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Long-run effects of capital market integration using OLG model

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Abstract

Buiter (1981) illustrates that in the OLG model, the ranking of stationary utility levels under autarky and openness, is ambiguous. We show that both countries increase their stationary utility levels only if the autarky capital-labor ratios are on opposite sides of the golden rule.

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1. Introduction

According to Diamond's (1965) OLG model it is possible to explain international capital movements from differences in time preferences. However the Buiter (1981) OLG model, shows that the ranking of stationary utility levels under autarky and openness is ambiguous. His demonstration is graphical and utility and production functions are not specified. Within the framework built by Buiter, we show that using an OLG model with a Cobb-Douglas production function and logarithmic utility it is possible to explicitly solve the model and determine the gains in well-being. These are positive for both countries only if the autarky capital-labor ratios of both countries are on opposite sides of the golden rule. We then generalize this result to all production and utility functions. Section 2, presents the OLG model in an open economy. Section 3, describes transitional dynamics and steady state. Section 4, determines the long-term gains of integration in the case of the log utility and Cobb -Douglas functions. Section 5, generalizes the result.

2. The OLG model in open economy

In the OLG model of Diamond (1965), agents live for two periods. Young people consume and save their labor income. The old consume their savings and the return on their savings.

Basic Assumptions:

$$c_t^y + s_t = w_t, \qquad c_{t+1}^o = (1 + r_{t+1})\, s_t \tag{1}$$

Logarithmic utility function:

$$V_t = \ln c_t^y + \frac{\ln c_{t+1}^o}{1+\rho} \tag{2}$$

A Cobb-Douglas production function, with $k_t = K_t/L_t$, the capital-labor ratio.

$$q_t = \Lambda k_t^\alpha \tag{3}$$

A world composed of two countries, "Tilde" and "Hat", such that each variable and parameter used a tilde or a hat to specify the country to which it refers. Variables have a double tilde or double hat in closed economy $(\tilde{\tilde{x}}, \hat{\hat{x}})$, a single tilde or hat in open economy $(\tilde{x}, \hat{x})$. It is necessary to obtain a steady state that the rate of population growth (n) is the same in each country. Assume for simplicity that countries have the same level of technology (Λ), the same depreciation rate of capital (δ). Suppose the depreciation rate for the period is unitary $(\delta = 1)$. This assumption is usual, and it simplifies the transitional dynamics in open economies. The countries differ in two respects. As in Buiter (1981), the two countries differ in the rate of time preference:

$$\tilde{\rho} > \hat{\rho} \tag{4}$$

The relative size of Hat Country in the world is represented by $\hat{\eta} = \hat{N}/(\tilde{N}+\hat{N})$, where N=L is the size of population. We assume that the populations are different sizes.

$$\tilde{\eta} \neq \hat{\eta} \tag{5}$$

In autarky, the wealth of each country is equal to its capital stock $A_t = K_t$, therefore dividing by N_t wealth per worker in each country is:

$$\hat{\hat{a}}_t = \hat{\hat{k}}_t \qquad \text{and} \qquad \tilde{\tilde{a}}_t = \tilde{\tilde{k}}_t \tag{6}$$

In each country, the increase in wealth is equal to the savings of the young, minus spending by the old, who consume all their wealth (they are egoistic). So, for the country Tilde: $\tilde{\tilde{A}}_{t+1} - \tilde{\tilde{A}}_t = \tilde{N}_t \tilde{\tilde{s}}_t - \tilde{\tilde{A}}_t$ or $\tilde{\tilde{A}}_{t+1} = \tilde{N}_t \tilde{\tilde{s}}_t$ or $(1+n)\tilde{\tilde{a}}_{t+1} = \tilde{\tilde{s}}_t$.

The equilibrium conditions of the capital market in each closed economy are:

$$\hat{\hat{s}}_t = (1+n)\hat{\hat{k}}_{t+1} \qquad \text{and} \qquad \tilde{\tilde{s}}_t = (1+n)\tilde{\tilde{k}}_{t+1} \tag{7}$$

Maximizing (2) under (1) youth savings are deducted:

$$\hat{\hat{s}}_t = \frac{\hat{\hat{w}}_t}{2+\hat{\rho}} \qquad \text{and} \qquad \tilde{\tilde{s}}_t = \frac{\tilde{\tilde{w}}_t}{2+\tilde{\rho}} \tag{8}$$

