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Convexity of the central bank's loss function and dependence between monetary instruments

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Abstract

In this paper we obtain conditions under which the central bank loss function is strictly convex in four different states of the economy: booming economy, recession, high inflation and high output. Moreover, we found that when inflation and output are linear functions of the monetary policy instrument, convexity is guaranteed for any of the four states mentioned. When we extend our analysis to the case of many instruments, we found that only linearity is not sufficient to guarantee the shape of loss function. Our results also provide conditions under which there exists dependence between instruments of monetary policy.

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1 Introduction

A fundamental assumption made by all literature which studies the central bank's behavior is that its loss function is concave both in inflation and output. In fact, the quadratic form adopted by the seminal paper of Barro and Gordon (1983) has been used by a variety of researches in the field. Examples arise in standard models of inflationary bias and reputation (e.g. Backus and Driffill, 1985; Ball, 1995; Canzoneri, 1985; Cukierman and Meltzer, 1986; Rogoff, 1985), in studies which consider asymmetric preferences for the central bank and persistence in output (e.g. Jonsson, 1997; Nobay and Peel, 2003; Ruge-Murcia, 2003), and even in those that follow the New Keynesian approach (e.g Clarida et al., 1999; Damjanovic et al., 2008; Sauer, 2010). Further, Woodford (2002) uses microeconomic foundations and shows that the central bank's loss function is in fact quadratic when it represents preferences of a representative household.

As the central bank does not affect inflation and output directly but through money supply, interest rate, among other variables, it is important to investigate under which conditions its loss functions has the desirable property of convexity with relation to its *instruments*. In this paper we consider four possible economy's states and obtain condition under current inflation and output functions in order to ensure that loss function behaves suitable, such that we can minimize it. With a single instrument, we found that when inflation and output are linear functions of the monetary policy instrument, convexity is guaranteed for any of the four states mentioned. When we extend our analysis to case of many instruments, we found that only linearity is not sufficient to guarantee the shape of the loss function. This last result is related to papers of optimal choice of instruments, like Atkeson et al. (2007); Collard and Dellas (2005); Friedman (1991); Goodhart et al. (2011); Poole (1970).

Besides the theoretical importance, our findings may also be useful for policymakers' decisions. Suppose, for instance, that a central banker believes that its loss function is strictly convex, when in fact it has a different shape. Thus, the choices made by the monetary authority would be based on a wrong model, what would imply wrong policies and possibly high inflation and low output. Indeed, when the loss function is not convex, there is no interior minimum, such that a wrong belief could substantially affect the economy.

This paper is divided in two sections besides this introduction. Section 2 introduces our framework and studies individually the cases of a single and several instruments. There is possible to find the differences between the two cases and the importance of the linearity in both settings. Section 3 concludes with a discussion about assumptions made throughout the paper and suggests few extensions. Appendix A presents some examples which illustrate our findings. Finally, appendix B shows the proof of our results.

2 Convexity of the central bank's loss function

2.1 The case of a single instrument

Consider the problem of the central bank of stabilizing both prices and output by minimizing deviations from inflation target and potential output. Its objective function may be expressed by $L(\pi, y; \pi^*, y^*) \in C^2$, where π and y are current inflation and output, respectively. The inflation target, π^*, and the potential output, y^*, are assumed to be exogenous parameters.

As the goal of the central bank is to minimize L, we make the following assumption.

Assumption 2.1 *$L(\pi, y; \pi^*, y^*)$ is strictly convex in π and y. In other words, there exists unique $\widehat{\pi}$ and $\widehat{y}$ such that $L(\widehat{\pi}, \widehat{y}) = \min_{\pi,y} L(\pi, y)$*[1].

An implication of assumption 2.1 is that $\frac{\partial^2 L}{\partial \pi^2} > 0$ e $\frac{\partial^2 L}{\partial y^2} > 0$[2]. Note that we assume that the central bank's loss of welfare varies at increasing rate according to current inflation and output individually raise. That indicates a large instability of L for high values of π and y.

