Economics Bulletin

Volume 34, Issue 3

Failures of reversal symmetry under two common voting rules

Ahmadou lô Gueye
Université Saint-Louis, Sénégal

Abstract

The aim of this paper is to study the likelihood of violations of reversal symmetry under two common voting rules, plurality and instant runoff. We show that plurality is much more sensitive to this phenomenon.

I am very grateful to Boniface Mbih for helpful comments and suggestions
Citation: Ahmadou lô Gueye, (2014) ''Failures of reversal symmetry under two common voting rules'', *Economics Bulletin*, Vol. 34 No. 3 pp. 1970-1975.
Contact: Ahmadou lô Gueye - losakhir@netcourrier.com.
Submitted: August 04, 2014. **Published:** September 13, 2014.

1 Introduction

Reversal symmetry is an axiom of social choice theory that prevents the selection of an alternative identified as best one and simultaneously as worst one. In other terms, a voting rule satisfying reversal symmetry does not select the same alternative when all individual preferences are reversed. This axiom was introduced in the social choice literature by Saari (1999). Unfortunately, some common voting rules do not satisfy reversal symmetry. Two of these rules are plurality and instant runoff. Plurality rule selects the alternative with the greatest number of first place positions in individual preferences, and instant runoff selects the alternative with a majority of first place positions, and if there is none, a runoff is organized between the two alternatives with the most first place positions. These two rules are probably the most used throughout the world for voting in committees, and they are also very usual in political elections.

In this paper, we evaluate how frequent they violate reversal symmetry. We consider the three-alternative case and use complete computer enumeration: for every value of the number of voters, we compute the total number of profiles at which reversal symmetry is violated, and the frequency of violation is obtained as the ratio between this number and the total number of profiles. We also provide limit values of these frequencies when the number of voters tends to infinity. These latter values are obtained by the use of the Fishburn-Gehrlein technique (also used by Mbih, Moyouwou and Picot (2008), among others).

The remainder of the paper is organized as follows: Notation and definitions are presented in Section 2; the results are discussed in Section 3; and Section 4 concludes the paper.

2 Notation and definitions

Consider a society N with n voters who have to choose one alternative out of a three-element set $A = \{a, b, c\}$. Each individual $i \in N$ reports a strict preference relation, that is a linear order (complete, asymmetric and transitive) on A. A *profile* is an $n-$tuple of individual preference relations. A *voting rule* selects a single alternative from each possible profile; we consider two voting rules, plurality and instant runoff, whose definitions have been given in the introduction above. For the sake of simplicity, we deliberately ignore the possibility of ties. Let π be a profile, π^{-1} the profile in which all individual preferences in π have been reversed, and F a voting rule, then, reversal symmetry is satisfied under F if for any possible profile π, $F(\pi) \neq F(\pi^{-1})$.

Tables 1 and 2 below illustrate a violation of reversal symmetry under plurality rule.

Example 1. Violation of reversal symmetry under plurality

Table 1. Profile π

	number of voters					
	10	30	10	25	5	20
preference order	a	a	b	b	c	c
	b	c	a	c	a	b
	c	b	c	a	b	a

Table 2. Profile π^{-1}

	number of voters					
	20	25	5	30	10	10
preference order	a	a	b	b	c	c
	b	c	a	c	a	b
	c	b	c	a	b	a

In profile π (Table 1), alternative a gets 40 votes, b 35 votes and c gets 25 votes. a is the selected outcome. In the reversed profile π^{-1} (Table 2), a, b, and c get 45, 35 and 20 votes, respectively; again, a is the selected outcome.

Note that in this example, instant runoff selects two different outcomes: there are 100 voters, and no alternative in profile π obtains more than 50 votes; a runoff between a and b leads to the victory of b. And in the reversed profile, now the runoff between a and b leads to the victory of a.

So, from this example, we conclude with a first observation:

Fact 1. *There are profiles at which plurality violates reversal symmetry, but instant runoff does not.*

Now, let us consider another example.