Maximizing producer profit under (3) yields the following factor prices:

$$\hat{\hat{w}}_t = (1-\alpha)\Lambda\hat{\hat{k}}_t^{\alpha} \qquad \text{and} \qquad \tilde{\tilde{w}}_t = (1-\alpha)\Lambda\tilde{\tilde{k}}_t^{\alpha} \tag{9}$$

$$\hat{\hat{R}}_t = \hat{\hat{r}}_t + \delta = \alpha\Lambda\hat{\hat{k}}_t^{\alpha-1} \qquad \text{and} \qquad \tilde{\tilde{R}}_t = \tilde{\tilde{r}}_t + \delta = \alpha\Lambda\tilde{\tilde{k}}_t^{\alpha-1} \tag{10}$$

Solving the recurrence equation obtained from (7),(8) and (9), steady state capital is obtained in autarky, and as $\tilde{\rho} > \hat{\rho}$:

$$\hat{\hat{k}} = \left(\frac{(1-\alpha)\Lambda}{(1+n)(2+\hat{\rho})}\right)^{\frac{1}{1-\alpha}} \qquad > \qquad \tilde{\tilde{k}} = \left(\frac{(1-\alpha)\Lambda}{(1+n)(2+\tilde{\rho})}\right)^{\frac{1}{1-\alpha}} \tag{11}$$

In steady state autarky, as Tilde Country is more impatient, the savings are lower $\tilde{\tilde{s}} < \hat{\hat{s}}$, capital is lower $\tilde{\tilde{k}} < \hat{\hat{k}}$, wages are lower $\tilde{\tilde{w}} < \hat{\hat{w}}$, production is lower $\tilde{\tilde{q}} < \hat{\hat{q}}$, the interest rate is higher $\tilde{\tilde{r}} > \hat{\hat{r}}$, and Tilde Country is poorer $\tilde{\tilde{a}} < \hat{\hat{a}}$.

In open economies (labor is immobile) agents are able to hold domestic and foreign securities[1]. E_t denotes the net amount of foreign securities owned. The wealth of a country is $A_t = K_t + E_t$ therefore dividing by N_t, wealth per worker in each country is:

$$\tilde{a}_t = \tilde{k}_t + \tilde{e}_t, \qquad \hat{a}_t = \hat{k}_t + \hat{e}_t \tag{12}$$

By hypothesis, the domestic and foreign securities are perfectly substitutable assets. Thus, they reported the same rate of return r. The global arbitrage implies that the world interest rate will be fixed "between" the rates of Hat Country and Tilde Country ($\tilde{\tilde{r}} > r > \hat{\hat{r}}$). Since $r > \hat{\hat{r}}$ the agents of Hat Country prefer to lend their savings in the global capital market (to agents of the Tilde Country), the Hat Country is a net lender ($\hat{e} > 0$). Conversely, as $\tilde{\tilde{r}} > r$ agents of Tilde Country are net borrowers $\tilde{e} < 0$. As there are only two countries, net lending by one is net borrowing by the other, so by construction: $\tilde{E}_t = -\hat{E}_t$ and expressed in per capita variables:

$$\tilde{e}_t < 0, \qquad \hat{e}_t > 0, \qquad \tilde{\eta}\tilde{e}_t = -\hat{\eta}\hat{e}_t \qquad \forall t \tag{13}$$

Since, at the instant of opening, the interest rate is the same in each country, as is the per capital-labor ratio, and the wage rate:

[1] Or government bonds, see Darreau and Pigalle (2013)

$$\tilde{r}_t = \hat{r}_t = r_t, \qquad \tilde{k}_t = \hat{k}_t = k_t, \qquad \tilde{w}_t = \hat{w}_t = w_t \tag{14}$$

This does not mean that the two countries have the same wealth and the same income. In open economies, it is important to distinguish between financial capital a_t and productive capital owned nationally k_t and to distinguish GDP ($q_t = Ak_t^\alpha$) and GNP ($y_t = q_t + R_t e_t$) where $R_t = r_t + \delta$. We deduce from the foregoing:

$$\hat{a}_t > \tilde{a}_t, \qquad \hat{q}_t = \tilde{q}_t, \qquad \hat{y}_t > \tilde{y}_t \tag{15}$$

The Hat Country with the lowest rate of time preference is always the richest at the steady state of an open economy.

