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Aggregating quasi-transitive preferences: a note

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Abstract

To examine the consequences of allowing individual to violate full rationality in collective decision making, this article discusses the possibility of aggregating quasi-transitive preferences in the Arrovian framework. Quasi-transitive valued aggregating functions are discussed and characterised. A characterisation of the weak Pareto extension rule is also achieved as a corollary.

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1. Introduction

A social aggregating function is a mapping from profiles of admissible individual preferences to a social preference over the set of alternatives. The original social welfare function focuses exclusively on aggregating individual transitive preferences into a transitive social preference. Arrow (1963) demonstrates in his seminal work that a social welfare function satisfying unrestricted domain, the Independence of Irrelevant Alternatives (IIA), and the Weak Pareto principle has to be dictatorial. Many authors focus on relaxing one of these axiomatic conditions. One particularly active branch has been to relax the collective rationality condition, namely the transitive valued property of the social welfare function. The result of relaxing transitive valued to quasi-transitive valued can be expressed in terms of the existence of an oligarchy, see Gibbard (1969) and Weymark (1984). Further relaxing of collective rationality to acyclic valued aggregating function is also fruitful. Blau and Deb (1977) and Kelsey (1985) show the existence of vetoer and veto group respectively.

Domain restriction is another typical rout in escaping Arrow's impossibility theorem. It is well known that by restricting permitted individual preferences to single peak domain, simple majority rule satisfies all other Arrovian axioms. In this article, instead of restricting individual preferences, we examine the consequences of expanding the Arrovian domain to allow individuals to have quasi-transitive preferences. Quasi-transitivity only requires the asymmetric part of the binary relation to be transitive. This version of consistent condition gains more support theoretically as well as experimentally (Quiin, 1990). This note examines the implication of individual intransitive indifferences on the preference aggregation.

This article is structured as follows. Section 2 introduces the framework. Section 3 discusses basic properties of quasi-transitive preferences. Section 4 presents and discusses results with quasi-transitive valued aggregating function. An axiomatization of the Weak Pareto extension rule is also provided here. Section 5 concludes.

2. framework

We begin with the space of social alternatives X and the set of individuals N. Let $n = |N|$ and $m = |X|$ denote the cardinality of N and X respectively. Throughout this article, n is assumed no less than 2 whereas m is assumed greater or equal to 4. Both n and m are assumed to be finite. A binary relation R, which is a subset of $X \times X$, is *transitive* if for all $x, y, z \in X$, $xRy \ \& \ yRz \Rightarrow xRz$. A binary relation is *quasi-transitive* if its asymmetric part P is transitive. For any integer t, a binary relation R contains cycle of order t if for some $x_1, x_2, \ldots, x_t$ in X we have $x_1Rx_2R\ldots Rx_tRx_1$. A binary relation is said to be acyclic if its asymmetric part contains no cycle of any order. Throughout this article, both individual and collective preferences are assumed to satisfy certain richness conditions, namely *Reflexivity* and *Completeness*[1]. Denote the set of all *transitive*, *quasi-transitive* and *acyclic* preferences on X as $O(X)$, $Q(X)$, and $A(X)$ respectively. For simplicity, we will use O, Q, and A without referring to the alternative space when there is no ambiguity.

Social aggregating functions generate collective binary relations from an n-tuples of individual binary relations like $p = (R_1, R_2, \ldots, R_n)$. A social aggregating function is said to be *transitive valued* (respectively, *quasi-transitive valued*, *acyclic valued*) if $f(p)$ is *transitive*

[1] *Reflexivity* requires for every $x \in X$ xRx. *Completeness* requires for any $x, y \in X$ xRy or yRx.

(respectively, *quasi-transitive*, *acyclic*) for any permitted profile p. An Arrovian social welfare function is a *transitive valued* social aggregating function with the domain restricted to O^n. In this article, we study social aggregating functions on the domain Q^n.