We also make another standard assumption, namely that the marginal utility of the central bank depends on the economy's state. If current inflation is below (above) its target, then an increase in π, *ceteris paribus*, increases (decreases) its welfare. Similarly, if the output is below (above) its potential level, then its marginal utility is positive (negative). By recalling that L is a loss (disutility) function, we can formally state it:

Assumption 2.2 *If $\pi > \pi^*$, then $\frac{\partial L}{\partial \pi} > 0$, and if $\pi < \pi^*$, then $\frac{\partial L}{\partial \pi} < 0$. In addition, if $y > y^*$, then $\frac{\partial L}{\partial y} > 0$, and if $y < y^*$, then $\frac{\partial L}{\partial y} < 0$.*

The set of assumptions 2.1-2.2 has been used by the leading models of central bank's otimization. Consider, for example, a modification of the seminal Barro and Gordon (1983), which has been the main functional form of L adopted by literature[3]:

$$L = \frac{\lambda}{2}(y - y^*)^2 + \frac{1}{2}(\pi - \pi^*)^2, \tag{2.1}$$

where λ measures the weight given by the central bank to output stabilization relative to inflation control. One can observe in (2.1) that, for all $(\pi, y) \in \mathbb{R}^2$ assumptions 2.1-2.2 are satisfied[4].

Even models that assume asymmetric preferences, like Nobay and Peel (2003), for instance, satisfy the above assumptions. Consider the loss function proposed by that paper

$$L = \frac{e^{\alpha(\pi-\pi^*)} - \alpha(\pi - \pi^*) - 1}{\alpha^2} + \frac{\lambda}{2}(y - y^*)^2, \tag{2.2}$$

where α is a constant and λ has the same interpretation of that in (2.1). By differentiating (2.2) we have assumptions 2.1 and 2.2 satisfied[5].

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[1]This assumption also ensure that minimum is interior.

[2]Indeed, assumption 2.1 is equivalent to state that the Hessian of L in π e y,

$$H_2 = \begin{bmatrix} \frac{\partial^2 L}{\partial \pi^2} & \frac{\partial^2 L}{\partial \pi \partial y} \\ \frac{\partial^2 L}{\partial y \partial \pi} & \frac{\partial^2 L}{\partial y^2} \end{bmatrix}$$

is positive. Therefore, $|H_1| = \frac{\partial^2 L}{\partial \pi^2} > 0$. Further, $|H_2| = \frac{\partial^2 L}{\partial \pi^2}\frac{\partial^2 L}{\partial y^2} - \left(\frac{\partial^2 L}{\partial \pi \partial y}\right)^2 > 0$, what implies that $\frac{\partial^2 L}{\partial y^2} > 0$.

[3]See the papers cited in introduction (section 1) and in the survey of Walsh (2010).

[4]Observe that $\frac{\partial L}{\partial y} = \lambda(y - y^*)$ and $\frac{\partial L}{\partial \pi} = (\pi - \pi^*)$, what satisfies assumption 2.2. Moreover, $\frac{\partial^2 L}{\partial y^2} = \lambda > 0$, $\frac{\partial^2 L}{\partial \pi^2} = 1$ and $\frac{\partial^2 L}{\partial \pi \partial y} = 0$, such that $\frac{\partial^2 L}{\partial y^2}\frac{\partial^2 L}{\partial \pi^2} - \left(\frac{\partial^2 L}{\partial \pi \partial y}\right)^2 = \lambda > 0$ implies that L is a strictly convex function in π and y, satisfying assumption 2.1.

[5]$\frac{\partial L}{\partial y} = \lambda(y - y^*) > 0$ if $y > y^*$ and $\frac{\partial L}{\partial y} < 0$ if $y < y^*$; and $\frac{\partial L}{\partial \pi} = \frac{e^{\alpha(\pi-\pi^*)}-1}{\alpha} > 0$ if $\pi > \pi^*$ and $\frac{\partial L}{\partial \pi} < 0$ if $\pi < \pi^*$. This satisfies assumption 2.2. In addition, $\frac{\partial^2 L}{\partial y^2} = \lambda > 0$, $\frac{\partial^2 L}{\partial \pi^2} = e^{\alpha(\pi-\pi^*)} > 0$ and $\frac{\partial^2 L}{\partial \pi \partial y} = 0$, such that $\frac{\partial^2 L}{\partial y^2}\frac{\partial^2 L}{\partial \pi^2} - \left(\frac{\partial^2 L}{\partial \pi \partial y}\right)^2 > 0$, what implies that L is strictly convex, satisfying assumption 2.1.