3. Steady state and transitional dynamic

In each country, the increase in wealth is equal to the savings of the young, minus spending by the old, who consume all their wealth (as they are egoistic). For Hat Country: $\hat{A}_{t+1} - \hat{A}_t = \hat{N}_t \hat{s}_t - \hat{A}_t$ or else $\hat{a}_{t+1} = \hat{k}_{t+1} + \hat{e}_{t+1} = \frac{\hat{s}_t}{1+n}$ Thus, the saving for each country is:

$$\tilde{s}_t = (1+n)(\tilde{k}_{t+1} + \tilde{e}_{t+1}) = \frac{\tilde{w}_t}{2+\tilde{\rho}} \quad and \quad \hat{s}_t = (1+n)(\hat{k}_{t+1} + \hat{e}_{t+1}) = \frac{\hat{w}_t}{2+\hat{\rho}} \tag{16}$$

The equilibrium condition of the capital market in an open economy becomes: $\tilde{N}_t \tilde{s}_t + \hat{N}_t \hat{s}_t = \tilde{K}_{t+1} + \tilde{E}_{t+1} + \hat{K}_{t+1} + \hat{E}_{t+1}$ which can be rewritten as:

$$\frac{\tilde{\eta}\tilde{s}_t}{1+n} + \frac{\hat{\eta}\hat{s}_t}{1+n} = \tilde{\eta}\tilde{k}_{t+1} + \tilde{\eta}\tilde{e}_{t+1} + \hat{\eta}\hat{k}_{t+1} + \hat{\eta}\hat{e}_{t+1} \tag{17}$$

Since, $\tilde{\eta}\tilde{e}_{t+1} = -\hat{\eta}\hat{e}_{t+1}$ and $\hat{k}_{t+1} = \tilde{k}_{t+1} = k_{t+1}$ (17) beccomes $\frac{\tilde{\eta}\tilde{s}_t}{1+n} + \frac{\hat{\eta}\hat{s}_t}{1+n} = k_{t+1}$, then, by replacing savings by its values, it becomes:

$$k_{t+1} = \frac{\tilde{\eta}(1-\alpha)\Lambda k_t^\alpha}{(1+n)(2+\tilde{\rho})} + \frac{\hat{\eta}(1-\alpha)\Lambda k_t^\alpha}{(1+n)(2+\hat{\rho})} \tag{18}$$

This is a recurrence equation that describes the dynamics of the capital in an open economy. At steady state $k_{t+1} = k_t = k^*$, by resolving the recurrence equation, steady state capital-labor ratio in an open economy is:

$$k^* = \left(\frac{\tilde{\eta}(1-\alpha)\Lambda}{(1+n)(2+\tilde{\rho})} + \frac{\hat{\eta}(1-\alpha)\Lambda}{(1+n)(2+\hat{\rho})} \right)^{\frac{1}{1-\alpha}} \tag{19}$$

The following figure illustrates the convergence and steady state. It assumes that before the opening, the two countries were at steady state.

For Hat Country, since it is a net lender $\hat{e} > 0$, its capital is lower in an open economy. Domestic agents spend a portion of their savings on investment outside the territory this leads to a reduction of productive capital in the country. For the Tilde Country, since it is a net borrower $\tilde{e} < 0$ its capital is higher in the open economy. Productive capital in the borrowing country is increased by the savings of foreigners coming to invest in the country. Fig.(1), illustrates the transitional dynamics. At time of opening (t) Hat Country lends $\hat{\eta}\hat{e}_t = -\tilde{\eta}\tilde{e}_t$ to Tilde Country. Hat Country reduces its capital stock, the rate of wages