Example 2. Violation of reversal symmetry under instant runoff

Table 3. Profile π'

	number of voters					
	4	0	0	3	2	0
preference order	a	a	b	b	c	c
	b	c	a	c	a	b
	c	b	c	a	b	a

Table 4. Profile π'^{-1}

	number of voters					
	0	3	2	0	0	4
preference order	a	a	b	b	c	c
	b	c	a	c	a	b
	c	b	c	a	b	a

In both profiles, instant runoff selects a, but plurality selects a in profile π' and c in profile π'^{-1}.

From which we conclude with the following further observation:

Fact 2. *There profiles at which instant runoff violates reversal symmetry, but plurality does not.*

Now, from Facts 1 and 2, when comparing plurality with instant runoff, we cannot conclude that one of the two rules is more sensitive to violations of reversal symmetry, since there is no inclusion relation between the sets of profiles at which these violations occur.

One way to go beyond these two facts is to evaluate how frequent violations of reversal symmetry occur under each rule. This is the purpose of the next section.

3 The likelihood of reversal symmetry

In order to compute the frequency of violation of reversal symmetry, we first introduce some further notations. Every possible preference order on A is a permutation of the three alternatives. There are 6 such permutations:

$$\begin{array}{lll} R_1 : abc & R_2 : acb & R_3 : bac \\ R_4 : bca & R_5 : cab & R_6 : cba \end{array}$$

For each of these preference orders, let n_k be the number of voters reporting preference order k, $k = 1, \ldots, 6$. Then, each profile can now be rewritten as a 6−component vector $v = (n_1, n_2, n_3, n_4, n_5, n_6)$. Such vectors are called anonymous voting profiles. And the likelihood of reversal symmetry is given by the following ratio:

$$\frac{\text{number of anonymous voting profiles at which reversal symmetry is violated}}{\text{total number of anonymous voting profiles}}.$$

By so doing, we compute this frequency under the hypothesis of impartial anonymous culture (IAC) as distinguished from other hypotheses, and especially from impartial culture (IC). For a further understanding of these models, see Gehrlein (2006) or Gehrlein and Lepelley (2012). Notice that the total number of voting situations is given by

$$\binom{n+5}{5} = \frac{1}{120}\left(n+5\right)\left(n+4\right)\left(n+3\right)\left(n+2\right)\left(n+1\right).$$

To see this, place $n+5$ empty cells in a horizontal row and place check marks in five of the cells. This defines an anonymous voting profile by letting n_1 be the number of cells to the left of the first check, n_2 the number of cells between the first and second checks, and so on up to n_6 the number of cells to the right of the fifth check mark.

It remains to compute the numerator of the above ratio. In order to do this, we subdivide the set of voting situations according to which alternative is chosen.

For plurality rule, we obtain:

$$S_1 : \begin{cases} n_1+n_2 > n_3+n_4 \\ n_1+n_2 > n_5+n_6 \\ n_4+n_6 > n_2+n_5 \\ n_4+n_6 > n_1+n_3 \\ n_1+n_2+n_3+n_4+n_5+n_6 = n \end{cases} , \quad S_2 : \begin{cases} n_3+n_4 > n_1+n_2 \\ n_3+n_4 > n_5+n_6 \\ n_2+n_5 > n_4+n_6 \\ n_2+n_5 > n_1+n_3 \\ n_1+n_2+n_3+n_4+n_5+n_6 = n \end{cases}$$

$$S_3 : \begin{cases} n_5+n_6 > n_1+n_2 \\ n_5+n_6 > n_3+n_4 \\ n_1+n_3 > n_4+n_6 \\ n_1+n_3 > n_2+n_5 \\ n_1+n_2+n_3+n_4+n_5+n_6 = n \end{cases}$$

The cardinality of each of these sets is given by the number of integer solutions of the corresponding set of inequalities. S_1,S_2 and S_3 are similar and then it suffices to compute the cardinality of either one of the sets and to multiply it by 3. The computations have been obtained by complete computer enumeration, using the Fishburn-Gehrlein technique. For each value of n (the number of voters), the frequencies are given in Table 5.

