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Asymmetric liquidity shocks and optimal monetary policy

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Abstract

This article develops an OLG model with random relocations of agents among more-than-two islands, wherein asymmetric liquidity shocks are observed. The model exhibits suboptimality of the Friedman rule. Furthermore, it is shown that there is no room for monetary policy to improve social welfare when the number of locations is extremely large. This article then shows that the discount window policy achieves an optimal allocation.

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1 Introduction

Since the Lehman Crisis in 2008, many economists have been studying financial crises. Most of these studies have been interested in the financial-intermediation role of banks and have tried to explain the relationship between bank runs and financial crises. Especially, Champ, Smith, and Williamson (1996), Smith (2002), and Haslag and Martin (2007) have incorporated the essence of Diamond and Dybvig (1983) into an overlapping generations (OLG) model and explored implications of monetary policies on the financial crises.[1] In their model, they concentrated their attention on the Friedman rule and showed its suboptimality as one of main results.

In their model, there are two islands between which there is no communication (spatial friction). In each island, there is a single bank as a coalition of agents. Furthermore, liquidity shocks are modeled by random relocation of agents. In order to analyze a symmetric situation, the size of liquidity shocks are common between two islands. Therefore, each bank faces common liquidity shocks in the existing models. However, liquidity shocks are usually asymmetric among banks. In this article, then, we modify the existing model to allow for asymmetric liquidity shocks among banks and reexamine optimality of the Friedman rule.

This article presents an OLG model with random relocations among more-than-two islands, numbered from 0 to $J \geq 1$. The fraction π of agents of the 0-th island are assumed to be randomly selected and equally distributed to other islands. At the same time, the fraction π/J of agents of the j-th island are randomly selected and move to the 0-th island. Because of the difference in the size of movers between 0-th island and other islands, this framework describes asymmetric liquidity shocks among islands. Furthermore, this paper defines monetary equilibrium (and, as its special case, monetary steady state) precisely.

In the model, we first verify the existence of monetary steady state and provide some sufficient condition for such existence. Given such existence of monetary steady state, this article shows suboptimality of the Friedman rule. Especially, we show that the optimal money growth rate is equal to one. This is an extension of the existing result obtained by Smith (2002) and Haslag and Martin (2007) to our framework. Furthermore, it is shown that, when the number of islands diverges, there is no room for monetary policy to improve social welfare. This is because the welfare loss by the liquidity shocks becomes relatively small when the number of islands increases. Finally, we show that the pair of the discount window and the Friedman rule achieves an optimal situation.

The organization of this paper is as follows: Section 2 describes the model considered in this article. Our model is an extension of that studied per Haslag and Martin (2007). Section 3 defines a monetary equilibrium precisely. Section 4 shows the existence of at least one monetary equilibrium. Section 5 reexamines optimality of the Friedman rule. Section 6 argues the effect of increasing in the number of islands and considers the discount window policy. Concluding remarks are provided in Section 7. Proofs are provided in the Appendix.

[1]See also Schreft and Smith (2002) and Matsuoka (2011) for an OLG model with spatial frictions.

2 Ingredients of the Model

We extend an overlapping generations model with spatial separation à la Smith (2002) and Haslag and Martin (2007) by allowing the number of islands to be more than two.[2]

Time is indexed by t and runs discretely from minus infinity to plus infinity. At each date, there exists a single perishable commodity, called the consumption good. As distinct locations, more-than-two islands exist, where there exits no communication among them. Islands are indexed and we denote by $\mathscr{J}$ the set of those indexes, where $\mathscr{J} = \{0, 1, \ldots, J\}$ for some integer $J \geq 1$. Furthermore, there exists a storage technology whereby one unit stored at date t generates $x > 1$ units of the consumption good at date $t + 1$. The gross return of the storage technology, x, is a known constant.[3]

At each date, one new generation, consisting of a continuum of ex-ante identical agents with a unit mass, appears on each island and lives for two periods. Agents born at date t are *young* at date t and *old* at date $t + 1$. They aim to maximize their utility $u(c_{t+1})$ derived from consumption, $c_{t+1} \geq 0$, at the second period of their lives, whereas they are endowed with $\omega > 0$ units of the consumption good at date t. This article specifies the utility function u by $u(c) = \ln c$ for each $c > 0$.[4]

Moreover, agents in the same cohort in each island $j \in \mathscr{J}$ are ex-ante identical but learn their types, $\theta \in \Theta := \{m, n\}$, at the end of the first period of their lives. Type m agents in each island at date t, called *movers*, move to other islands at date $t+1$, whereas type n agents, called *nonmovers*, stay in the same island. It is assumed that, in each island $j \in \mathscr{J}$, movers cannot receive the return of the storage investment, whereas nonmovers can do so. Especially, we assume that movers in island 0 are equally distributed to islands $1, \ldots, J$, whereas movers in island $k \in \{1, \ldots, J\}$ move to island 0. We also assume that the fraction of movers in island 0 is given by $\pi \in (0, 1)$ and that in island $k \in \{1, \ldots, J\}$ is given by $\mu = \pi/J$. Figure 1 illustrates an example of the relocation mechanism in the case that $J = 2$.