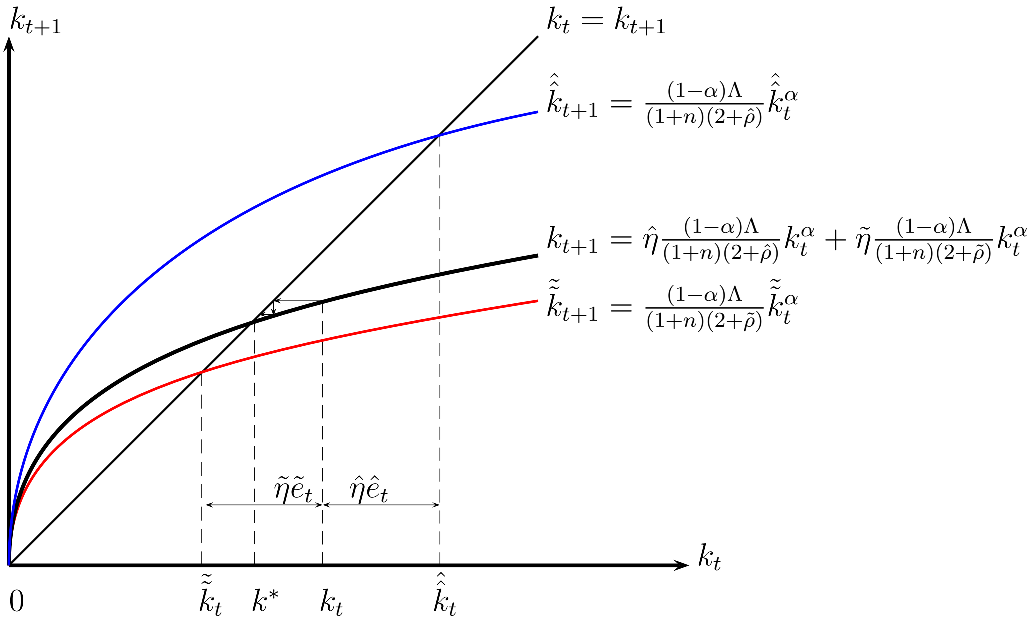

Figure 1: Closed economies and open economy

and therefore their savings in the next period. The reverse is done for Tilde Country[2] . Given the concavity of production function, Hat Country's saving decreases more than Tilde Country's saving increases. Globally, total saving decreases and therefore capital stock for the next period decreases $k_{t+1} < k_t$. The capital-labor ratio converges to k^* as shown in Fig.(1). We see from the figure that the capital-labor ratio of the open world is below the average capital-labor ratio of the two countries in autarky.

It is shown in Appendix B that $\hat{\hat{k}} > \left(\hat{\eta}\hat{\hat{k}} + \tilde{\eta}\tilde{\tilde{k}}\right) > k^* > \tilde{\tilde{k}}$, $\hat{\hat{q}} > q^* > \left(\hat{\eta}\hat{\hat{q}} + \tilde{\eta}\tilde{\tilde{q}}\right) > \tilde{\tilde{q}}$ and $R^* = \tilde{\tilde{R}}\hat{\hat{R}}/\left(\tilde{\eta}\hat{\hat{R}} + \hat{\eta}\tilde{\tilde{R}}\right)$.

Proposition 1 *: The average capital-labor ratio is smaller in an open economy.*

Proposition 2 *:The average output per head is greater in an open economy if $\alpha \in]0, 1/2[$, smaller if $\alpha \in]1/2, 1[$.*

Proposition 3 *: The interest rate in an open economy is the weighted harmonic mean of autarky interest rates.*

Our model reproduces all the findings of Buiter to which we add three additional propositions that Buiter cannot give without specifying a production function. Buiter's Proposition 8, only indicates that the common steady-state open-economy capital-labor ratio lies between the two autarkic capital-labor ratios.

[2] In the short-run the effect of openness on welfare is positive in Hat Country where the first generation enjoys both, a high salary of autarky and high interest rates of the open economy. They are negative in the country Tilde. See Buiter (1981) for the analysis of short-run effects.

4. Long-Runs Gains from integration

We now study the long-term gains from integration. As Buiter, we will distinguish three cases according to the position in autarky of capital-labor ratios relative to the capital-labor ratio of the golden rule. It is well known that the golden rule is independent of the utility function.

By maximizing consumption at steady state ($\max c = f(k) - (n+\delta)\,k$), we obtain the golden rule: $f'(k) = R^{gold} = n + \delta$ or $r^{gold} = n$. This implies, using the Cobb-Douglas production function and $\delta = 1$ that:

$$R^{gold} = 1 + n \Rightarrow k^{gold} = \left(\frac{\alpha\Lambda}{1+n}\right)^{\frac{1}{1-\alpha}} \tag{20}$$

In a competitive economy where the utility function is a logarithmic function, the capital-labor ratio at steady state is given by (11). Comparing (20) and (11) the specification of functions implies:

$$\tilde{\tilde{k}} \quad \text{or} \quad \hat{\hat{k}} \quad \begin{smallmatrix}<\\>\end{smallmatrix} \quad k^{gold} \Leftrightarrow \alpha \begin{smallmatrix}>\\<\end{smallmatrix} \frac{1-\alpha}{2+\rho} \tag{21}$$

The left side of (21) is the share of capital and the right side is the savings rate of the country (see Appendix C). We can now determine the effects on long-term welfare, of integration, with our specification of production function using the Cobb-Douglas and the utility function as a logarithmic function. By introducing in utility function (2) the competitive equilibrium values of consumption and factor prices [3] we can write the steady state utility as a function of capital-labor ratio at steady state:

$$V = ln\left(\frac{1+\rho}{2+\rho}\right) - \frac{ln(2+\rho)}{1+\rho} + \frac{2+\rho}{1+\rho} ln((1-\alpha)\Lambda) + \frac{1}{1+\rho} ln(\alpha\Lambda) + \frac{2+\rho}{1+\rho}\left(\alpha - \frac{1-\alpha}{2+\rho}\right) lnk$$

$$\text{With} \frac{\partial V}{\partial k} \begin{smallmatrix}>\\<\end{smallmatrix} 0 \qquad \text{if} \qquad \alpha \begin{smallmatrix}>\\<\end{smallmatrix} \frac{1-\alpha}{2+\rho} \tag{22}$$

Taking into account that an open economy decreases the capital-labor ratio of the Hat Country and increases the capital-labor ratio of Tilde Country we obtain the following results from (21) and (22) in the three cases described by Buiter:

- Case 1. Both countries are in dynamic inefficiency in autarky ($\tilde{\tilde{k}} > k^{gold}$ and $\hat{\hat{k}} > k^{gold}$): Hat Contry have a higher and Tilde Contry have a lower stationary utility level (ambiguous in Buiter).

