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A straightforward proof of Arrow's theorem

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Abstract

We present a straightforward proof of Arrow's Theorem. Our approach avoids some of the complexities of existing proofs and is meant to be transparent and easily followed.

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1 Introduction

There are a number of alternative proofs of Arrow's Theorem in the literature, but many of them involve considerable complexity. The original approach in Arrow (1951) (and refined by others) involves defining semi-decisive groups, proving a "field expansion lemma" establishing that semi-decisive groups are decisive, proving a "group contraction lemma" establishing that a decisive group contains a proper decisive group, and concluding that there is a single decisive individual. A more recent approach pioneered by Barbera (1980) and sharpened by Geanakoplos (2005) and Reny (2001) involves identifying a voter that is pivotal for a given pair of alternatives, meaning that changing just this voter's preference over the pair of alternatives results in a change in the social preference over the pair. Although in principle different pairs of alternative could give rise to different pivotal voters, the proof is completed by showing that in fact a unique individual is pivotal for all pairs of alternatives.

In this proof, we combine these two approaches in order to capitalize on the strengths of each while also eliminating the complexities inherent in each approach. Specifically, in the proof we identify a single individual and show that she is decisive over various alternatives, but we do not need to specify groups or distinguish between semi-decisive and decisive groups. This single decisive individual is identified as being pivotal over two specific alternatives and then shown to be decisive over all alternatives, but we do not need to consider how different alternatives can correspond to different pivotal individuals. Thus, our proof is straightforward as it gives an uncomplicated argument that relies on no additional complexities to achieve the result.

Other elementary proofs of Arrow's Theorem exist, such as Dardanoni (2001), although these proof often impose additional assumptions such as linear orders in order to simplify the presentation. While straightforward, the proof below is still fully general in that individuals can have indifferences in their rankings and no restrictions are made about the number of alternatives or individuals. Finally, Yu (2012) is perhaps the most closely related paper in the literature. Yu presents a short proof of Arrow's Theorem that also combines the two common approaches above. But his proof focuses more on the identities of the pivotal voters for various pairs of alternatives, while the present paper requires only that a single individual be identified. Thus, although Yu's proof is shorter, the current paper is more transparent by relying solely on profile comparisons to establish the existence of a dictator.

2 The Model

Here we briefly review the standard framework for Arrovian preference aggregation. Let A be a set containing at least three *alternatives* and let N be a finite set of *individuals*. The *preferences* of individuals are weak orders; that is, individual i's preference is given by a reflexive, complete, and transitive binary relation R_i on A. We denote the set of all possible weak orders on A by $\mathcal{R}$ and a preference profile by $(R_1, \dots, R_n) \in \mathcal{R}^n$. We generate strict preference P_i in the usual way: for all $x, y \in A$, xP_iy if and only if not yR_ix.

A *social preference function*, given by $R : \mathcal{R}^n \to \mathcal{R}$, assigns a weak order to every possible preference profile.[1] This weak order represents the preference of society as a whole and we refer to it as the social preference given by a preference profile. A social preference function satisfies *Unanimity* if for all $x, y \in A$, xP_iy for all $i \in N$ implies xPy. A pair of profiles $(R_1, \dots, R_n)$ and

[1] The assumption that the domain of R is all possible preference profiles is called Universal Domain in the literature.

$(R'_1, \ldots, R'_n)$ *rank* x *and* y *the same* if, for all $i \in N$, xR_iy if and only if xR'_iy. A social preference function satisfies *Independence of Irrelevant Alternatives* (IIA) if for all $x, y \in A$ and all profiles $(R_1, \ldots, R_n)$ and $(R'_1, \ldots, R'_n)$ that rank x and y the same, xRy if and only if $xR'y$.

For a given social preference function, an individual i is *decisive for some* x *over some* $y \neq x$ if xP_iy implies xPy. An individual i is a *dictator* if she is decisive for every x over every $y \neq x$.