In order to close this section, we introduce a durable and intrinsically useless object, called money, to the economy. Money is issued by the central bank and its per-capita money stock in period t is denoted by M_t, which is common to all islands except for island 0. The per-capita money stock in island 0 is assumed to be given by JM_t. The stock of money in islands $1, \ldots, J$ follows the equation, $M_t = \sigma_t M_{t-1}$ for each date t, where $\sigma_t > 0$ is the growth rate of money and chosen by the central bank. Young agents in island $k \in \{1, \ldots, J\}$ at date t receive the newly issued money, $Z_t := M_t - M_{t-1} = (\sigma_t - 1)M_t/\sigma_t$,

[2]When the number of locations is two, our model degenerates into that studied per Haslag and Martin (2007).

[3]In this article, the storage technology is explicitly assumed to be unable to be scrapped. This captures the feature that the investment cannot be liquidate prematurely.

[4]In the existing literature, it is standard to specify the utility function by the constant relative risk aversion (RRA) utility functions with coefficient of RRA lying on $(0, 1)$, not being greater than or equal to one. This is because, if the RRA aversion is greater than [equal to] 1, the model produces the counterintuitive result that bank reserves increase [are constant] when inflation increases. However, as adopted by Smith (2002), a utility function with the logarithmic form is a technical specification to make the argument simpler.

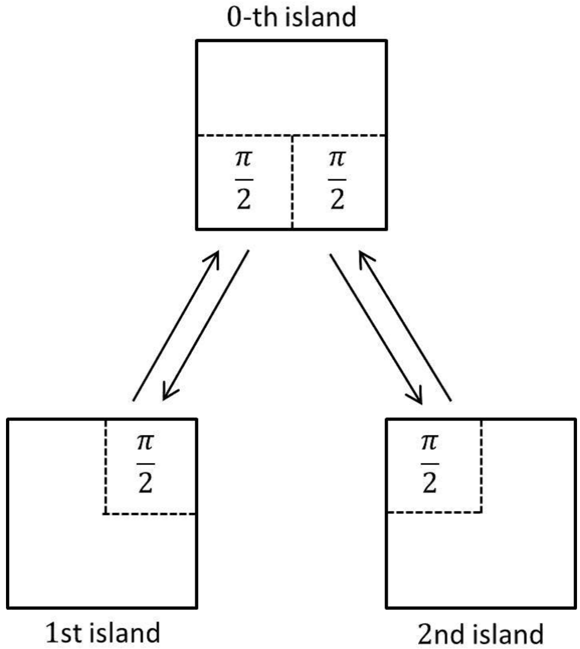

Figure 1: *An* Example of the Relocation Mechanism: The Case with $J = 2$

and those in island 0 receive JZ_t as lump-sum money tax/transfer when they are young. For each date t, we denote by P_t^j the nominal price of the consumption good in island $j \in \mathscr{J}$. Especially, this paper considers a situation where nominal prices in islands $1, \ldots, J$ coincide with each other, i.e.: $P_t^j \equiv Q_t$ for each $j \in \{1, \ldots, J\}$, because islands $1, \ldots, J$ face an identical situation. We also denote by ρ_t^0 and ρ_t the per-capita real money balance in island 0 and other islands, respectively, i.e.: $\rho_t^0 := JM_t/P_t^0$ and $\rho_t := M_t/Q_t$.

3 Definition of Equilibria

Suppose now that, at each date t, the young agents in the same island $j \in \mathscr{J}$ establish a bank and deposit all of their after-tax/transfer endowment, $\omega + \tau_t^j$, with the bank, where τ_t^j is defined by $JZ_t/P_t^0 = [(\sigma_t - 1)/\sigma_t]\rho_t^0$ if $j = 0$ and otherwise by $Z_t/P_t^j = [(\sigma_t - 1)/\sigma_t]\rho_t$. It is assumed that movers in each island $j \in \mathscr{J}$ loose their connection to their banks in their second period.

In each island $j \in \mathscr{J}$, the bank established in the island at date t enters into local spot markets at dates t and $t+1$ and is assumed to behave as if it is a price-taker. At date t, the bank chooses its portfolio in a local spot market, i.e.: the amount of investment in the storage technology and money. Its balance sheet constraint is given by

$$s_t^j + \frac{m_t^j + n_t^j}{P_t^j} \leq \omega + \tau_t^j, \tag{1}$$

where s_t^j and $(m_t^j + n_t^j)/P_t^j$ are the amount of investment in the storage technology and money, respectively.[5]

[5]Interpretations of m_t^j and n_t^j are given after Eq.(3).