- Case 2. Both countries are in dynamic efficiency in autarky ($\tilde{\tilde{k}} < k^{gold}$ and $\hat{\hat{k}} < k^{gold}$): Hat Country have a lower (ambiguous in Buiter) and Tilde Country have a higher stationary utility level.

- Case 3. Countries are in autarky on opposite sides of the golden rule (($\tilde{\tilde{k}} < k^{gold}$ and $\hat{\hat{k}} > k^{gold}$): Hat Country have a higher (ambiguous in Buiter) and Tilde Country have a higher (ambiguous in Buiter) stationary utility level.

[3] $c_t^y = w_t - s_t, c_{t+1}^o(1 + r_{t+1})s_t$, $w_t = (1-\alpha)\Lambda k_t^{\alpha}, (1 + r_{t+1}) = \alpha\Lambda k_{t+1}^{\alpha-1}$, and $s_t = w_t/(2+\rho)$

Proposition 4 *: With the Cobb-Douglas production function and log-utility function, the two countries have highers stationary utility levels only if the autarky capital-labor ratios are on opposite sides of the golden rule.*

Using the Cobb-Douglas production function and log-utility function we avoid any ambiguity highlighted by Buiter. One could discuss whether this result does not come from the neutralization of substitution effects related to the use of the log-utility function. In particular, we might ask whether Hat Country does not benefit from higher interest rate. The following section shows that Proposition 4 remains true regardless of the utility and production functions.

5. Generalization

We generalize the previous result to all separable utility functions and all production functions with constant returns to scale with the OLG model.

Hypothesis 1: *We retain the general specification of separable utility function and constraints of the OLG model to express the utility in terms of factor prices:*

$$U\left(c^y, c^o\right) = u(c^y) + \frac{1}{1+\rho}u(c^o) \quad \text{with} \quad u'(c) > 0 \text{ and } u''(c) < 0$$

$$c^y = w - s(w,r) \quad \text{and} \quad c^o = (1+r)s(w,r)$$

According to the implicit function theorem in the plane (w,r) the slope of the indifference curve (IC) is[4]:

$$\frac{dw}{dr} = -\frac{\frac{\partial U}{\partial r}}{\frac{\partial U}{\partial w}} = -\frac{-\frac{\partial u}{\partial c^y}\frac{\partial s}{\partial r} + \frac{1}{(1+\rho)}\frac{\partial u}{\partial c^o}\left(s + (1+r)\frac{\partial s}{\partial r}\right)}{\frac{\partial u}{\partial c^y}\left(1 - \frac{\partial s}{\partial w}\right) + \frac{1}{(1+\rho)}\frac{\partial u}{\partial c^o}\frac{\partial s}{\partial w}(1+r)} = -\frac{s}{1+r} \tag{23}$$

Hypothesis 2: *We retain the general form of a production function with constant returns to scale and we use the condition of competitive factor prices :*

$$F(K,L) = Lf(k) \quad \text{with} \quad f'(k) > 0 \quad \text{and} \quad f''(k) < 0 \quad \text{thus} \quad w = f(k) - (r+\delta)k$$

Samuelson (1962) showed that in the plane (w,r) the slope of the Factor-Price Frontier (FPF) is :

$$\frac{dw}{dr} = -k \tag{24}$$

In autarky and at steady state in the OLG model, the equilibrium condition of the capital-labor ratio market is:

$$s = (1+n)k \tag{25}$$

From the three previous equations we can deduce that (In autarky and at steady state) the slopes of the indifference curve and the FPF are equal if:

$$\text{IC slope} = -\frac{1+n}{1+r}k = -k = \text{FPF slope} \tag{26}$$

[4]Dividing by $\frac{\partial u}{\partial c^o}$ and using $\frac{u'(c^y)}{u'(c^o)} = \frac{(1+r)}{(1+\rho)}$ we get the result.