3 Proof

Here we give our proof of Arrow's Theorem:

Arrow's Theorem *If a social preference function satisfies Unanimity and IIA, then some individual is a dictator.*

Proof: Fix two distinct alternatives a and b. The proof consists of the following 7 steps, the first two of which follow the line of proof in Yu (2012).

Step 1: Identify individual i^*. Fix the following two profiles

$$\begin{array}{ccc} R_1 & \cdots & R_n \\ \hline a & \cdots & a \\ b & \cdots & b \\ \vdots & \cdots & \vdots \end{array} \qquad \text{and} \qquad \begin{array}{ccc} R_1 & \cdots & R_n \\ \hline b & \cdots & b \\ a & \cdots & a \\ \vdots & \cdots & \vdots \end{array}$$

where the dotted ranges represent the other alternatives in fixed but arbitrary locations in the two profiles. By Unanimity, the left hand profile must have aPb and the right hand profile must have bPa. Now transform the left hand profile into the right profile by switching a and b one individual at a time, starting with individual 1 and holding all other alternatives fixed. Let individual i^* be the individual for which the social preference changes from aPb to something else for the first time. To be more concrete, individual i^* is defined as the individual for which the following two things are true. For the profile

$$\begin{array}{ccc} R_1 \ \cdots \ R_{i^*-1} & R_{i^*} & R_{i^*+1} \ \cdots \ R_n \\ \hline b & a & a \\ a & b & b \\ \vdots & \vdots & \vdots \end{array} \tag{1}$$

the social preference is aPb and for the profile

$$\begin{array}{ccc} R_1 \ \cdots \ R_{i^*-1} & R_{i^*} & R_{i^*+1} \ \cdots \ R_n \\ \hline b & b & a \\ a & a & b \\ \vdots & \vdots & \vdots \end{array} \tag{2}$$

the social preference is bRa.

Step 2: For all $c \neq a, b$, individual i^* is decisive for b over c. The profile

$$\begin{array}{ccc} R_1 \ \cdots \ R_{i^*-1} & R_{i^*} & R_{i^*+1} \ \cdots \ R_n \\ \hline b & a & a \\ c & b & b \\ a & c & c \\ \vdots & \vdots & \vdots \end{array} \tag{3}$$

ranks a and b the same as profile (1), so by IIA we must have aPb. Moreover, by Unanimity we have bPc and therefore aPc must hold by transitivity of social preference.

Next consider the set of profiles of the following form

$$\begin{array}{ccc} \underline{R_1 \ \cdots \ R_{i^*-1}} & \underline{R_{i^*}} & \underline{R_{i^*+1} \ \cdots \ R_n} \\ b/c & b & a \\ a & a & b/c \\ \vdots & c & \vdots \\ \vdots & \vdots & \vdots \end{array} \tag{4}$$

where the notation b/c means that the alternatives b and c can be ranked arbitrarily in the indicated spot by profiles in the set (so that, for each individual, b can be ranked immediately above c or vice versa or b and c can be indifferent). Every such profile ranks a and b the same as profile (2) and ranks a and c the same as profile (3), so by IIA we must have $bRaPc$, which implies bPc. But now note that every profile with $bP_{i^*}c$ ranks b and c the same as some profile in the set given by (4) and so by IIA we conclude that individual i^* is decisive for b over c.

Step 3: For all $c \neq a, b$, individual i^* is decisive for a over c. Consider the set of profiles

$$\begin{array}{ccc} \underline{R_1 \ \cdots \ R_{i^*-1}} & \underline{R_{i^*}} & \underline{R_{i^*+1} \ \cdots \ R_n} \\ a/c & a & a/c \\ b & b & b \\ \vdots & c & \vdots \\ \vdots & \vdots & \vdots \end{array} \tag{5}$$

As i^* is decisive for b over c, we have bPc and by Unanimity we have aPb, so it must be that aPc for all profiles of this form. Again, every profile with $aP_{i^*}c$ ranks a and c the same as some profile in this set and so by IIA we conclude that individual i^* is decisive for a over c.