The bank also chooses a schedule of returns on deposits at date $t+1$. For each $j \in \mathscr{J}$, let $d_t^{j,m}$ and $d_t^{j,n}$ denote the gross real return rates offered to movers and nonmovers, born in period t, in island j, respectively. The returns of investments of the bank must meet the total return of deposits. This is captured by

$$(\omega + \tau_t^j)d_t^{j,m}\pi_j + (\omega + \tau_t^j)d_t^{j,n}(1-\pi_j) \leq xs_t^j + \mathscr{R}_{t+1}^j \frac{n_t^j}{P_t^j} + \mathscr{E}_{t+1}^j \frac{m_t^j}{P_t^j}, \tag{2}$$

where $\mathscr{R}_{t+1}^j := P_t^j/P_{t+1}^j$,

$$\mathscr{E}_{t+1}^j := \begin{cases} \dfrac{P_t^0}{Q_{t+1}} & \text{if} \quad j = 0, \\ \dfrac{Q_t}{P_{t+1}^0} & \text{otherwise,} \end{cases}$$

and π_j is equal to π if $j = 0$ and otherwise μ. The bank also faces a *liquidity constraint.* Because movers loose their connection to their banks in their second period, they withdraw their money after they learn their types. At the beginning of the second period, therefore, the bank in island $j \in \mathscr{J}$ must have sufficient liquidity in order to meet the needs of movers:

$$(\omega + \tau_t^j)d_t^{j,m}\pi_j \leq \mathscr{E}_{t+1}^j \frac{m_t^j}{P_t^j}. \tag{3}$$

Given this liquidity constraint, one can interpret m_t^j and n_t^j as the money holdings for movers and nonmovers, respectively.

As the objective function, the bank in island $j \in \mathscr{J}$ at date t is assumed to adopt

$$U^i(c_{t+1}^{j,m}, c_{t+1}^{j,n}) = u(c_{t+1}^{j,m})\pi_j + u(c_{t+1}^{j,n})(1-\pi_j),$$

where $c_{t+1}^{j,\theta}$ are consumptions of agents, whose type is $\theta \in \Theta$, in island j at date $t+1$. We then define an equilibrium with circulating money, wherein islands $1, \ldots, J$ are treated equally: A *monetary equilibrium given* $\{\sigma_t\}_{t=-\infty}^{\infty}$ is defined by $\{(s_t^0, s_t, \rho_t^0, \rho_t)\}_{t=-\infty}^{\infty}$ of storage investments $s_t^0, s_t \geq 0$ in island 0 and other islands and per-capita real money balances $\rho_t^0, \rho_t > 0$, satisfying that $\rho_t^0/\sigma_t, \rho_t/\sigma_t < \omega$, in island 0 and other islands such that there exists $\{d_t^{j,m}, d_t^{j,n}, m_t^j, n_t^j\}_{t=-\infty}^{\infty}$ satisfying that, at each date t, (i) for each $j \in \mathscr{J}$, $(d_t^{j,m}, d_t^{j,n}, s_t^j, m_t^j, n_t^j)$ maximizes $U((\omega + \tau_t^j)d_t^{j,m}, (\omega + \tau_t^j)d_t^{j,n})$ subject to (1), (2), and (3) given $P_\tau^0 = M_\tau/\rho_\tau^0$ and $Q_\tau = M_\tau/\rho_\tau$ for $\tau = t, t+1$, (ii) $n_t^0 + \sum_{k=1}^J m_t^k = JM_t$, and (iii) $m_t^0 + n_t^0 = JM_t$ and $m_t^k + n_t^k = M_t$ for each $k \in \{1, \ldots, J\}$, where $s_t^j = s_t^0$ if $j = 0$ and otherwise $s_t^j = s_t$. In this definition, condition (i) describes the optimization problem of banks, condition (ii) represents the ex-post money stock in island 0 is stationary, and condition (iii) represents the money-market clearing condition. It is also a *monetary steady state given* $\sigma_t = \sigma$ for some given constant $\sigma > 0$ if $(s_t^0, s_t, \rho_t^0, \rho_t) \equiv (s^0, s, \rho^0, \rho)$ for each t.

