j^i(e(\mathbf{x},\mathbf{p})) - c_k^i(e(\mathbf{x},\mathbf{p}))\right] + c_i^j(e(\mathbf{x},\mathbf{p})) + c_i^k(e(\mathbf{x},\mathbf{p})).$$

The way that we construct $e(\mathbf{z},\mathbf{s})$ is to ensure

$$c_j^i(e(\mathbf{z},\mathbf{s})) = \lambda c_j^i(e(\mathbf{x},\mathbf{p})) + (1-\lambda) c_j^i(e(\mathbf{y},\mathbf{q}))$$

for each i and j. To achieve this, we need two conditions for each player i:

$$\begin{cases} s_j^i z_i = \lambda p_j^i x_i + (1-\lambda) q_j^i y_i \\ \left(1-s_j^i\right) z_i = \lambda\left(1-p_j^i\right) x_i + (1-\lambda)\left(1-q_j^i\right) y_i \end{cases}$$

by which we obtain

$$z_i = \lambda x_i + (1-\lambda) y_i \quad \text{and} \quad s_j^i = \frac{\lambda p_j^i x_i + (1-\lambda) q_j^i y_i}{\lambda x_i + (1-\lambda) y_i}.$$

Observe that $z_i \in [\delta m, \delta M]$ and $s_j^i \in [0,1]$, thus $e(\mathbf{z},\mathbf{s})$ is well-defined. It is easy to verify that with this configuration, we have $u_i(e(\mathbf{z},\mathbf{s})) = \lambda u_i(e(\mathbf{x},\mathbf{p})) + (1-\lambda) u_i(e(\mathbf{y},\mathbf{q}))$.

References

Banks, J. S. and J. Duggan (2000) "A Bargaining Model of Collective Choice" *American Political Science Review* **94**, 73-88.

Baliga, S. and R. Serrano (1995) "Multilateral Bargaining with Imperfect Information" *Journal of Economic Theory* **67**, 578-589.

Baron, D. P. and J. A. Ferejohn (1989) "Bargaining in Legislature" *American Political Science Review* **83**, 1181-1206.

Baron, D. P. and E. Kalai (1993) "The Simplest Equilibrium of a Majority-Rule Division Game" *Journal of Economic Theory* **61**, 290-301.

Eraslan, H. (2002) "Uniqueness of Stationary Equilibrium Payoffs in the Baron-Ferejohn Model" *Journal of Economic Theory* **103**, 11-30.

Eraslan, H. and A. Merlo (2002) "Majority Rule in a Stochastic Model of Bargaining" *Journal of Economic Theory* **103**, 31-48.

Herings, P.J.J., A. Meshalkin and A. Predtetchinski (2013) "Subgame Perfect Equilibria in Majoritarian Bargaining" Working Paper, Maastricht University.

Herrero, M. (1985) "A Strategic Bargaining Approach to Market Institutions" Ph.D. Thesis, London School of Economics.

Osborne, M. and A. Rubinstein (1990) *Bargaining and Markets*, Academic Press: San Diego.

Sutton, J. (1986) "Non-cooperative Bargaining Theory: An Introduction" *Review of Economic Studies* **53**, 709-724.