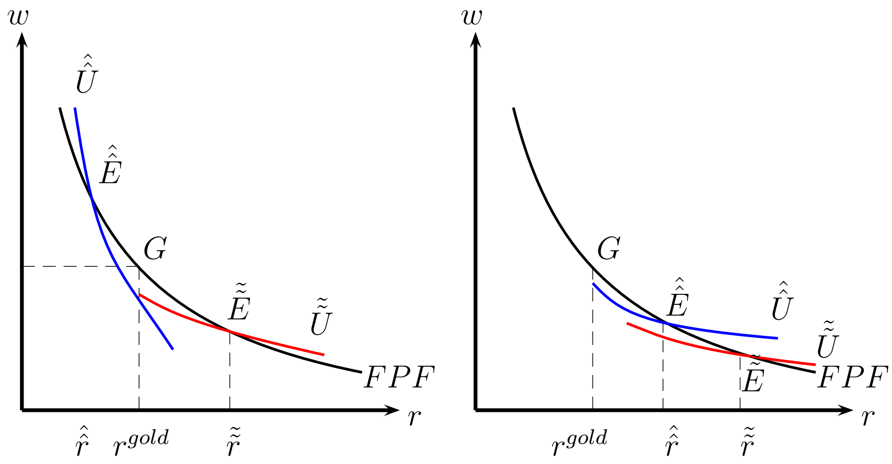

Figure 2: Indifference Curve and Factor-Price Frontier

That is if country is at golden rule ($r = n$). Outside the golden rule, if ($r < n$) then IC slope < FPF slope, if ($r > n$) then IC slope > FPF slope.

So in the plane (w, r), at left (right) of the golden rule, the slope of IC is lower (upper) the slope of the FPF. It follows therefore, that the stationary utility increases for a country at left (right) of the golden rule only if the capital-labor ratio decreases (increases). Since we have shown that during the capital market integration the capital-labor ratio increases for Tilde Country and decreases for Hat Country, the stationary utility level of Tilde Country and Hat Country can increase only if, in autarky, Tilde Country is on the right and Hat Country is on the left of the golden rule.

Proposition 5 *: In the OLG model, both countries increase their stationary utility levels only if the autarky capital-labor ratios are on opposite sides of the golden rule.*

Figure 2 illustrates this result. In autarky and at steady state, for given prices $(\hat{\hat{w}}, \hat{\hat{r}})$, the Hat Country maximizes its utility at point $\hat{\hat{E}}$. On the left graph, the slope of IC is less than the slope of FPF. The utility may be greater for a lower capital-labor ratio. The situation is the opposite for Tilde Country. Its utility may be greater for an upper capital-labor ratio.

The capital market integration increases the utility of both countries only if the two countries are on either side of the golden rule, not where the capital-labor ratio of the two countries is below or above the golden rule. The right graph of figure 2, shows that a lower capital-labor ratio decreases Hat Contry steady state utility. Thus both figures 5 and 6 are impossible in Buiter (p 792) (see Appendix D).

The message of this general result is important. The capital market integration necessarily reduces the capital of the most patient country. In the OLG model, the most patient country can only gain from long-term capital market integration if it is in autarky in dynamic inefficiency. Which may be the case of for China at present. The corollary is that, in the long run, a country in efficiency cannot gain from investing capital in foreign markets. Maddison (1982) stressed that the slowdown in economic growth in the U.K. at the end of the 19^{th} century was due to the fact that it had a massive foreign capital investment which was as high as its domestic investment.

6. Conclusion

In the OLG model, countries do not consistently gain well-being in the long run, when they integrate their capital markets. We have shown, in the OLG model, both countries will only gain in the particular case where they are both in autarky on opposite sides of the golden rule. More specifically the most patient country, net lender, can only gain if it is in autarky in dynamic inefficiency, with too much capital.

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Appendix A

- **Consumption in an open economy** (for Tilde Country): In an open economy savings equals $\tilde{s}_t = (1+n)\tilde{a}_{t+1}$ Consumption of young is:

$$\tilde{c}_t^y = w_t - s_t = (1-\alpha)\Lambda k_t^\alpha - (1+n)\tilde{a}_{t+1} \tag{27}$$

Consumption of old at date $t+1$: $\tilde{c}_t^o = (1+r_{t+1})(1+n)\tilde{a}_{t+1}$ and at t: $\frac{\tilde{c}_t^o}{1+n} = (1+r_t)\tilde{a}_t$
Total comsumption per capita of workers at t is:

$$\tilde{c}_t = (1-\alpha)\Lambda k_t^\alpha - (1+n)\tilde{a}_{t+1} + (1+r_t)\tilde{a}_t \tag{28}$$