We can easily verify that, at each monetary equilibrium, the constraints (1) and (2) hold with equality because of strict monotonicity of u. Therefore, one should remark that, at the equilibrium, the good-market clearing condition for the whole economy, in addition to the money-market one, holds, i.e.: we can observe that, at a monetary equilibrium,

$$\sum_{i\in\mathscr{J}}\left[c^{j,m}_{t+1}\pi_j + c^{j,n}_{t+1}(1-\pi_j) + s^j_{t+1}\right] = (1+J)\omega + x\sum_{i\in\mathscr{J}} s^j_t$$

for each $j \in \mathscr{J}$ and each t, where $c^{j,\theta}_{t+1} = (\omega + \tau^j_t)d^{j,\theta}_t$ for each $\theta \in \Theta$ and each t.

4 Existence of Monetary Steady State

In order to argue monetary policy, we should ensure that at least one monetary equilibrium. Before exploring such existence, this section observes a basic property on the equilibrium rates of return. As noted at the last paragraph in the previous section, the constraints (1) and (2) hold with equality at each monetary equilibrium. Then, the bank in island j at date t faces the constraint that

$$(\omega+\tau^j_t)\left[d^{j,m}_t\pi_j + d^{j,n}_t(1-\pi_j)\right] = x(\omega+\tau^j_t) + \left(\mathscr{R}^j_{t+1} - x\right)\frac{n^j_t}{P^j_t} + \left(\mathscr{E}^j_{t+1} - x\right)\frac{m^j_t}{P^j_t} \quad (4)$$

in addition to its liquidity constraint, Eq.(3). We can therefore obtain that, at each monetary equilibrium, both $\mathscr{R}^j_{t+1} \leq x$ and $\mathscr{E}^j_{t+1} \leq x$ hold. This is because the bank in island j chooses ∞ as n^j_t if $\mathscr{R}^j_{t+1} > x$ and as m^j_t if $\mathscr{E}^j_{t+1} > x$. Furthermore, we can observe that $\left(\mathscr{R}^j_{t+1} - x\right) n^j_t/P^j_t = 0$ and $\left(\mathscr{E}^j_{t+1} - x\right) m^j_t/P^j_t \leq 0$. The former observation holds immediately if $\mathscr{R}^j_{t+1} = x$ and, if $\mathscr{R}^j_{t+1} < x$, follows immediately from the fact that the bank prefers to invest in the storage investment rather than money holdings for nonmovers and chooses zero as n^j_t. The latter also follows immediately from the fact that $\mathscr{E}^j_{t+1} \leq x$. Remark that, in a monetary equilibrium, it is allowed that $\left(\mathscr{E}^j_{t+1} - x\right) m^j_t/P^j_t < 0$ because of the liquidity constraint, Eq.(3). If $\mathscr{R}^j_{t+1} < x$, the bank prefers to invest in the storage investment rather than money holdings for movers and chooses m^j_t as small as possible. However, the liquidity constraint prevents the value of m^j_t from being zero (and holds with equality). Therefore, it might hold that $\left(\mathscr{E}^j_{t+1} - x\right) m^j_t/P^j_t < 0$.

Obviously, money holdings for movers [nonmovers] and the storage investment are completely substitutable when $\mathscr{E}^j_{t+1} = x$ [$\mathscr{R}^j_{t+1} = x$]. In such a case, monetary equilibrium might be indeterminate. In order to avoid such indeterminacy, we explore an equilibrium with $\mathscr{E}^j_{t+1} < x$ and $\mathscr{R}^j_{t+1} < x$. Under such conditions, as argued in the previous paragraph, the money holding for nonmovers, n^j_t, becomes zero and the liquidity constraint, Eq.(3), holds with equality.[6] Since Eq.(1) also holds with equality, the objective of the bank at island j is now to choose m^j_t in order to maximize

$$u\left(\frac{1}{\pi_j}\mathscr{E}^j_{t+1}\frac{m^j_t}{P^j_t}\right)\pi_j + u\left(\frac{x}{1-\pi_j}\left[\omega + \tau^j_t - \frac{m^j_t}{P^j_t}\right]\right)(1-\pi_j).$$

[6]Therefore, the constraints of each bank in island $j \in \mathscr{J}$ can be rewritten as (a) the balance sheet constraint (1) with equality, (b) the liquidity constraint with equality: $(\omega+\tau^i_t)d^{j,m}_t\pi_j = \mathscr{E}^j_{t+1}m^j_t/P^j_t$, and (c) constraints about returns to nonmovers: $(\omega+\tau^i_t)d^{j,n}_t(1-\pi_j) = xs^j_t$, where the equality in (c) follows from Eq.(4) and (b). This is a standard set of constraints in the existing literature.