As an auxiliary step of introducing axiomatic properties, we define the power structure of aggregating functions. We denote the social binary relation generated by the aggregating function by $R = f(p)$ without subscript. P and I denote the asymmetric and symmetric part respectively. A coalition $L \subseteq N$ is *decisive* for x against y, denoted by $L \in D(x,y)$, if xP_iy for every $i \in L$ implies xPy socially. If $L \in D(x,y)$ for all $x, y \in X$ then we say L is a decisive group. A coalition $L \subseteq N$ has veto power for x against y, denoted by $L \in V(x,y)$, if xP_iy for every $i \in L$ implies xRy socially. If $L \in V(x,y)$ for all $x, y \in X$ then we say L is a veto group (vetoer if L is a singleton set). An oligarchy is a decisive coalition in which every member is a vetoer. A social aggregating function is oligarchical if there is an oligarchy coalition.

Similarly, $L \subseteq N$ is said to be *indifference decisive* for x against y, denoted by $L \in ID(x,y)$, if xI_iy for every $i \in L$ implies xIy socially. L is an indifference decisive group if $L \in ID(x,y)$ for all $x, y \in X$. Further, a coalition $L \subseteq N$ is *almost indifference decisive* for x against y, denoted by $L \in AID(x,y)$, if xI_iy for every $i \in L$ and xP_jy for every $j \notin L$ implies xIy socially.

A coalition $L \subseteq N$ has *strong veto* power for x against y, denoted by $L \in SV(x,y)$, if xR_iy for every $i \in L$ implies xRy socially. L is a strong veto group if $L \in SV(x,y)$ for all $x, y \in X$. Indifference decisive says the coalition can impose its indifference preference on society. Strong veto says the coalition can prevent strict preference against the alternative they believe to be at least as good. It is straightforward to check that decisive implies veto whereas strong veto implies both indifference decisive and veto.

We are now ready to introduce properties on aggregating functions. A social aggregating function satisfies *Weak Pareto* (respectively, *Pareto Indifference*, *URR*) if N is a decisive group (respectively, indifference decisive group, strong veto group). It is clear from the definition that URR implies Pareto Indifference whereas other axioms are independent.

A social aggregating function satisfies *independence of irrelevant alternatives (IIA)* if for any $x, y, \in X$ and any profiles p, p', $xR_iy \Leftrightarrow xR'_iy$ implies $xRy \Leftrightarrow xR'y$. *Neutrality* requires that for any $x, y, z, w \in X$ and any profile p, p', if $xR_iy \Leftrightarrow zR'_iw$ then $xRy \Leftrightarrow zR'w$. *Anonymity* requires that for any permutation $\sigma : N \leftrightarrow N$ and any profile p, $f(p) = f(\sigma(p))$.

3. Quasi-transitive preferences

Quasi-transitivity imposes transitivity on the asymmetric part of a binary relation but put no restriction on the symmetric part. Therefore, xPy & yIz only imply xRz. In terms of preference cycles, transitivity prevents preference cycles which contains strict preferences whereas acyclicity prevents cycles consists of strict preferences alone. Quasi-transitivity lies between transitivity and acyclicity in terms of restrictions on preference cycles by preventing cycles contains zero or one indifference. In other words, it allows preference cycles of any order with at least two indifferences.

While experimental evidence is the main rationale behind allowing individuals to possess quasi-transitive preferences, the reason of imposing quasi-transitivity as a collective rationality requirement is completely different. Plott (1973) shows that a choice function satisfying the generalised Condorcet property[2] is path independent if and only if it can be rationalised by a quasi-transitive binary relation. However, a quasi-transitive valued social aggregating function on the domain of quasi-transitive preferences has an immediate implication which is rather disturbing. A social aggregating function $f\colon Q^n \to Q$ cannot satisfy *Strong Pareto Principle*[3] in general. This annoying fact can be illustrated by the following example.

Example 1.
xP_1y, yI_1z, xI_1z
xI_2y, yP_2z, xI_2z

If the society consisting of two individuals shows such preferences, then Strong Pareto gives xPy, yPz, xIz which violates quasi-transitivity. Furthermore, it has an additional implication on the power structure when *Neutrality* is imposed. We state it as a Lemma.