The central bank does not choose directly the level of current inflation and output. Instead, it affects those variables by using monetary policy's instrument. Initially, we consider the case in which the single instrument avaiable is the change in money supply, Δm. Therefore, we have that both inflation and output are functions of Δm and Ω, a vector of exogenous parameters. Vector Ω may still include stochastic terms, but this change does not affect our results. Formally:

$$\pi = \pi(\Delta m; \Omega) \tag{2.3}$$

and

$$y = y(\Delta m; \Omega). \tag{2.4}$$

As a further assumption, we assume that $\pi, y \in C^2$.

As it is standard in literature, we assume that money supply affects positively current inflation. We also make the strong assumption that output is positively affected by money supply even in the short run. Although Phillips curve with rational expectations states that only unexpected inflation impacts output, we make that cited assumption in order to have a channel from monetary policy to real activity. As we will see, it is possible to relax this requirement in some extent without any change in our results.

Assumption 2.3 *The change in the money supply affects positively both current inflation and output. Formally,* $\frac{\partial \pi}{\partial \Delta m} > 0$ *and* $\frac{\partial y}{\partial \Delta m} > 0$.

Definitions below let us state our first results about convexity of the central bank's loss function. Notice that each state defined below may be understood through assumption 2.2.

Definition 2.4 *An economy is:*

1. *in recession when both current inflation is lower than its target and output is lower than its potential level. That is,* $\pi(\Delta m; \Omega) < \pi^*$ *and* $y(\Delta m; \Omega) < y^*$*;*

2. *booming when both current inflation is higher than its target and output is higher than its potential level. That is,*$\pi(\Delta m; \Omega) > \pi^*$ *and* $y(\Delta m; \Omega) > y^*$*;*

3. *with high inflation when both current inflation is higher than its target and output is lower than its potential level. That is,* $\pi(\Delta m; \Omega) > \pi^*$ *and* $y(\Delta m; \Omega) < y^*$*; and*

4. *with high output when both current inflation is lower than its target and output is higher than its potential level. That is,* $\pi(\Delta m; \Omega) < \pi^*$ *and* $y(\Delta m; \Omega) > y^*$.

Proposition 2.5 *Consider that central bank's loss function* $L(\pi, y; \pi^*, y^*)$ *and the current inflation and output functions,* $\pi(\Delta m; \Omega)$ *and* $y(\Delta m; \Omega)$*, respectively. Suppose that assumptions 2.1 and 2.2 are satisfied. Then, we have the following sufficient conditions for L being strictly convex in* Δm*:*

i. $\frac{\partial^2 \pi}{\partial (\Delta m)^2} \geq 0$ *and* $\frac{\partial^2 y}{\partial (\Delta m)^2} \geq 0$*, if the economy is booming;*

ii. $\frac{\partial^2 \pi}{\partial (\Delta m)^2} \leq 0$ *and* $\frac{\partial^2 y}{\partial (\Delta m)^2} \leq 0$*, if the economy is in recession;*

iii. $\frac{\partial^2 \pi}{\partial (\Delta m)^2} \geq 0$ *and* $\frac{\partial^2 y}{\partial (\Delta m)^2} \leq 0$*, if the economy is with high inflation;*

iv. and $\frac{\partial^2 \pi}{\partial(\Delta m)^2} \leq 0$ *and* $\frac{\partial^2 y}{\partial(\Delta m)^2} \geq 0$*, if the economy is with high output.*

Observe that proposition 2.5 indicates some kind of asymmetry in central bank's preferences. When $\pi > \pi^*$, as in the cases in which the economy is booming and with high inflation, one of the sufficient conditions which enssure strict convexity of L is increasing marginal effect of the change in money supply. On the other hand, when $\pi < \pi^*$, it is sufficient that the effect of the change in money supply increases at decreasing rate. This means that, in this situation, the "power" of the instrument Δm must be decreasing as its level increases. In a certain way, the difference in the required response of the instrument for ensuring convexity of L may be seen as an asymmetry between high and low inflation (relative to its target).