Combining the first order conditions and the money market clearing conditions, we obtain a system of equations characterizing monetary equilibrium given $\{\sigma_t\}_{t=-\infty}^{\infty}$:

$$0 = \frac{1}{\sigma_t}\rho_{t+1}Ju'\left(\frac{1}{\sigma_t\pi}\rho_{t+1}J\right) - x\rho_t^0 u'\left(\frac{x}{1-\pi}\left[\omega - \frac{\rho_t^0}{\sigma_t}\right]\right) \tag{5}$$

in island 0 and

$$0 = \frac{1}{\sigma_t}\rho_{t+1}^0 u'\left(\frac{1}{\sigma_t\mu}\frac{\rho_{t+1}^0}{J}\right) - x\rho_t Ju'\left(\frac{x}{1-\mu}\left[\omega - \frac{\rho_t}{\sigma_t}\right]\right) \tag{6}$$

in island $j \neq 0$ (and the balance sheet constraint (1) with equality for determinations of s^0 and s).

In order to guarantee the existence of at least one monetary equilibrium, we assume that the central bank chooses a constant σ and explore a monetary steady state. Then, we can obtain the following proposition:

Proposition 1 *A unique monetary steady state* $(s^0(\sigma), s(\sigma), \rho^0(\sigma), \rho(\sigma))$ *given* σ *exists. Furthermore, (a) it is characterized by*

$$(s^0(\sigma), s(\sigma), \rho^0(\sigma), \rho(\sigma)) = \left(\frac{(1-\pi)\omega}{1-\pi+\pi/\sigma}, \frac{(1-\mu)\omega}{1-\mu+\mu/\sigma}, \frac{\pi\omega}{1-\pi+\pi/\sigma}, \frac{\mu\omega}{1-\mu+\mu/\sigma}\right),$$

(b) $s^0 = s$ *and* $\rho^0 = \rho$ *when* $J = 1$*, and (c)* $s \uparrow \omega$ *and* $\rho \downarrow 0$ *as* $J \uparrow \infty$.

As shown by (a), we can obtain equilibrium outcomes by a closed form. The claim (b) says that the equilibrium outcomes in two islands are identical to each other when $J = 1$, i.e.: the number of islands is equal to two. This is a standard environment in the existing literature on OLG models with spatial separation. The claim (c) is a remarkable result in this article. When the number of islands increases, the left-hand side of the liquidity constraint (3) for the bank in each island $j \neq 0$ becomes small. In such a case, therefore, the liquidity constraint for the bank in each island $j \neq 0$ is relaxed. Then, it is shown that banks in islands $j \neq 0$ lose their incentive to hold money when the number of islands increases. However, one should note that, even in the limit that $J \uparrow \infty$, money still circulates in this economy, whereas the real money balances in island $j \neq 0$ tends to zero. This may look like a *fallacy of composition*. The rate of return of money for movers, $\mathscr{E}_{t+1}^j$ is calculated by, as $J \uparrow \infty$,

$$\mathscr{E}_{t+1}^j = \begin{cases} \frac{1}{\sigma}\frac{1-\pi+\pi/\sigma}{1-\mu+\mu/\sigma} & \rightarrow \quad \frac{1}{\sigma}(1-\pi+\pi/\sigma) & \text{if} \quad j = 0, \\ \frac{1}{\sigma}\frac{1-\mu+\mu/\sigma}{1-\pi+\pi/\sigma} & \rightarrow \quad \frac{1}{\sigma}\frac{1}{1-\pi+\pi/\sigma} & \text{if} \quad j \neq 0, \end{cases}$$

which is a positive and finite value in each case.

We should also verify that both $\mathscr{E}_{t+1}^j < x$ and $\mathscr{R}_{t+1}^j < x$ hold at the above monetary steady state given σ. The following proposition provides a sufficient condition for ensuring such a situation.

Proposition 2 *When x is greater than but sufficiently close to one and J is extremely large, both $\mathscr{E}^j_{t+1} < x$ and $\mathscr{R}^j_{t+1} < x$ hold at the monetary steady state given $\sigma > \sigma_0(x)$, where*

$$\sigma_0(x) := \frac{1-\pi+\sqrt{(1-\pi)^2+4x\pi}}{2x},$$

which is less than one if x is sufficiently close to one.

This is a rather technical statement. Combining Propositions 1 and 2, however, we might say that a desirable monetary steady state exists for σ, the range of which includes a neighborhood of one.

