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A note on justifiable preferences over opportunity sets

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Abstract

A ranking over opportunity sets is justifiable if there exists a binary relation on the set of alternatives, such that one opportunity set is at least as good as the second, if and only if there exists at least one alternative in the first set which is at least as good as any alternative of the two sets combined. This note characterizes (reflexive and complete) opportunity sets rankings which can be justified by acyclic binary relations – the broadest possible class of justifiable rankings.

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1 Introduction

Many individual and collective choice problems naturally possess two stages. For example, when one chooses a meal, he/she has to choose a restaurant first and subsequently chooses a meal from the menu of the chosen restaurant. In the first stage, choices over opportunity sets (menus) are made and these choices effectively constrain the opportunity of the second stage. It is then of interest to investigate how opportunity sets are evaluated. There are two approaches addressing this issue. One way is to emphasize the intrinsic value of flexibility provided by opportunity sets, see Dowding and Van Hees (2009) for a survey.

The other route, the classical indirect utility approach, evaluates opportunity sets on strictly instrumentalist ground, i.e. the utility of the "best" alternative in an opportunity set determines its value. Lahiri (2003) extends this approach to ordinal preferences: a preference over opportunity sets is *justifiable* if there exists a binary relation on the set of alternatives, such that one opportunity set is at least as good as the second, if and only if there is at least one alternative in the first set which is at least as good as any alternative of the two sets combined. Lahiri (2003) characterizes transitive and quasi-transitive justifiable rankings. The objective of this note is to extend Lahiri's (2003) work to explore the necessary and sufficient conditions for rankings over opportunity sets to be justifiable by acyclic binary relations, which is the broadest possible class of justifiable rankings.

The remainder of this note is structured as follows. Section 2 introduces framework. Section 3 characterizes acyclic justifiability. Section 4 concludes.

2 Framework

We begin with a finite set of alternatives X. For any set S, let $Pow(S)$ denote the collection of all nonempty subsets of S.

A binary relation $\succeq$ on a set S is a subset of $S \times S$. A binary relation $\succeq$ is transitive if $(x, y), (y, z) \in \succeq$ implies $(x, z) \in \succeq$. Given a binary relation $\succeq$, let $\succ$ denote its asymmetric part, i.e. $\succ = \{(x, y) \in \succeq : (y, x) \notin \succeq\}$. A binary relation is quasi-transitive if its asymmetric part is transitive. Given a natural number m, a binary relation is said to have cycle of order m if there exist $x_1, \dots, x_m \in S$ such that $(x_1, x_2), (x_2, x_3), \dots, (x_{m-1}, x_m), (x_m, x_1) \in \succeq$. A binary relation is said to be *acyclic* if its asymmetric part contains no cycle of any order. Given a binary relation $\succeq$ and a set S, let $G_{\succeq}(S) = \{x \in S : \forall y \in S, (x, y) \in \succeq\}$, i.e. the set of greatest elements in S in terms of $\succeq$.

A binary relation $\succeq$ on X is said to be

(a) reflexive, if for all $x \in X$, $(x, x) \in \succeq$;

(b) complete, if for all $x, y \in X$ & $x \neq y$, either $(x, y) \in \succeq$ or $(y, x) \in \succeq$;

(c) an abstract game, if it is both reflexive and complete.

A binary relation $\succeq^*$ on $Pow(X)$ is said to be

(a) reflexive, if for all $A \in Pow(X)$, $(A, A) \in \succeq^*$;

(b) complete, if for all $A, B \in Pow(X)$ & $A \neq B$, either $(A, B) \in \succeq^*$ or $(B, A) \in \succeq^*$;

(c) a preference over opportunity sets (POS), if it is both reflexive and complete.

The objective of this note is to explore the necessary and sufficient conditions for a POS to be guided by an acyclic abstract game.

3 Characterizing acyclic justifiability

Lahiri (2003) calls a POS $\succeq^*$ *justifiable* if there exists an abstract game $\succeq$ such that

$$A \succeq^* B \Leftrightarrow A \cap G_{\succeq}(A \cup B) \neq \emptyset \tag{J}$$

In words, A is at least as good as B if and only if A contains at least one alternative which achieves the highest value in two sets combined.

In order to characterize justifiability, Lahiri (2003) introduces the following conditions. $\succeq^*$ is said to satisfy Concordance (C) if

$$A \succeq^* B \Rightarrow A \succeq^* A \cup B \tag{C}$$

Concordance says for instance if the opportunity to consume tea is at least as good as the opportunity to consume coffee, then the former should be at least as good as the opportunity to consume either tea or coffee. $\succeq^*$ is said to satisfy Monotonicity (M) if

$$A \subseteq B \Rightarrow B \succeq^* A \tag{M}$$

Theorem 1 (Lahiri (2003, theorem 1))**.** *A POS $\succeq^*$ is justifiable by a transitive abstract game $\succeq$ if and only if $\succeq^*$ satisfies C, M, and transitivity.*

Lahiri (2003) introduces following conditions to characterize quasi-transitive justifiability. $\succeq^*$ satisfies Strict Concatenation (SC) if

$$[A \succ^* C \ \ \& \ \ B \succ^* D] \Rightarrow A \cup B \succ^* C \cup D \tag{SC}$$

Strict Concatenation says for instance if tea is preferred to coffee and apple juice to orange juice, then the opportunity to consume either tea or apple juice is preferred to the opportunity to consume either coffee or orange juice.