Lemma 1. *If an aggregating function $f\colon Q^n \to Q$ satisfies Neutrality, then for any distinct pair of alternatives $x, y \in X$ and any $L \subseteq N$, $L \in D(x,y) \cap ID(x,y) \Rightarrow L \in SV(x,y)$.*

Proof. Assume for some distinct $x, y \in X$ and $L \in D(x,y) \cap ID(x,y)$, we have to prove $L \in SV(x,y)$. This is equivalent to $\{\forall i \in L, xR_iy\} \Rightarrow xRy$. Assume $\forall i \in L, xR_iy$, consider a third alternative z such that $\forall i \in L, xR_iy, xI_iz, zP_iy$. This is possible because we only require individual preference to be quasi-transitive. Since $L \in D(x,y)$, by Neutrality we have zPy socially. Since $L \in ID(x,y)$, by Neutrality we have xIz socially. Because the aggregating function is quasi-transitive valued, we got xRy socially. Again by Neutrality, this has nothing to do with the position of z. Therefore we have $L \in SV(x,y)$ □

As Weak Pareto, Pareto Indifference, and URR are all requirements of power structure regarding the set N itself, a straightforward corollary follows.

Corollary 1. *If the aggregating function $f\colon Q^n \to Q$ satisfies Neutrality, then Weak Pareto principle and Pareto Indifference imply URR.*

One more thing to note here is that if the coalition consists of only one person, then *veto* and *indifference decisive* imply *strong veto* for this coalition. If *Neutrality* is satisfied, the reverse is also true. Proof is obvious hence omitted here.

It is well known that a social welfare function satisfies *Neutrality* if and only if it satisfies *Pareto Indifference* and IIA. For aggregating function $f\colon Q^n \to Q$, *Neutrality* still implies *Pareto Indifference* and IIA while the reverse is not necessarily true. Instead, we have the following result.

Lemma 2. *An aggregating function $f\colon Q^n \to Q$ satisfies Neutrality if it satisfies Weak Pareto principle and IIA.*

Proof. Let $f\colon Q^n \to Q$ satisfies Weak Pareto and IIA. Consider two pair of distinct alternatives $(x,y), (w,v)$ and two profiles $p = (R_1, \ldots, R_n)$ and $p' = (R'_1, \ldots, R'_n)$. Assume $xR_iy \Leftrightarrow wR'_iv$. We want to prove $xPy \Leftrightarrow wP'v$ and $xIy \Leftrightarrow wI'v$. Due to symmetry, it suffices to prove $\Rightarrow$. Since the case $(x,y) = (w,v)$ is directly implied by IIA, we assume $(x,y) \neq (w,v)$ here. Partition N into three groups according to profile p

[2]Generalised Condorcet property says that if an alternative wins every pairwise comparisons, it should be chosen when choice is made from the whole set.

[3]Strong Pareto is stronger than URR. In addition to URR, Strong Pareto requires that if at least one individual has strict preference xP_iy then society will also has strict preference xPy.

and p' with $N_1 = \{i \in N : xP_iy \ \ \& \ \ wP'_iv\}$, $N_2 = \{i \in N : xI_iy \ \ \& \ \ wI'_iv\}$, and $N_3 = \{i \in N : yP_ix \ \ \& \ \ vP'_iw\}$ respectively.

(1) We first consider the case $\{x, y\} \cap \{w, v\} = \emptyset$. If xPy, consider the following profile p''.

$i \in N_1$	$i \in N_2$	$i \in N_3$
$wP''_ixP''_iyP''_iv$	$wP''_ixI''_iyP''_iv$ wI''_iv	$yP''_ivP''_iwP''_ix$

By IIA, $xP''y$. By Weak Pareto, $wP''x$ and $yP''v$. By quasi-transitive valued, $wP''v$. By IIA $wP'v$.

Now assume xIy, then by IIA $xI''y$. Combining with $wP''x$ and $yP''v$ we have $wR''v$ by quasi-transitivity. By IIA, $wR'v$. Again, consider the following profile p'''

$i \in N_1$	$i \in N_2$	$i \in N_3$
$xP'''_iwP'''_ivP'''_iy$	$xP'''_iwI'''_ivP'''_iy$ xI'''_iy	$vP'''_iyP'''_ixP'''_iw$

Similar argument will give $vR'w$ which leads to $wI'v$ when combining with $wR'v$.

(2) The second case is when $x = w$ (hence also referred as x) and $y \neq v$. If xPy, consider profile p^*

$i \in N_1$	$i \in N_2$	$i \in N_3$
$xP^*_iyP^*_iv$	$xI^*_iyP^*_iv$ xI^*_iv	$yP^*_ivP^*_ix$

We have xP^*y by IIA and yP^*v by Weak Pareto, hence xP^*v by quasi-transitivity and $xP'v$ by IIA.