In fact, the same asymmetry may be found when we analyse the behavior of output: if it is above its potential level, in order to ensure convexity of L, one of the sufficient conditions is that change in money supply has increasing marginal effect on y; and if it is below its potential level, then it is sufficient that marginal effect of Δm on y decreases as the change in money supply raises. Although our result is about other type of asymmetry, it may be considered a theoretical background for models of Nobay and Peel (2003) and Ruge-Murcia (2003), for instance.

Corollary 2.6 *If both* $\pi(\Delta m; \Omega)$ *and* $y(\Delta m; \Omega)$ *are linear functions of* Δm*, then the loss function* $L(\pi, y; \pi^*, y^*)$ *is strictly convex in* Δm *for any of the four economy's state.*

The result of corollary 2.6 has been extensively used by literature in an implicit way. Consider again the model of Barro and Gordon (1983), followed by the most of later research. Their functions equivalent to our $\pi(\Delta m; \Omega)$ and $y(\Delta m; \Omega)$ are:

$$\pi = \Delta m + u \tag{2.5}$$

and

$$y = y_n + a(\pi - \pi^e) + v, \tag{2.6}$$

where y_n and a are positive exogenous parameters, and u and v are stochastic shocks. Equation (2.6) is a Lucas supply function, whereas (2.5) shows the direct relationship between money supply and inflation. A simple substitution of (2.5) into (2.6) is enough for we verify that both are linear functions of Δm.

It is important to observe that corollary 2.6 is just a result of sufficiency. Thus, it is possible that there exist non-linear functions $\pi(\Delta m; \Omega)$ and $y(\Delta m; \Omega)$ that make $L(\pi, y; \pi^*, y^*)$ strictly convex in any of the economy's states. However, it does not seem a simple task finding such functions. We show in appendix A two examples of that difficulty. Example A.1 replaces (2.6) by a convex Phillips curve, such that convexity of loss function depends on the parameters of the model. In turn, example A.2 assumes that inflation is a non-linear function of the change in money supply and finds a non-convex L for some of the economy's state.

2.2 Several instruments

All the analysis conducted throughout section 2.1 was based on the assumption that the central bank uses only a single instrument of monetary policy. It is more realistic, however, to consider the case in which the monetary authority uses a set of avaiable instruments. For the sake of generality, now we assume that the central bank manages two different instruments, a_1 and a_2. In fact, it is important to stress out that conclusions of this section are independent of which instruments we are dealing. The only requirement is that these instruments be exogenous, such that the monetary rule cannot be affected by inflation and output. Thus, our analysis does not apply to DSGE models (e.g Clarida et al., 1999).

In this new setting, inflation and output may also be affected by both a_1 and a_2, such that $\pi(a_1, a_2; \Omega)$ and $y(a_1, a_2; \Omega)$. Suppose that a_1 behaves like Δm, such that it satisfies assumption 2.3. Still, we need to add an assumption in order to include the effect of the new instrument in the model. We assume a_2 impacts negatively both inflation - through decreases in aggregate demand, for instance - and output - through increase in the cost of capital and then decrease in investment.

Assumption 2.7 *The new instrument a_2 affects negatively both inflation and output. Formally, $\frac{\partial \pi}{\partial a_2} < 0$ and $\frac{\partial y}{\partial a_2} < 0$.*

In order to guarantee the convexity of L when there are more than one instrument of monetary policy, it does not suffices that second derivatives are positive. Now we have to analyse the Hessian with relation to a_1 and a_2, and to impose conditions over the behavior of the mixed derivatives.

Proposition 2.8 *Consider the central bank's loss function $L(\pi, y; \pi^*, y^*)$ and the current inflation and output functions, $\pi(a_1, a_2; \Omega)$ and $y(a_1, a_2; \Omega)$, respectively. Suppose that assumptions 2.1, 2.2, 2.3 and 2.7 are satisfied. Then, the following conditions are sufficient to ensure the convexity of L in a_1 e a_2:*

I. when the economy's is booming:

i. $\frac{\partial^2 \pi}{\partial (a_1)^2}, \frac{\partial^2 y}{\partial (a_1)^2}, \frac{\partial^2 \pi}{\partial (a_2)^2}, \frac{\partial^2 y}{\partial (a_2)^2} \geq 0;$

ii. $0 \leq \frac{\partial^2 \pi}{\partial a_1 \partial a_2} \leq \sqrt{\frac{\partial^2 \pi}{\partial (a_2)^2} \frac{\partial^2 \pi}{\partial (a_1)^2}}$ *and* $0 \leq \frac{\partial^2 y}{\partial a_1 \partial a_2} \leq \sqrt{\frac{\partial^2 y}{\partial (a_2)^2} \frac{\partial^2 y}{\partial (a_1)^2}};$

iii. $\frac{\partial^2 \pi}{\partial a_1 \partial a_2} \frac{\partial^2 y}{\partial a_1 \partial a_2} \leq \min\{A, B, C\};$