$\succeq^*$ satisfies Weak Expansion (WE) if

$$\{x\} \succeq^* A \ \ \& \ \ \{x\} \succeq^* B \Rightarrow \{x\} \succeq^* A \cup B \tag{WE}$$

Weak Expansion is a limited extension of SC to weak preferences.

For the purpose of this note, we decompose Lahiri's (2003) Strict Monotonicity into two conditions: Expansion Monotonicity (EM) and Contraction Monotonicity (CM).

$$A \succeq^* B \Rightarrow A \cup C \succeq^* B \tag{EM}$$

$$A \succeq^* B \Rightarrow A \succeq^* B \setminus C \tag{CM}$$

Expansion Monotonicity says if A is at least as good as B then adding more alternatives into A should not affect the ranking. Contraction Monotonicity says if A is at least as good as B then subtracting alternatives from B should not affect the ranking. Lahiri's (2003) monotonicity is defined on asymmetric part of $\succeq^*$, i.e. $\succ^*$. We connect our definition with the original one by the following observation. The proof is straightforward hence omitted.[1]

Lemma 1. *1. EM is equivalent to the following:*

$$A \succ^* B \Rightarrow A \succ^* B \setminus C$$

2. CM is equivalent to the following:

$$A \succ^* B \Rightarrow A \cup C \succ^* B$$

Proposition 1. *If a POS $\succeq^*$ is induced by an abstract game $\succeq$ via* (J)*, then $\succeq^*$ satisfies CM.*

Proof. $A \succeq^* B$
$\Rightarrow A \cap G_{\succeq}(A \cup B) \neq \emptyset$ by (J)
$\Rightarrow A \cap G_{\succeq}(A \cup (B \setminus C)) \neq \emptyset$
$\Rightarrow A \succeq^* B \setminus C$ by (J)
□

The next theorem is a variation of Lahiri's (2003, theorem 3) characterization of quasi-transitive justifiability.

Theorem 2. *A POS $\succeq^*$ is justifiable by a quasi-transitive abstract game $\succeq$ if and only if $\succeq^*$ satisfies WE, SC, EM, CM, and quasi-transitivity.*

Proof. In light of Lahiri's (2003, theorem 3) and proposition 1, it is left to establish the necessity of EM. Actually, it suffices to show that for any $x \notin A$, $A \succeq^* B$ implies $A \cup \{x\} \succeq^* B$. Suppose $A \succeq^* B$. By (J), there exists $y \in A$ such that $\forall z \in A \cup B, y \succeq z$.

Case 1: If $y \succeq x$, then $\forall z \in A \cup B \cup \{x\}, y \succeq z$. By (J), $A \cup \{x\} \succeq^* B$.

Case 2: If $x \succ y$, then $\forall z \in A \cup B \cup \{x\}, x \succeq z$ by quasi-transitivity. By (J), $A \cup \{x\} \succeq^* B$.
□

In this note, we explore necessary and sufficient conditions for acyclic justifiability. First, observe that any justifying $\succeq$ must be acyclic.

Proposition 2 (Lahiri (2003, proposition 2))**.** *A POS $\succeq^*$ is justifiable if and only if it is justifiable by an acyclic abstract game $\succeq$.*

Observe also that if $\succeq^*$ is induced by $\succeq$ via (J), then $\succeq^*$ is also acyclic. Recall that given reflexivity and completeness, acyclicity of $\succeq$ is necessary and sufficient for the following: $G_{\succeq}(S) \neq \emptyset$ for all $S \subseteq X$ (see, e.g., Kreps 1988).

[1] Lahiri's (2003) Strict Monotonicity is slightly weaker than EM and CM combined.

Proposition 3. *If a POS $\succeq^*$ is induced by an abstract game $\succeq$ via* (J)*, then $\succeq^*$ is acyclic.*

Proof. Suppose that $\succeq^*$ is justified by $\succeq$. By proposition 2, $\succeq$ is acyclic. Towards a contradiction, assume there exist a natural number m and $A_1, \dots, A_m \in Pow(X)$ such that $A_1 \succ^* A_2 \succ^* \cdots \succ^* A_m \succ^* A_1$. By (J),

$$\forall x \in A_2, \exists y \in A_1 \cup A_2, s.t.\ y \succ x$$

$$\forall x \in A_3, \exists y \in A_2 \cup A_3, s.t.\ y \succ x$$

$$\vdots$$

$$\forall x \in A_m, \exists y \in A_{m-1} \cup A_m, s.t.\ y \succ x$$

$$\forall x \in A_1, \exists y \in A_m \cup A_1, s.t.\ y \succ x$$

Therefore, $\forall x \in A_1 \cup \cdots \cup A_m$, there exists $y \in A_1 \cup \cdots \cup A_m$ such that $y \succ x$. This is equivalent to $G_{\succeq}(A_1 \cup \cdots \cup A_m) = \emptyset$, which is impossible since $\succeq$ is acyclic. □

To achieve our characterization, we relax EM to Weak Expansion Monotonicity (WEM):

$$[A \succeq^* B \ \ \& \ \ C \subseteq B] \Rightarrow A \cup C \succeq^* B \tag{WEM}$$

Weak Expansion Monotonicity says if A is at least as good as B then adding elements of B into A should not affect the ranking. We are now ready for our main theory.

