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# A note on Cournot equilibrium with positive price

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## Abstract

Consider an oligopoly in which firms compete in quantity, the market inverse demand is strictly decreasing (on the set of quantities for which the price is positive), twice differentiable and log-concave, and each of the firms has nondecreasing, twice differentiable cost of production (not necessarily convex). We extend previous literature on the existence of Cournot equilibrium by showing that, under additional mild assumptions, Cournot equilibrium with positive price is unique. This also holds if the costs are piecewise differentiable, nondecreasing, and convex with a finite number of kinks. Furthermore, if at least one firm incurs positive cost whenever the industry aggregate output implies zero market price, then the equilibrium is unique and the corresponding price is positive.

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# 1 Introduction

We study the existence and uniqueness of Cournot equilibrium. Our conditions are weaker than the ones appearing in the literature, at the expense of multiple equilibrium points. However, we show that among these equilibrium points only one has a positive price. If we add the requirement that at least one firm produces at a positive cost whenever the industry aggregate output implies zero market price, then the equilibrium is unique and the equilibrium price is positive. Moreover, existence and uniqueness of equilibrium with a positive price is preserved if costs are piecewise differentiable, nondecreasing, and convex with a finite number of kinks. Let the market inverse demand be given by $P(Q) = \max\{0, \widehat{P}(Q)\}$. To show the existence and uniqueness of Cournot equilibrium with positive price, we require that (i) $\widehat{P}(\cdot)$ is log-concave, strictly decreasing, and twice differentiable, (ii) the cost function of each firm is nondecreasing and twice differentiable, and (iii) $\widehat{P}'(Q) - c_j''(q_j) < 0$ for all industry aggregated output $Q$ and all individual firm's output $q_j$, where $c_j(\cdot)$ is firm $j$'s cost function. If in addition, there exists a firm $j$ such that $c_j(q_j) > 0$ whenever $P(Q) = 0$, then Cournot equilibrium is unique and the market price is positive.

Several other papers have addressed the uniqueness of Cournot equilibrium. We weaken the assumptions that profits are concave and marginal costs are strictly positive (Szidarovszky and Yakowitz (1977), Gaudet and Salant (1991), and Van Long and Soubeyran (2000)). Furthermore, we do not require convex costs as in Szidarovszky and Yakowitz (1977) and Van Long and Soubeyran (2000). Kolstad and Mathiesen (1987) provide necessary and sufficient conditions for the existence of a unique Cournot equilibrium. Some of their assumptions, however, are not globally stated and they require certain properties to hold at all equilibrium points. Their regularity conditions require (i) the Jacobian of the marginal profits for the firms with positive output to be nonsingular at every equilibrium point, and (ii) all Cournot equilibria to be non degenerate (that is, firms producing zero at equilibrium, have marginal costs greater than the equilibrium price).

Our result still holds even if "smooth" requirements on costs are dropped. This observation is motivated by the conspicuous relevance of non differentiable costs to applied problems. To the best of our knowledge the only paper addressing the issues of existence and uniqueness of Cournot equilibrium with non differentiable costs is Szidarovszky and Yakowitz (1982). They require, however, concavity of the inverse demand. Our result strengths theirs by allowing more general inverse demands.

# 2 Setup and Results

Let $N = \{1, \ldots, n\}$ be the set of firms producing a homogeneous good in a market with inverse demand given by $P(Q) = \max\{0, \widehat{P}(Q)\}$. For each $j \in N$, let $c_j(\cdot)$ be the cost function of firm $j$. The profit of firm $j$ when producing $q_j$ units of the good is

$$\Pi_j(q_1, \ldots, q_n) = q_j P(Q) - c_j(q_j). \tag{1}$$

where $Q = \sum_{i=1}^{n} q_i$. Firms are assumed to choose production levels simultaneously and independently. Given a profile $(q_i)_{i \in N}$ of outputs, each firm receives a payoff given by (1).

We refer to this game as *Cournot game* and denote it by $G$. A *Cournot equilibrium* is a Nash equilibrium of $G$.