Step 4: For all $c \neq a, b$, individual i^* is decisive for c over a. The profile

$$\begin{array}{ccc} \underline{R_1 \ \cdots \ R_{i^*-1}} & \underline{R_{i^*}} & \underline{R_{i^*+1} \ \cdots \ R_n} \\ b & c & c \\ c & a & a \\ a & b & b \\ \vdots & \vdots & \vdots \end{array} \tag{6}$$

ranks a and b the same as profile (1), so by IIA we must have aPb. Moreover, by Unanimity we have cPa and therefore cPb must hold.

All profiles of the form

$$\begin{array}{ccc} \underline{R_1 \ \cdots \ R_{i^*-1}} & \underline{R_{i^*}} & \underline{R_{i^*+1} \ \cdots \ R_n} \\ b & c & a/c \\ a/c & b & b \\ \vdots & a & \vdots \\ \vdots & \vdots & \vdots \end{array} \tag{7}$$

rank a and b the same as profile (2) and rank b and c the same as profile (6), so by IIA we must have $cPbRa$, which implies cPa for all such profiles. Once again, every profile with $cP_{i^*}a$ ranks a and c the same as some profile in the set given by (7) and so by IIA we conclude that individual i^* is decisive for c over a.

Step 5: For all $c \neq a, b$, individual i^* is decisive for c over b. Consider the set of profiles

$$\begin{array}{ccc} \underline{R_1 \ \cdots \ R_{i^*-1}} & \underline{R_{i^*}} & \underline{R_{i^*+1} \ \cdots \ R_n} \\ a & c & a \\ b/c & a & b/c \\ \vdots & b & \vdots \\ \vdots & \vdots & \vdots \end{array} \tag{8}$$

As i^* is decisive for c over a, we have cPa and by Unanimity we have aPb, so it must be that cPb for all profiles of this form. Again, every profile with $cP_{i^*}b$ ranks b and c the same as some profile in this set and so by IIA we conclude that individual i^* is decisive for c over b.

Step 6: Individual i^* is decisive for a over b and for b over a. Consider the set of profiles

$$\begin{array}{ccc} \underline{R_1 \ \cdots \ R_{i^*-1}} & \underline{R_{i^*}} & \underline{R_{i^*+1} \ \cdots \ R_n} \\ a/b & a & a/b \\ \vdots & c & \vdots \\ \vdots & b & \vdots \\ \vdots & \vdots & \vdots \end{array} \tag{9}$$

As i^* is decisive for a over c and for c over b, we have $aPcPb$ and therefore aPb for all such profiles. Every profile with $aP_{i^*}b$ ranks a and b the same as some profile in this set and so by IIA we conclude that individual i^* is decisive for a over b. The argument for b over a is the same, swapping the position of a and b.

Step 7: Individual i^* is a dictator. We must show that individual i^* is decisive for every x over every $y \neq x$. The only case remaining is $x \neq a, b$ and $y \neq a, b$. Consider profiles of the form

$$\begin{array}{ccc} \underline{R_1 \ \cdots \ R_{i^*-1}} & \underline{R_{i^*}} & \underline{R_{i^*+1} \ \cdots \ R_n} \\ x/y & x & x/y \\ a/b & a & a/b \\ \vdots & b & \vdots \\ \vdots & y & \vdots \\ \vdots & \vdots & \vdots \end{array} \tag{10}$$

As i^* is decisive for x over a, for a over b and for b over y, we have $xPaPbPy$ and therefore xPy for all such profiles. Every profile with $xP_{i^*}y$ ranks x and y the same as some profile in this set and so by IIA we conclude that individual i^* is decisive for x over y. This shows that individual i^* is a dictator and completes the proof.

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