Assume xIy, we have xI^*y by IIA. Then xR^*v by quasi-transitivity and $xR'v$ by IIA.

Then consider profile p^{**} as follows.

$i \in N_1$	$i \in N_2$	$i \in N_3$
$xP^{**}_ivP^{**}_iy$	$xI^{**}_ivP^{**}_iy$ xI^{**}_iv	$vP^{**}_iyP^{**}_ix$

Similarly, we have $vR'x$. Therefore $xI'v$.

(3) The case $x \neq w$ and $y = v$ is symmetric to the second case. The proof is therefore omitted here.

(4) The last case is when $x = v$ and $y = w$. This can be achieved by considering a sequence of pairs $(x, y), (x, z), (y, z), (y, x)$ by using the results of case two and case three.

□

4. Quasi-transitive valued aggregating function

In this section, we discuss and characterise the group of aggregating function $f : Q^n \to Q$ which satisfies *IIA* and *Weak Pareto principle*. We first state the classic oligarchy result from Weymark (1984). Although the result is derived with $f : O^n \to Q$, it also apply with $f : Q^n \to Q$.

Theorem 1. *(Weymark, 1984) For any aggregating function $f : Q^n \to Q$ satisfying IIA and Weak Pareto principle, there exists a unique oligarchy.*

Extending the domain from orderings to quasi-transitive preferences has further implication on the power structure of the aggregating function: the oligarchy coalition will possess the power of strong veto. Further, there is at least one person in this oligarchy possessing indifference decisive power by himself. We prove this result through several lemmas. In light of lemma 2, we have *Neutrality* throughout this section, hence use the concept about power structure without referring to particular pair of alternatives. The following lemma says *indifference decisive* is equivalent to *almost indifference decisive* in the presence of IIA and Weak Pareto.

Lemma 3. *For an aggregating function $f: Q^n \to Q$ satisfying IIA and Weak Pareto, $L \subseteq N$ is indifference decisive if and only if it is almost indifference decisive.*

Proof. Only if part is obvious by definition. We prove the *if* part. Note that *Neutrality* is implied by IIA and Weak Pareto by Lemma 2 and in turn implies Pareto Indifference.

Step 1: We first prove if L is almost indifference decisive, then $\{\forall i \in L, xI_iy \;\; \& \;\; \forall j \in N \setminus L, xR_jy\} \Rightarrow xIy$. Assume $\forall i \in L, xI_iy \;\; \& \;\; \forall j \in N \setminus L, xR_jy$, consider a third alternative z. Individual preferences are listed in the table below.

$i \in L$	$i \in N \setminus L$
xI_iy	xR_iy
yI_iz	zP_iy
zP_ix	zP_ix

By almost indifference decisive, yIz and by Weak Pareto, zPx. By quasi-transitive valued, yRx. By Corollary 1, we have URR. By URR we got xRy since no one strictly prefer y to x. yRx and xRy then gives xIy. By IIA, it has nothing to do with the position of z. Hence, we have proved that if L is almost indifference decisive, then $\{\forall i \in L, xI_iy \;\; \& \;\; \forall j \in N \setminus L, xR_jy\} \Rightarrow xIy$.

Step 2: Assume $\forall i \in L, xI_iy$, break the rest of the people down to three groups with xIy, xPy, and yPx respectively. Consider the following profile.

$i \in L$	$i \in N \setminus L(1)$	$i \in N \setminus L(2)$	$i \in N \setminus L(3)$
xI'_iy	xI'_iy	xP'_iy	yP'_ix
zP'_iy	zP'_iy	zP'_iy	zP'_iy
xI'_iz	xI'_iz	zP'_ix	zP'_ix

By Weak Pareto, we got zPy. By step 1, we got xIz. By quasi-transitive valued, xRy. By IIA, we have xRy regardless of the position of z.

Consider a second profile as follows.