- **Trade balance** (for Tilde Country):
By definition : $\tilde{Z}_t = (\tilde{X}_t - \tilde{M}_t) = \tilde{Q}_t - \tilde{C}_t - \tilde{I}_t = \tilde{N}_t f(k_t) - (\tilde{c}_t^y \tilde{N}_t + \tilde{c}_t^o \tilde{N}_{t-1}) - (\tilde{K}_{t+1} - \tilde{K}_t + \delta \tilde{K}_t)$. Dividing by $\tilde{N}_t$: $\tilde{z}_t = f(k_t) - \left(\tilde{c}_t^j + \frac{\tilde{c}_t^v}{1+n}\right) - ((1+n)k_{t+1} - (1-\delta)k_t)$

$$\tilde{z}_t = f(k_t) - \tilde{c}_t - ((1+n)k_{t+1} - (1-\delta)k_t) \tag{29}$$

- **Current account:** By definition $\tilde{CC}_t = \tilde{Z}_t + R_t\tilde{E}_t = \tilde{Z}_t + R_t(\tilde{A}_t - \tilde{K}_t)$. Dividing by $\tilde{N}_t$:

$$\tilde{cc}_t = \tilde{z}_t + R_t(\tilde{a}_t - k_t) \tag{30}$$

We can also write the current account as the excess of savings relative to investment:

$$\tilde{cc}_t = ((1+n)\tilde{a}_{t+1} - (1-\delta)\tilde{a}_t) - ((1+n)k_{t+1} - (1-\delta)k_t) \tag{31}$$

It can also be written as the increase in the holding of securities: $\tilde{CC}_t = \tilde{E}_{t+1} - (1-\delta)\,\tilde{E}_t$

$$\tilde{cc}_t = (1+n)\tilde{e}_{t+1} - (1-\delta)\tilde{e}_t \tag{32}$$

- **At steady state we obtain:**

  (30) becomes:
  $$\tilde{cc} = \tilde{z} + R\tilde{e} \tag{33}$$

  (31) becomes:
  $$\tilde{cc} = (n+\delta)(\tilde{a} - k) \tag{34}$$

  (29) becomes:
  $$\tilde{z} = f(k) - \tilde{c} - (n+\delta)k \tag{35}$$

  Using (33) and (34) $\tilde{z}$ becomes:
  $$\tilde{z} = (n-r)\tilde{e} \tag{36}$$

  (28) becomes:
  $$\tilde{c} = \Lambda k^{\alpha} - (n+\delta)k + (r-n)\tilde{e} = \Lambda k^{\alpha} - (n+\delta)k - \tilde{z} \tag{37}$$

  and finally :
  $$c = \hat{\eta}\hat{c} + \tilde{\eta}\tilde{c} = \Lambda k^{\alpha} - (n+\delta)k \tag{38}$$

Appendix B

- **Proposition 1:** *The capital-labor ratio decreases in an open economy.*

  Show that $\left(\hat{\eta}\hat{k} + \tilde{\eta}\tilde{k}\right) > k^*$ where $\left(\hat{\eta}\hat{k} + \tilde{\eta}\tilde{k}\right)$ is the world average capita-labor ratio in autarky. Assuming $X = \left(\frac{(1-\alpha)\Lambda}{(1+n)(2+\rho)}\right)$, since $\alpha \in]0,1[$, we have $\frac{1}{1-\alpha} > 1$ and the function $X^{\frac{1}{1-\alpha}}$ is convex and therefore $\hat{\eta}\hat{X}^{\frac{1}{1-\alpha}} + \tilde{\eta}\tilde{X}^{\frac{1}{1-\alpha}} > \left(\hat{\eta}\hat{X} + \tilde{\eta}\tilde{X}\right)^{\frac{1}{1-\alpha}}$.

- **Proposition 2:** *Output per capita increases in global open economy if* $0 < \alpha < 1/2$ *and decrease if* $1/2 < \alpha < 1$.

  We will show that $q^* > \hat{\eta}\hat{q} + \tilde{\eta}\tilde{q}$ where $\hat{\eta}\hat{q} + \tilde{\eta}\tilde{q}$ is the average output per worker in autarky and $q = k^{\alpha}$. Since $0 < \alpha < 1/2$, we have $0 < \frac{\alpha}{1-\alpha} < 1$ and the function $X^{\frac{\alpha}{1-\alpha}}$ is concave and thus $\left(\hat{\eta}\hat{X} + \tilde{\eta}\tilde{X}\right)^{\frac{\alpha}{1-\alpha}} > \hat{\eta}\left(\hat{X}\right)^{\frac{\alpha}{1-\alpha}} + \tilde{\eta}\left(\tilde{X}\right)^{\frac{\alpha}{1-\alpha}}$. Conversely if $1/2 < \alpha < 1$ the function is convex and the average per worker world production decreases.