II. when the economy's in recession:

i. $\frac{\partial^2 \pi}{\partial (a_1)^2}, \frac{\partial^2 y}{\partial (a_1)^2}, \frac{\partial^2 \pi}{\partial (a_2)^2}, \frac{\partial^2 y}{\partial (a_2)^2} \leq 0;$

ii. $-\sqrt{\frac{\partial^2 \pi}{\partial (a_2)^2} \frac{\partial^2 \pi}{\partial (a_1)^2}} \leq \frac{\partial^2 \pi}{\partial a_1 \partial a_2} \leq 0$ *and* $-\sqrt{\frac{\partial^2 y}{\partial (a_2)^2} \frac{\partial^2 y}{\partial (a_1))^2}} \leq \frac{\partial^2 y}{\partial a_1 \partial a_2} \leq 0;$

iii. $\frac{\partial^2 \pi}{\partial a_1 \partial a_2} \frac{\partial^2 y}{\partial a_1 \partial a_2} \leq \min\{A, B, C\};$

III. when the economy's is with high inflation:

i. $\frac{\partial^2 \pi}{\partial (a_1)^2}, \frac{\partial^2 \pi}{\partial (a_2)^2} \geq 0, \frac{\partial^2 y}{\partial (a_1)^2}, \frac{\partial^2 y}{\partial (a_2)^2} \leq 0;$

ii. $0 \leq \frac{\partial^2 \pi}{\partial a_1 \partial a_2} \leq \sqrt{\frac{\partial^2 \pi}{\partial (a_2)^2} \frac{\partial^2 \pi}{\partial (a_1)^2}}$ *and* $-\sqrt{\frac{\partial^2 y}{\partial (a_2)^2} \frac{\partial^2 y}{\partial (a_1)^2}} \leq \frac{\partial^2 y}{\partial a_1 \partial a_2} \leq 0;$

iii. $\frac{\partial^2 \pi}{\partial a_1 \partial a_2} \frac{\partial^2 y}{\partial a_1 \partial a_2} \geq \max\{-A, B, C\};$

IV. when the economy's is with high output:

i. $\frac{\partial^2\pi}{\partial(a_1)^2}, \frac{\partial^2\pi}{\partial(a_2)^2} \leq 0, \frac{\partial^2 y}{\partial(a_1)^2}, \frac{\partial^2 y}{\partial(a_2)^2} \geq 0$;

ii. $-\sqrt{\frac{\partial^2\pi}{\partial(a_2)^2}\frac{\partial^2\pi}{\partial(a_1)^2}} \leq \frac{\partial^2\pi}{\partial a_1\partial a_2} \leq 0$ *and* $0 \leq \frac{\partial^2 y}{\partial a_1\partial a_2} \leq \sqrt{\frac{\partial^2 y}{\partial(a_2)^2}\frac{\partial^2 y}{\partial(a_1)^2}}$;

iii. $\frac{\partial^2\pi}{\partial a_1\partial a_2}\frac{\partial^2 y}{\partial a_1\partial a_2} \geq \max\{-A, B, C\}$;

Where $A = \sqrt{\frac{\partial^2\pi}{\partial(a_2)^2}\frac{\partial^2\pi}{\partial(a_1)^2}\frac{\partial^2 y}{\partial(a_2)^2}\frac{\partial^2 y}{\partial(a_1)^2}}$, $B = \frac{\partial^2 y}{\partial(a_2)^2}\frac{\partial^2\pi}{\partial(a_1)^2}$ *and* $C = \frac{\partial^2\pi}{\partial(a_2)^2}\frac{\partial^2 y}{\partial(a_1)^2}$.