We make the following assumptions:

**Assumption 1.** *$\widehat{P}(\cdot)$ is a strictly decreasing, twice differentiable log-concave function and* $\lim_{Q\to\infty} P(Q) = 0$.

**Assumption 2.** *For each $j \in N$, $c_j(\cdot)$ is twice differentiable with $c'_j(q) \geq 0$ for all $q \in \mathbf{R}_+$.*

**Assumption 3.** *For each $j \in N$ and $(q, Q) \in \mathbf{R}^2_+$, $\widehat{P}'(Q) - c''_j(q) < 0$.*

*Remark* 1. Under Assumption 1 there exists a unique $0 < Q^0 \leq +\infty$ such that $\widehat{P}(Q^0) = 0$. As noted in Amir (1996) the log-concavity assumption relaxes the so called Novshek's condition, $P'(Q) - QP''(Q) \leq 0$ for all $Q \in [0, Q^0)$ (Novshek (1985)). Assumption 3 is standard in the literature and can be interpreted as relaxing the requirement that costs are convex.

Without loss of generality we normalize $c_j(0) = 0$ and assume $P(0) > c'_j(0)$ for all $j \in N$. For instance, if the last inequality does not hold for some firm, then by Assumption 3 this firm will optimally produce zero.

The following is a useful observation.

**Lemma 1.** *Suppose assumptions 1-3 hold. Then, for each $j \in N$ and each profile of outputs $q_{-j} = (q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_n)$, the function given by $q_j\widehat{P}(Q) - c_j(q_j)$ is quasiconcave in $q_j$.*

*Proof.* Observe that, under assumptions 1-3,

$$f(q_j) \equiv \frac{c'_j(q_j) - q_j\widehat{P}'(Q)}{\widehat{P}(Q)}$$

is strictly increasing. Since $q_j\widehat{P}(Q) - c_j(q_j)$ is increasing whenever $f(q_j) \leq 1$ and decreasing whenever $f(q_j) \geq 1$, it follows that it is single-peaked and therefore quasiconcave. □

The next proposition is the main result of this paper.

**Proposition 1.** *Suppose assumptions 1-3 hold. Then $G$ has a unique Cournot equilibrium with positive price.*

Before turning to the proof of the above proposition, we illustrate the importance of Assumption 3 and prove an auxiliary result.

*Example* 1. Suppose $N = \{1, 2\}$ and $P(Q) = \max\{0, 10 - Q\}$. Define the strictly decreasing function $f : \mathbf{R}_+ \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1 + 4(1-x)^2 & \text{if } x \leq 1 \\ 1 - 4(x-1)^2 & \text{if } x > 1. \end{cases}$$

Each firm $j \in N$ produces $q_j$ units of output at cost

$$c(q_j) = 10q_j - q_j^2 - \int_0^{q_j} f(x)\, dx.$$

It can be easily checked that all assumptions, except Assumption 3, hold. Clearly, $(q_1, q_2) = (1, 1)$, $(q_1, q_2) = (3/4, 5/4)$, and $(q_1, q_2) = (5/4, 3/4)$ are equilibrium points, each of them associated with a positive price.

Let $\widehat{G}$ be the *auxiliary Cournot game* having $\widehat{P}(\cdot)$ as inverse demand function. The proof of Proposition 1 relies on

**Proposition 2.** *The game $\widehat{G}$ has a unique equilibrium $q^* = (q_i^*)_{i\in N}$ and $\widehat{P}(Q^*) > 0$.*

*Proof.* For each $j \in N$ and every $(q_j, Q) \in \mathbf{R}_+^2$, define

$$F_j(q_j, Q) = \widehat{P}(Q) + q_j \widehat{P}'(Q) - c_j'(q_j).$$

In addition, for each $j \in N$, let

$$S_j = \big\{Q \in \mathbf{R}_+ : \exists q_j \geq 0 \text{ such that } F_j(q_j, Q) = 0\big\}.$$

It can be shown that $S_j = [0, Q_j]$ for some $Q_j \leq +\infty$ and that $F_j(\cdot, Q)$ is strictly decreasing on $S_j$. It follows that for each $Q \in S_j$ the solution to $F_j(q_j, Q) = 0$ is unique. For each $Q \in S_j$, let $q_j(Q)$ denote this solution and put $q_j(Q) = 0$ for $Q \notin S_j$. By the Implicit Function Theorem and assumptions 1-3, $q_j(Q)$ is decreasing on $S_j$ and continuous on $\mathbf{R}_+$.