5 The Optimum Quantity of Money

We now examine the optimal monetary policy. This section first defines and characterizes golden rule optimality as an optimality criterion for stationary feasible allocations. A *stationary feasible allocation*, which is also *symmetric* with respect to islands $j \neq 0$, is defined by $(c^{0,m}, c^{0,n}, s^0, c^m, c^n, s)$ satisfying that $s^0, s \in [0, \omega]$ and

$$[c^m\pi + c^{0,n}(1-\pi)] + J[c^{0,m}\mu + c^n(1-\mu)] \leq (1+J)\omega + (x-1)[s^0 + Js], \tag{7}$$

which is a resource constraint for the whole economy, not for each island. It is said to be *interior* if $(c^{0,m}, c^{0,n}, c^m, c^n) \gg 0$ and, in this article, *golden rule optimal* (GRO) if it maximizes the equally-weighted sum of lifetime utility functions in all islands,

$$V^J(c^{0,m}, c^{0,n}, c^m, c^n) := \frac{1}{1+J}[u(c^{0,m})\pi + u(c^{0,n})(1-\pi)] + \frac{J}{1+J}[u(c^m)\mu + u(c^n)(1-\mu)],$$

in the space of all stationary feasible allocations. Furthermore, we say that a money growth rate σ is *first best* if the allocation at the monetary steady state given σ is GRO. We can then characterize GRO completely:

Proposition 3 *An interior stationary feasible allocation $(c^{0,m}, c^{0,n}, s^0, c^m, c^n, s)$ is golden rule optimal if and only if $c^{0,m} = c^{0,n} = c^m = c^n = x\omega$ and $s^0 = s = \omega$.*

At an interior GRO allocation, therefore, all of the initial endowment are invested in the storage technology and (idiosyncratic) liquidity shocks are fully insured.

Consider now optimality of monetary steady state. By Proposition 1, the consumption allocation at the monetary steady state is given by

$$(c^{0,m}(\sigma), c^{0,n}(\sigma), c^m(\sigma), c^n(\sigma)) := \left(\frac{\omega}{\sigma+(1-\sigma)\mu}, \frac{\sigma x\omega}{\sigma+(1-\sigma)\pi}, \frac{\omega}{\sigma+(1-\sigma)\pi}, \frac{\sigma x\omega}{\sigma+(1-\sigma)\mu}\right).$$

Applying Proposition 3 to this allocation, we obtain the next proposition:

Proposition 4 *There exists no first-best money growth rate σ.*

In other words, the allocation at the monetary steady state given any σ cannot be GRO. Because the central bank cannot reach the first-best situation, it should now consider a "second-best" situation. Here, we assume that the central bank chooses σ in order to maximize the welfare at the monetary steady state,

$$W_J(\sigma) := V^J(c^{0,m}(\sigma), c^{0,n}(\sigma), c^m(\sigma), c^n(\sigma)),$$

where V^J is defined as above. A monetary growth rate σ is now *second best* if it is a solution of the above optimization problem of the central bank. The following proposition characterizes the second-best money growth rate.

Proposition 5 *A second-best money growth rate exists uniquely and is given by* $\sigma_{**} = 1$.

The monetary steady state and its consumption allocation given σ_{**} is calculated by

$$(s^0_{**}, s_{**}, \rho^0_{**}, \rho_{**}) := (s^0(\sigma_{**}), s(\sigma_{**}), \rho^0(\sigma_{**}), \rho(\sigma_{**})) = ((1-\pi)\omega, (1-\mu)\omega, \pi\omega, \mu\omega)$$

and

$$(c^{0,m}_{**}, c^{0,n}_{**}, c^m_{**}, c^n_{**}) := (c^{0,m}(\sigma_{**}), c^{0,n}(\sigma_{**}), c^m(\sigma_{**}), c^n(\sigma_{**})) = (\omega, x\omega, \omega, x\omega)\,.$$

Actually, this does not imply a GRO allocation.

As a corollary of the previous proposition, we can claim suboptimality of the Friedman rule. Define the gross nominal interest rate in island $j \in \mathscr{J}$ at date $t{+}1$ by $I^j_{t+1} := x/\mathscr{R}^j_{t+1}$. Then, the Friedman rule in the *monetary steady state* corresponds to the money growth rate σ_f such that $1 = I^j_{t+1} = x\sigma_f$, i.e.: $\sigma_f = 1/x$. Because $x > 1$, $\sigma_f < 1 = \sigma_{**}$. Therefore, the money growth rate σ_f corresponding to the Friedman rule is neither first nor second best.

6 Discount Window Policy

As shown in Propositions 4 and 5, the central bank cannot achieve a first-best situation by controlling σ, i.e.: there exists a limit in an improvement of efficiency by the choice of money growth rate. Such a limit of controlling σ can be observed even when the number of islands is extremely large.

Proposition 6 $\lim_{J\uparrow\infty} W_J(\sigma) = \ln(x\omega)$.