- **Proposition 3:** *The open economy interest rate is the weighted harmonic average interest rate of autarky.*

  In autarky for Hat country: $\hat{\hat{R}} = \alpha\Lambda\hat{\hat{k}}^{\alpha-1} = \alpha\Lambda\left(\hat{X}\right)^{\frac{\alpha-1}{1-\alpha}}$

  In open economy: $R = \alpha\Lambda k^{\alpha-1} = \alpha\Lambda\left(\tilde{\eta}\tilde{X} + \hat{\eta}\hat{X}\right)^{\frac{\alpha-1}{1-\alpha}}$

$$R = \alpha\Lambda\left(\tilde{\eta}\frac{\alpha\Lambda}{\tilde{\tilde{R}}} + \hat{\eta}\frac{\alpha\Lambda}{\hat{\hat{R}}}\right)^{-1} = \alpha\Lambda\left(\frac{\tilde{\tilde{R}}\hat{\hat{R}}}{\tilde{\eta}\alpha\Lambda\hat{\hat{R}} + \hat{\eta}\alpha\Lambda\tilde{\tilde{R}}}\right) = \frac{\tilde{\tilde{R}}\hat{\hat{R}}}{\tilde{\eta}\hat{\hat{R}} + \hat{\eta}\tilde{\tilde{R}}}$$

Appendix C

- **The saving rate:** Young's savings is: $s_t = \frac{w_t}{2+\rho} = \frac{1-\alpha}{2+\rho}y_t = \zeta.y_t$. The overall savings for the period $t, t+1$ is: $S_t = N_t s_t - (K_t - \delta K_t)$ whether $S_t = N_t\zeta y_t - \zeta Y_{t-1} + \delta\varsigma Y_{t-1}$ therefore $\frac{S_t}{Y_t} = \zeta\frac{Y_t}{Y_t} - \zeta\frac{Y_{t-1}}{Y_t} + \delta\zeta\frac{Y_{t-1}}{Y_t}$ therefore $\frac{S_t}{Y_t} = \zeta - \zeta\frac{1}{(1+n)} + \delta\zeta\frac{1}{(1+n)}$. Macroeconomic saving rate is:

$$e = \zeta\left(1 - \frac{1}{(1+n)} + \frac{\delta}{(1+n)}\right) = \zeta\left(\frac{\delta+n}{1+n}\right)$$

  Assuming $\delta = 1$, the saving rate is: $e = \zeta = \frac{(1-\alpha)}{2+\rho}$

Appendix D

The case of Fig.6 in Buiter (p. 792) correspond to the following figure in our plan (w, r). We showed that this case is impossible. One country (Hat) having a capital-labor ratio lower than the golden rule capital-labor ratio cannot have at steady state an IC slope less than the slope of the FPF as shown in the following figure.

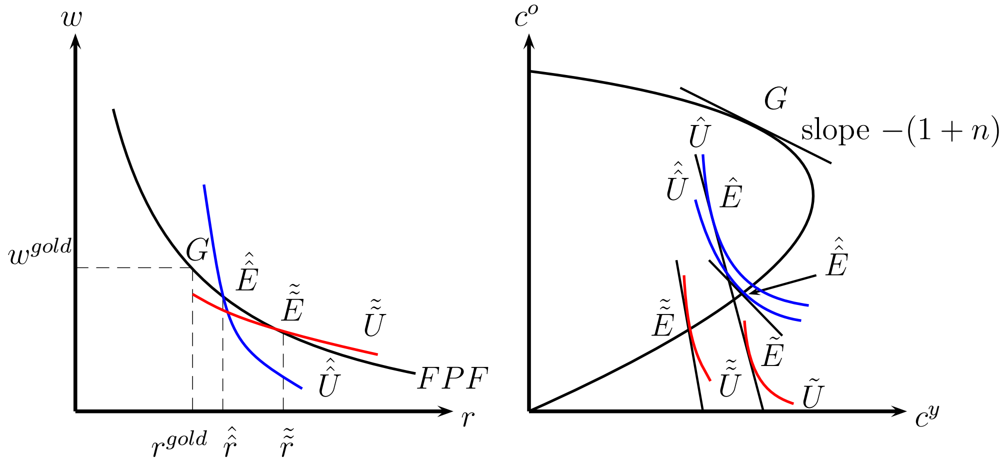

Figure 3: Impossible case of Buiter

On its Fig.6 (p. 792), Buiter draws a constraint whose slope is dependent on r and the intercept depends on w. But there is a link between r and w (our FPF) which Buiter ignores. The strong increase in r that Buiter represents, implies a strong decrease in w and therefore the intercept of the constraint can be as high as Buiter illustrated.