Let $b = \sum_{i=1}^n q_i(0)$, and define $H : [0, b] \to [0, b]$ by $H(Q) = \sum_{i=1}^n q_i(Q) - Q$. Since $H(\cdot)$ is continuous, strictly decreasing, $H(0) = b \geq 0$, and $H(b) \leq 0$, there exists $Q^* \in [0, b]$ such that $H(Q^*) = 0$. That is, $\sum_{i=1}^n q_i(Q^*) = Q^*$. It can be easily checked that $q_j(Q^*)$ satisfies $F_j(q_j(Q^*), Q^*) \leq 0$. Using Lemma 1 we conclude that $q^* = \big(q_i(Q^*)\big)_{i\in N}$ is an equilibrium of $\widehat{G}$.

To see that the equilibrium is unique, suppose $\tilde{q} = (\tilde{q}_i)_{i\in N}$ is also an equilibrium and define the set $J = \{j \in N : \tilde{q}_j > 0\}$. For each $j \in J$, $F_j(\tilde{q}_j, \widetilde{Q}) = 0$ and $\widetilde{Q} \in S_j$. Therefore, $q_j(\widetilde{Q}) = \tilde{q}_j$. If $j \notin J$, $\tilde{q}_j = 0$ and $F_j(0, \widetilde{Q}) \leq 0$. Since $F_j(\cdot, \widetilde{Q})$ is strictly decreasing, $F_j(q_j, \widetilde{Q}) < 0$, for all $q_j > 0$. Thus, $\widetilde{Q} \geq Q_j$ and $q_j(\widetilde{Q}) = 0 = \tilde{q}_j$. It then follows that, for each $j \in N$, $q_j(\widetilde{Q}) = \tilde{q}_j$ and $H(\widetilde{Q}) = 0 = H(Q^*)$. Since $H(\cdot)$ is strictly decreasing, it must be $\widetilde{Q} = Q^*$ and $\tilde{q}_j = q_j(\widetilde{Q}) = q_j(Q^*) = q_j^*$, for each $j \in N$.

One can easily verify that $\widehat{P}(Q^*) > 0$. This observation concludes the proof of the proposition. □

We can now prove Proposition 1.

*Proof of Proposition 1.* Let $q^*$ be the unique equilibrium of $\widehat{G}$. Let $q_j \geq 0$. Suppose first that $q_j + \sum_{i\neq j} q_i^* \leq Q^0$. Since $\widehat{P}(Q^*) > 0$, $\Pi_j(q^*) = \widehat{\Pi}_j(q^*) \geq \widehat{\Pi}_j(q_j, q_{-j}^*) = \Pi_j(q_j, q_{-j}^*)$. Suppose next that $q_j + \sum_{i\neq j} q_i^* > Q^0$. Since $Q^* < Q^0$, it follows that $q_j > q_j^*$ and, by Assumption 2, $c_j(q_j) \geq c_j(q_j^*)$. Thus, $\Pi_j(q^*) \geq -c_j(q_j) = \Pi_j(q_j, q_{-j}^*)$. That is, $q^*$ is an equilibrium of the Cournot game.