More specifically, the numerator of the ratio above constitutes a linear system S that defines a parametrized rational convex polytope $S(n)$ of dimension 5 in $\mathbb{R}^6$. The problem then amounts to counting the number of integer lattice points inside $S(n)$. The solution of this problem consists (1) in subdividing the polytope into a union of simpler ones (S_1, S_2, and S_3 here), (2) in determining the limits of variation of each variable n_k and then (3) in computing each piece by multiple summation. This process leads to periodic polynomials where each polynomial is valid for some specific values of n. Because of the very high periodicity of the polynomials for the problem studied in this paper, we skip the details of our computations and we only provide numeric values of the frequencies of reversal symmetry. For similar details on a problem with a smaller periodicity - the Condorcet's paradox - the interested reader can refer to Gehrlein and Fishburn (1976).

Similarly, for instant runoff, a subdivision of the set of anonymous voting profiles has been obtained, from which we have computed the frequencies in Table 5. We distinguish six cases, two leading to the selection of each of the three alternatives. For example, for the selection of alternative a, the two disjoint subsets are:

$$T_{11}: \begin{cases} n_1 + n_2 > n_5 + n_6 \\ n_3 + n_4 > n_5 + n_6 \\ n_1 + n_2 + n_5 > n_3 + n_4 + n_6 \\ n_4 + n_6 > n_2 + n_5 \\ n_1 + n_3 > n_2 + n_5 \\ n_4 + n_5 + n_6 > n_1 + n_2 + n_3 \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \end{cases} , \quad T_{12}: \begin{cases} n_1 + n_2 > n_3 + n_4 \\ n_5 + n_6 > n_3 + n_4 \\ n_1 + n_2 + n_3 > n_4 + n_5 + n_6 \\ n_4 + n_6 > n_1 + n_3 \\ n_2 + n_5 > n_1 + n_3 \\ n_3 + n_4 + n_6 > n_1 + n_2 + n_5 \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \end{cases}$$

Complete computer enumeration gives the results in Table 5. It clearly appears that for any value of n (except for $n = 5$), plurality rule violates reversal symmetry more often that instant runoff. As the number of voters tends to infinity (that is in large electorates) both frequencies rise, and at the limit ($n = \infty$), the probability of violation is slightly over 9% for plurality, while it is slightly less than 3% for instant runoff.

Table 5. Frequencies of violations of reversal symmetry.

n	plurality	instant runoff	n	plurality	instant runoff
4	0.023809	0.000000	16	0.057497	0.007961
5	0.000000	0.023810	17	0.053315	0.022329
6	0.025974	0.000000	18	0.060002	0.009451
7	0.037879	0.007576	19	0.062112	0.020610
8	0.020979	0.004662	20	0.057933	0.011067
9	0.041958	0.020979	21	0.064032	0.023168
10	0.044955	0.001998	24	0.066540	0.012833
11	0.037088	0.001403	33	0.072678	0.024684
12	0.048966	0.001083	42	0.076296	0.017731
13	0.053221	0.018908	51	0.078902	0.025776
14	0.045666	0.006708	60	0.080727	0.020427
15	0.056115	0.020511	∞	0.0925926	0.028356

4 Conclusion

The aim of this paper was to evaluate the possibility of violations of reversal symmetry under plurality and instant runoff. The results presented in this paper are a preliminary step to a more complete evaluation of the frequency of this phenomenon. Many directions are possible for further investigation, including the study of other probability models (impartial culture, maximal culture, etc. see Regenwetter et al. 2006) and the study of other voting rules, for a more complete comparison.

References

Gehrlein WV (2006), *Condorcet's paradox*. Springer, Berlin Gehrlein WV (1976).

Gehrlein WV, Fishburn (1976), "Condorcet's paradox and anonymous preference profiles" *Public Choice* 26, 1-18.

Gehrlein WV, Lepelley D (2012) "The value of research based on simple assumptions on voters' preferences", in Felsenthal D, Machover M (eds). *Electoral Systems*, Springer, Berlin.

Mbih B, Moyouwou I, Picot (2008), "Pareto violations of parliamentary voting rules" *Economic Theory* 34, 331-358.

Saari DG, (1999), "Explaining all three-alternative voting outcomes" *Journal of Economic Theory* 87, 313-355.

Regenwetter M, Grofman B, Marley AAJ, Tsetlin IM (2006) *Behavioral social choice: probabilistic models, statistical inference, and applications.* Cambridge University Press, Cambridge