This proposition says that the welfare function of the central bank is independent of the value of σ when J increases boundlessly. Actually, in the limit that $J \uparrow \infty$, the consumption allocation at the monetary steady state given σ becomes

$$(c^{0,m}(\sigma), c^{0,n}(\sigma), c^m(\sigma), c^n(\sigma)) := \left(\frac{\omega}{\sigma}, \frac{\sigma x\omega}{\sigma + (1-\sigma)\pi}, \frac{\omega}{\sigma + (1-\sigma)\pi}, x\omega\right).$$

This consumption allocation can be identical with that at the monetary steady state given $\sigma_{**} = 1$ if the central bank chooses one as σ. However, because of Proposition 6, we cannot claim that such a choice about σ increases the welfare. In other words, there is no room for monetary policy, such as controlling the money growth rate, to improve social welfare when the number of islands is extremely large. A natural question here is whether there exist other policies implementing a first-best situation.

In an OLG model with random relocations of agents between two islands, Haslag and Martin (2007, Proposition 5) have shown that an efficient allocation can be achieved if the central bank can make loans and follow the Friedman rule. We finally reexamine their results in our framework. The timing of the central bank loans are as follows: At date t, banks established at the date can borrow money from the central bank for movers born at the date. Then, at date $t+1$, banks sell the consumption good to agents who moved from the other islands in order to obtain money for repaying the central bank loan. It is assumed that the central bank loans are made at a net interest rate of zero. Also, it is assumed that the central bank imposes the upper bound $\bar{\ell}^j$ on the real central bank loans of the bank at island j.

Let ℓ_t^j denote the amount of money of bank at island j as the loan received from the central bank at date t. Then, the constraints faced by the bank at island j are given by

$$(\omega+\tau_t^i)d_t^{j,m}\pi_j \le \mathscr{E}_{t+1}^j \frac{m_t^j}{P_t^j} + \mathscr{E}_{t+1}^j \frac{\ell_t^j}{P_t^j}, \tag{8}$$

$$(\omega+\tau_t^i)d_t^{j,n}(1-\pi_j) \le x s_t^j - \mathscr{R}_{t+1}^j \frac{\ell_t^j}{P_t^j} = x(\omega+\tau_t^j) - x\frac{m_t^j}{P_t^j} - \mathscr{R}_{t+1}^j \frac{\ell_t^j}{P_t^j}, \tag{9}$$

which are constraints on payments for movers and nonmovers, and

$$\frac{\ell_t^j}{P_t^j} \le \bar{\ell}^j \tag{10}$$

in addition to Eq.(1), where the last terms in the right-hand sides of the first and second inequalities represent the borrowing of the bank from the central bank and the repayment of the bank to the central bank, respectively.[7]

Because this article considers a monetary equilibrium such that $\mathscr{R}_{t+1}^j \le x$ as argued in Section 4, banks can relax constraint Eq.(9) by increasing ℓ_t^j and decreasing m_t^j. This does not affect the right-hand side of Eq.(8) because the discount window loans ℓ_t^j are perfect substitute for the money holding m_t^j. Therefore, it is always optimal for banks to borrow as much as possible such that Eq.(10) holds with equality. This observation also implies that, if the central bank increases $\bar{\ell}^j$, banks borrow more from the central bank, decrease money holdings, and increase the amount of the storage investment, i.e.: it follows that $m_t^j \downarrow 0$ and $s_t^j \uparrow \omega$ when $\bar{\ell}^j$ increases. This remains true if $\mathscr{R}_{t+1}^j < x$ or $1/\sigma < x$ in the monetary steady state. As a summary of this argument, we provide the following proposition.

Proposition 7 *The money growth rate σ_f corresponding to the Friedman rule can be first best if the central bank can make sufficiently large loans.*

[7] See also Footnote 4.

This result implies that the combination of the discount window policy and the Friedman rule is optimal even under the presence of asymmetric liquidity socks. It reinforces Proposition 5 of Haslag and Martin (2007).

7 Concluding Remarks

This article has modified an overlapping generations (OLG) model with spatial separation by allowing that the number of islands is more than two and asymmetric liquidity shocks are observable. We have precisely defined monetary equilibrium and argued its existence. The model exhibits subptimality of the Friedman rule and optimality of the combination of the Friedman rule and the discount window. These observations (partially) strengthen the results of the existing literature (Smith, 2002; Haslag and Martin, 2007; Matsuoka, 2011, for example). Furthermore, it has been shown that, if the number of islands is extremely large, any monetary policies controlling money growth rates have no room in order to improve social welfare. This is because an increase in the number of islands relaxes the liquidity constraints of some banks.