Let us next prove uniqueness. Assume $\tilde{q} = (\tilde{q}_i)_{i\in N}$ is also an equilibrium of $G$ and $P(\widetilde{Q}) > 0$. We will show that $\tilde{q}$ is also an equilibrium of $\widehat{G}$. Let $q_j \geq 0$. Suppose first that $q_j + \sum_{i\neq j} \tilde{q}_i \leq Q^0$. Since $P(\widetilde{Q}) > 0$, $\widehat{\Pi}_j(\tilde{q}) = \Pi_j(\tilde{q}) \geq \Pi_j(q_j, \tilde{q}_{-j}) = \widehat{\Pi}_j(q_j, \tilde{q}_{-j})$. Suppose next that $q_j + \sum_{i\neq j} \tilde{q}_i > Q^0$. Then $q_j > \tilde{q}_j$, and $c_j(q_j) \geq c_j(\tilde{q}_j)$. Hence, $\widehat{\Pi}_j(\tilde{q}) \geq \widehat{\Pi}_j(q_j, \tilde{q}_{-j})$. That is, $\tilde{q}$ is an equilibrium of $\widehat{G}$. By Proposition 2, $\tilde{q} = q^*$, and the proof of Proposition 1 is complete. □

*Remark* 2. The result is intact even if Assumption 2 does not hold for some firms, provided that for those firms costs are piecewise differentiable, nondecreasing, and convex with a finite number of kinks.[1] (Of course, in this case, Assumption 3 does not hold either.) Let us outline the proof of this claim. Suppose that, for each $j \in N$, $c_j(\cdot)$ satisfies the latter requirements. Equilibrium existence follows by Novshek (1985). Suppose there are two equilibrium points with positive price. Starting with firm 1, and proceeding firm by firm, one at a time, "smooth" the firm's cost such that assumptions 2 and 3 hold and the firm's best response is unchanged at each equilibrium.[2] Clearly, at each step the equilibrium points are preserved. However, after the process terminates, Proposition 1 applies, which leads to a contradiction. Szidarovszky and Yakowitz (1982) prove uniqueness of Cournot equilibrium allowing for non differentiable costs. However, the authors require concavity of inverse demand. This remark, together with Proposition 3 below, relaxes this condition.

As the next result shows, equilibrium uniqueness can be obtained with an additional assumption.

**Proposition 3.** *Suppose that assumptions 1-3 hold and there exists $j \in N$ such that $c_j(q_j) > 0$ whenever $P(Q) = 0$. Then $G$ has a unique equilibrium $q^*$ and $P(Q^*) > 0$.*

Observe that uniqueness is obtained without the standard assumption that marginal costs are strictly positive. Also note that the additional assumption is essential. For instance, if $N = \{1, 2, 3\}$, $\widehat{P}(Q) = 2 - Q$, and $c_j(q_j) = 0$ for $q_j \in [0, 1]$ and $c_j(q_j) = q_j - 1$ for $q_j > 1$, then $(q_i^*)_{i \in N} = (1, 1, 1)$ is a Cournot equilibrium, but $P(Q^*) = 0$.

*Proof of Proposition 3.* Suppose to the contrary that $\tilde{q}$, with $P(\widetilde{Q}) = 0$, is an equilibrium of $G$. Hence, it must be $\Pi_j(\tilde{q}) \geq \Pi_j(0, \tilde{q}_{-j})$ for each $j \in N$. However, this inequality implies that, for each $j \in N$, $c_j(\tilde{q}_j) \leq 0$, a contradiction. □

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[1] Observe that, under the listed assumptions, for each $j \in N$, $q_j \widehat{P}(Q) - c_j(q_j)$ is still quasiconcave in $q_j$.

[2] Let $q^*$ be an equilibrium point and suppose that, for some $j \in N$, $(q_j^*, c_j(q_j^*))$ is at a kink of the graph of $c_j(\cdot)$. Then, for $\varepsilon > 0$ small enough, we replace $c_j(\cdot)$ by a twice differentiable function $\tilde{c}_j(\cdot)$ satisfying the following properties: (i) $\tilde{c}_j(q_j) = c_j(q_j)$ for all $q_j \notin [q_j^* - \varepsilon, q_j^* + \varepsilon]$, and (ii) $\tilde{c}_j'(q_j^*) = \mathrm{MR}_j(q_j^*, q_{-j}^*)$, where the constant $\mathrm{MR}_j(q_j^*, q_{-j}^*)$ is firms $j$'s marginal revenue at the equilibrium point $q^*$.

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