As an auxiliary step, we define the base relation induced by $\succeq^*$. For any POS $\succeq^*$, the base relation $\succeq^B$ induced by $\succeq^*$ is defined by

$$\succeq^B = \{(x, y) \in X \times X : (\{x\}, \{y\}) \in \succeq^*\}$$

Theorem 3. *A POS $\succeq^*$ is justifiable by an abstract game $\succeq$ if and only if it satisfies C, WE, WEM, CM, and SC. In such cases, $\succeq$ is acyclic.*

Proof. $\Rightarrow$: In light of Lahiri (2003, proposition 11, 13, and 15) and proposition 1, It is left to establish WEM. Suppose $\succeq^*$ is justifiable by $\succeq$.

WEM: $A \succeq^* B$

$\Rightarrow A \cap G_{\succeq}(A \cup B) \neq \emptyset$ — by (J)

$\Rightarrow (A \cup C) \cap G_{\succeq}(A \cup C \cup B) \neq \emptyset$ — by $C \subseteq B$

$\Rightarrow A \cup C \succeq^* B$ — by(J)

$\Leftarrow$: We show that $\succeq^*$ can be justified by its associate base relation $\succeq^B$.

Suppose $A \succeq^* B$ and assume towards a contradiction that $A \cap G_{\succeq^B}(A \cup B) = \emptyset$. Then

$$\forall x \in A, \exists y(x) \in A \cup B, s.t.\ y(x) \succ^B x$$

By definition of $\succeq^B$,

$$\forall x \in A, \exists y(x) \in A \cup B, s.t.\ \{y(x)\} \succ^* \{x\}$$

By (SC),

$$\{y(x) : x \in A\} \succ^* A$$

By (CM) and lemma 1,

$$A \cup B \succ^* A$$

This is a contradiction because $A \succeq^* B$ implies $A \succeq^* A \cup B$ by (C).

Now suppose $A \cap G_{\succeq^B}(A \cup B) \neq \emptyset$. Then there exists $x \in A$ such that

$$\forall y \in A \cup B, x \succeq^B y$$

By definition of $\succeq^B$,

$$\forall y \in A \cup B, \{x\} \succeq^* \{y\}$$

By (WE),

$$\{x\} \succeq^* A \cup B$$

By (WEM),

$$A \succeq^* A \cup B$$

By (CM), $A \succeq^* B$.

We complete the proof by showing that the acyclicity of $\succeq^B$ is also guaranteed by SC. Assume there exist a natural number m and $x_1, \ldots, x_m \in X$ such that $x_1 \succ^B \cdots \succ^B x_m \succ^B x_1$. By definition of $\succeq^B$,

$$\{x_1\} \succ^* \{x_2\}$$

$$\vdots$$

$$\{x_{m-1}\} \succ^* \{x_m\}$$

$$\{x_m\} \succ^* \{x_1\}$$

By (SC), $\{x_1, \ldots, x_m\} \succ^* \{x_1, \ldots, x_m\}$, violating reflexivity.

□

We conclude this section by providing an example of justifiable POS which cannot be justified by any quasi-transitive abstract game.

Example 1. *Let* $X = \{x, y, z\}$. *Let POS* $\succeq^*$ *be defined as follows:* $\succeq^* = Pow(X) \times Pow(X) \setminus \{(\{y\}, \{x\}), (\{y, z\}, \{x\}), (\{z\}, \{y\}), (\{y\}, \{x, y\}), (\{z\}, \{x, y\}), (\{y, z\}, \{x, y\}), (\{y\}, \{x, z\}), (\{y, z\}, \{x, z\}), (\{z\}, \{y, z\}), (\{y\}, \{x, y, z\}), (\{z\}, \{x, y, z\}), (\{y, z\}, \{x, y, z\})\}$

It is verifiable that this POS can only be justified by the following abstract game: $X \times X \setminus \{(y, x), (z, y)\}$, which is not quasi-transitive.

4 Concluding remarks

Lahiri (2003) characterizes transitive and quasi-transitive justifiability. In this note, we extend Lahiri's (2003) work to characterize acyclic justifiability, completing the axiomatization of justifiable POS with major coherence conditions.

All the characterizing results rely on the revealed base relation: rankings over singleton sets. The base relation can be constructed because we assume completeness, i.e. rankings over all pairs of opportunity sets are observable. When completeness is dropped, there exists $\succeq^*$ which satisfies all axiomatic conditions (vacuously) but is not justifiable. There remains much scope in developing theories without completeness.

References

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