In order to close this paper, we mention the reserve-to-deposit ratio of the bank at island j, γ_t^j, defined by $\rho_t^0/(\omega+\tau_t^0)$ if $j=0$ and otherwise $\rho/(\omega+\tau_t^j)$. By an easy calculation, one can find that $\gamma_t^0=\pi$ and $\gamma_t^j=\mu$ for $j\neq 0$. Therefore, in our setting, the reserve-to-deposit ratios are independent of the value of σ or the nominal interest rate. This is because the logarithmic utility function neglects the wealth effect. Therefore, the reserve-to-deposit ratios are also independent of inflation, which might be a counterintuitive result.[8] The reexamination of our results in the economy, wherein the coefficient of the constant relative risk aversion is less than one, remains an important future research.

Appendix: Proofs of Propositions

Proof of Proposition 1. By Eqs.(5) and (6) in addition to the logarithmic utility function, a monetary steady state (s^0, s, ρ^0, ρ) given σ is characterized by

$$\pi\left[\omega-\frac{\rho^0}{\sigma}\right]=(1-\pi)\rho^0 \quad \text{and} \quad \mu\left[\omega-\frac{\rho}{\sigma}\right]=(1-\mu)\rho$$

and the balance sheet constraints (1) with equality. By solving these equations, we obtain (a). The statements of (b) and (c) follows immediately from (a) with the fact that $\mu=\pi/J\downarrow 0$ as $J\uparrow\infty$. Q.E.D.

Proof of Proposition 2. By Proposition 1, both $\mathscr{E}_{t+1}^j<x$ and $\mathscr{R}_{t+1}^j<x$ hold at the monetary steady state given σ if and only if it holds that (a.1) $(1-\mu)x\sigma^2-(1-\pi-\mu x)\sigma-\pi>0$, (a.2) $(1-\pi)x\sigma^2-(1-\pi x-\mu)\sigma-\mu>0$, and (b) $1/\sigma<x$. Here, we consider the case that $J\to\infty$. Then, (a.1) and (a.2) are rewritten as $x\sigma^2-(1-\pi)\sigma-\pi>0$ and

[8]See also Footnote 3.

$[(1-\pi)x\sigma + \pi x - 1]\sigma > 0$. By solving each of these equalities, in addition to (b) and $\sigma > 0$, with respect to σ, we obtain four inequalities such that

$$\sigma > \sigma_0(x) := \frac{1-\pi+\sqrt{(1-\pi)^2+4x\pi}}{2x}$$

and

$$\sigma > \frac{1-\pi x}{(1-\pi)x}$$

with $\sigma > 1/x := f(x)$ and $\sigma > 0$. Note that $\sigma_0(1) = 1 = f(1)$. Because

$$\sigma_0'(x) = \frac{\pi\left[(1-\pi)^2+4\pi x\right]^{-1/2} - \left[1-\pi-\sqrt{(1-\pi)^2+4\pi x}\right]}{2x^2},$$

it follows that $\sigma_0'(1) = -1+\pi/(1+\pi) < 0$ and $\sigma_0'(1) > -1 = f'(1)$. Therefore, there exists some sufficiently small $\delta > 0$ such that $0 < \sigma_0(x) < 1$, $-1 < \sigma_0'(x) < 0$, and $\sigma_0(x) > 1/x$ for any $x \in X :=]1, 1+\delta[$.

Combining the last inequality with the inequality that $1/x > (1-\pi x)/x(1-\pi)$, we can conclude that the solution of the system of inequalities (a.1), (a.2), and (b) is $\sigma > \sigma_0(x)$ when x is greater than but sufficiently close to 1 and J is extremely high. Q.E.D.

Proof of Proposition 3. Because of concavity of the objective function V^J, an interior CGRO allocation can be characterized by first-order conditions, which can be written as $u'(c^{0,m}) = u'(c^{0,n}) = u'(c^m) = u'(c^n) = \lambda$ and $s^0 = s = \omega$, where λ is a Lagrange multiplier for the resource constraint (7). Proposition 3 follows immediately from this observation. Q.E.D.

Proof of Proposition 4. One can easily verify that there exists no σ which implies that $c^{0,m}(\sigma) = c^{0,n}(\sigma) = c^m(\sigma) = c^n(\sigma) = x\omega$. This implies Proposition 4. Q.E.D.

Proof of Proposition 5. We can obtain that

$$(1+J)W_J'(\sigma) = \left(\frac{1}{\sigma}-1\right)\pi\left[\frac{1-\pi}{\pi+(1-\pi)\sigma}+\frac{1-\mu}{\mu+(1-\mu)\sigma}\right]\left\{\begin{array}{c}>\\=\\<\end{array}\right\}0 \quad \text{if} \quad \sigma\left\{\begin{array}{c}<\\=\\>\end{array}\right\}1.$$

Therefore, $\sigma_{**} = 1$ is a unique second-best money growth rate. Q.E.D.

Proof of Proposition 6. Proposition 6 follows immediately from the fact that $\mu \downarrow 0$ as $J \uparrow \infty$. Q.E.D.

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