Observe that item (i) of each one the economy's state is equivalent to result of proposition 2.5: whenever current inflation is above (below) its target, it is necessary that both instruments have increasing (decreasing) marginal effect on the price level. Similar conclusion may be found by analysing the output's behavior. Furthermore, a pattern arises from proposition 2.8: sufficient conditions for strict convexity require positive (negative) mixed derivatives of π whenever $\pi > \pi^*$ ($\pi < \pi^*$).

One of the differences between propositions 2.5 and 2.8 is in the inclusion of constraints over the sign and magnitude of mixed derivatives. Consider the case of booming economy, for example. Besides the increasing marginal effects of a_1 and a_2 on π and y, in order to ensure strict convexity of L is also necessary that mixed derivatives are positive and bounded from above. In addition, there must exist an upper bound for the product of the mixed derivatives as well. In other words, if the marginal effect of the each instrument on π e y is increasing - measured by the "own" second derivatives - but bounded when they are combined - mixed derivatives and its product -, then L is strictly convex.

However, now unlike the univariate case, linearity of $\pi(a_1, a_2; \Omega)$ and $y(a_1, a_2; \Omega)$ (see corollary 2.6) does not suffice to ensure strict convexity for any of the economy's state. In fact, with two instruments, it is necessary a further assumption, as stated below.

Corollary 2.9 *Suppose that assumptions 2.1, 2.2, 2.3 and 2.7 are satisfied. In addition, the following conditions are true:*

i. $\pi(a_1, a_2; \Omega)$ *and* $y(a_1, a_2; \Omega)$ *are linear functions of both* a_1 *and* a_2;

ii. and $\frac{\frac{\partial\pi}{\partial a_1}}{\frac{\partial\pi}{\partial a_2}} \neq \frac{\frac{\partial y}{\partial a_1}}{\frac{\partial y}{\partial a_2}}$.

Then L *is strictly convex in both* a_1 *and* a_2 *in any state of the economy.*

Condition (ii) of corollary 2.9 states that the marginal rate of substitution of inflation (MRS) between a_1 and a_2 must be different than the MRS of output between the same two instruments. In order to better understand the importance of condition (ii), we present an example of the optimization problem of the central bank with a slight modification in appendix A. There one can see that when MRSs are equal and both functions are linear in the instruments, there are infinite solutions for the minimization. Indeed, under such conditions, the system of first order derivatives is linearly dependent and the Hessian is non-negative. Therefore, the central bank is not able to choose independently the level of both instruments. Instead, when it decides the level of a_1, market adjusts a_2, and vice-versa.

The result found in example A.3 may be generalized for any finite number $n \geq 2$ of monetary policy's instruments.

Proposition 2.10 *Consider the central bank's loss function $L(\pi, y; \pi^*, y^*)$ and current inflation and output functions $\pi(a_1, a_2, ..., a_n; \Omega)$ and $y(a_1, a_2, ..., a_n; \Omega)$, respectively, where $a_1, a_2, ..., a_n$ are n avaiable monetary policy's instruments. Let both $\pi(a_1, a_2, ..., a_n; \Omega)$ and $y(a_1, a_2, ..., a_n; \Omega)$ be linear functions of $a_1, a_2, ..., a_n$. Furthermore, suppose that*

$$\frac{\frac{\partial \pi}{\partial a_k}}{\frac{\partial \pi}{\partial a_j}} = \frac{\frac{\partial y}{\partial a_k}}{\frac{\partial y}{\partial a_j}}$$

for all $k \neq j$. Then $L(\pi, y; \pi^*, y^*)$ is (not strictly) convex in $a_1, a_2, ..., a_n$.

Proposition 2.10 states that monetary policy's instruments are dependent when both inflation and output are linear functions of those intruments and there is equality of MRS for all of them. Remember that means that the central bank is not able to choose independently the level of each instrument, such that it chooses values for some a_k, with $k < n$, and the other a_j, with $j = n - k$, adjust according the economy's structure.

The bad news is that proposition 2.10 just provides sufficient conditions for *dependence* between instruments. A direct extension of corollary 2.9 for n instruments would give us conditions for independence. However, in a linear setting, it does not suffices to require that MRSs are different, because although this would rule out the possibility of multiplicity between the Hessian's rows, it would not ensure the absence of any other linear combination between them. Nevertheless, in some cases we may handle that limitation, as appendix A shows, by presenting an example (A.4) with two independent instruments.