$i \in L$	$i \in N \setminus L(1)$	$i \in N \setminus L(2)$	$i \in N \setminus L(3)$
xI''_iy	xI''_iy	xP''_iy	yP''_ix
yP''_iz	yP''_iz	yP''_iz	yP''_iz
xI''_iz	xI''_iz	xP''_iz	xP''_iz

Again, by Weak Pareto, yPz. By step 1, xIz. By quasi-transitive valued, yRx. By IIA, yRx is independent of the position of z. Combining xRy and yRz we have xIy, which proved the Lemma.

□

The next lemma shows the contraction property of indifference decisiveness.

Lemma 4. *For any aggregating function $f: Q^n \to Q$ satisfying IIA and Weak Pareto, if $L \subseteq N$ is indifference decisive and $|L| \geq 2$, then $\exists L' \subset L$ which is indifference decisive.*

Proof. Consider an indifference decisive coalition L and partition it into $L_1 \cap L_2 = \emptyset$ and $L_1 \cup L_2 = L$ with the following profile.

$i \in L_1$	$i \in L_2$	$i \in N \setminus L$
xI_iy	xI_iy	yP_ix
xI_iz	zP_ix	zP_ix
yP_iz	zI_iy	yP_iz

Assume, without loss of generality, L_2 is not indifference decisive, hence not almost indifference decisive by Lemma 3. We then have $\neg zIy$. By Corollary 1, we have URR which gives yRz. Therefore we have yPz from $\neg zIy$ and yRz. By indifference decisive of L, xIy. By quasi-transitive valued, xRz. Again by URR, we have zRx which leads to xIz when combining with xRz. Therefore, L_1 is almost indifference decisive. By lemma 3, it is indifference decisive. Consequently, either L_1 or L_2 is indifference decisive. □

We are now ready to state our main theorem. It says that the unique oligarchy coalition also possesses the power of strong veto. Further, there exits an subgroup of this oligarchy in which every individuals possess the power of strong veto.

Theorem 2. *For any aggregating function $f\colon Q^n \to Q$ satisfying IIA and Weak Pareto principle, there exists a unique oligarchy $L \subseteq N$. Further, there exists a nonempty subset L' of L such that $\forall i \in L'$, $\{i\}$ is a strong Veto coalition.*

Proof. The existence of a unique oligarchy $L \subseteq N$ is guaranteed by Theorem 1. By Pareto Indifference, which implied by Neutrality and in turn implied by IIA and Weak Pareto, N is indifference decisive. By Lemma 4, there is at least one individual i such that $\{i\}$ is indifference decisive. Denote the group of individuals with this power as L'. Observe that $L' \subseteq L$ otherwise the decisiveness of L and the indifference decisiveness of $\{i\}$ will contradict each other. Further, these individuals have strong veto power because they possess veto and indifference decisive power simultaneously. □

With this theorem, we provide an axiomatization of the Weak Pareto extension rule as a corollary.

Definition 1 (Weak Pareto extension rule)**.** *The Weak Pareto extension rule is a collective choice rule (aggregating function) such that:*

$$\forall x, y \in X, xRy \Leftrightarrow \neg[\forall i \in N, yP_ix]$$

Corollary 2. *An aggregating function $f\colon Q^n \to Q$ is Weak Pareto extension rule if and only if it satisfies Anonymity, IIA, and Weak Pareto principle.*

Proof. Only if part is obvious by the definition of the Weak Pareto extension rule. We prove the *if* part. Assume $f\colon Q^n \to Q$ satisfies Anonymity, IIA, and Weak Pareto principle. By Theorem 2, there exists $i \in N$ such that $\{i\}$ is strong veto coalition. By Anonymity, everyone has strong veto power.

Assume xRy, by Weak Pareto we have $\neg[\forall i \in N, yP_ix]$.

Conversely, assume $\neg[\forall i \in N, yP_ix]$, which is equivalent to $\exists i \in N, xR_iy$ in the presence of *completeness*. Since everyone possess the power of strong veto, we got xRy. Therefore, $\forall x, y \in X, xRy \Leftrightarrow \neg[\forall i \in N, yP_ix]$. □

5. Concluding Remarks

This article shows how the the domain expansion of the aggregating function from orderings to quasi-transitive preferences affects the structure of possible quasi-transitive value aggregating rules. In general, the possible aggregating functions significantly shrink in number comparing to the case of aggregating orderings. There remains much scope in extending this work to social choice functions.

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