3 Concluding remarks

Our conclusions are based on some assumptions which deserve some attention. First, recall that one implicit requirement we made was the instruments must be exogenous, what implies that our results are not applicable to DSGE models. Still, assumptions 2.2 is satisfied by any central bank which has inflation and output stabilization as its goals, such that it does not seem strong. In turn, assumption 2.3 is not strong as well, because money supply is always assumed to affect inflation by literature. On the other hand, one can discuss whether output is positively affected by nominal variables. Nevertheless, it does not make sense to assume that output is negatively affected by them, and if we impose $\frac{\partial y}{\partial \Delta m} = 0$, our results do not change. The same reasoning may be applied to the analysis of the validity of assumption 2.7. Finally, our results do change when we modify assumption 2.1. However, by relaxing such requirement we may make the optimization problem have no minimum, such that analysis of the central bank's behavior lose much of its intuition and applicability.

There are many potential extensions in this field. For example, it is of interest conducting similar analysis and obtaining conditions under which the central bank's loss function is quasi-convex. Moreover, relaxing the underlying assumption of differentiability used in all our results may bring different insights for policymakers. Further, one can consider how the conclusions change when we consider more than two goals for the central bank (a target of exchange rate, for instance). Finally, we must try to find *necessary* and sufficient conditions for convexity, such that we can state results about *independence* between monetary instruments rather than dependence ones. The reason is that for the central bank is more important to know which instruments can in fact use than those which are not under its control.

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A Examples

Example A.1 *We follow the basic model of Barro and Gordon (1983), such that the central bank's loss function is given by (2.1) and inflation function is given by (2.5). Here we adopt the convex Phillips curve proposed by Schaling (1999), as presented by Semmler and Zhang (2004):*

$$y = y^* - \frac{\pi - \beta r}{\phi(\psi\pi + \beta r\psi - \theta)}, \tag{A.1}$$

where β measures the sensitiveness of inflation to changes in real interest rate r, ψ is an index of curvature, θ measures the sensitiviness of inflation to changes in unemployment and ϕ is a parameter of an auxiliar linear function used in the construction of (A.1). We assume that all parameters are strictly positive and $1 > \psi \geq 0$[6].

By substituting (2.5) into (A.1), and both into (2.1), we are able to optimize L. Second order condition is given by:

$$\frac{\partial^2 L}{\partial(\Delta m)^2} = \frac{\lambda\theta}{\phi^2}\left[\frac{2\psi(\Delta m + \beta r) + \theta}{(\psi\Delta m + \beta r\psi - \theta)^3}\right] + 1.$$

Suppose now that the economy's output is high, such that $y - y^* > 0$. Hence,

$$y - y^* = \frac{-(\Delta m + \beta r)}{\phi(\psi\Delta m + \beta r\psi - \theta)} > 0, \tag{A.2}$$

what implies that $\psi\Delta m + \beta r\psi - \theta < 0$. Therefore, by (A.2), it is not possible to state that L is convex. In fact, in this case, the sign of $\frac{\partial^2 L}{\partial(\Delta m)^2}$ depends on the parameters' values.

Example A.2 *Let the loss function be given by (2.1) and assume now that output function is (2.6) while the inflation function is given by*

$$\pi = \ln \Delta m. \tag{A.3}$$

[6]The model of Schaling (1999), as presented by Semmler and Zhang (2004), uses the Phillips curve without expectations $\pi = -\beta r - \theta\mu$, where μ is the unemployment. Through Okun's law $g = -\mu$, where $g = y - y^*$ is the output gap, authors use the function $f(g) = \frac{\phi g}{1-\psi\phi g}$ in order to rewrite the Phillips curve as $\pi = -\beta r - \theta\Pi(\mu)$, where $\Pi(\mu) = \frac{\phi\mu}{1+\psi\phi\mu}$. Expression (A.1) just uses Okun's law to replace μ in the above Phillips curve.

Note that, in this simple example, we restrict the function's domain to $\mathbb{R}^*_+$*, such that* $\Delta m > 0$*: the central bank can only choose strictly positive changes in the money supply. In addition, assumption 2.3 is satisfied, that is, both inflation and output respond positively to increases in the money supply. However, those increases occur at decreasing rates, because* $\frac{\partial^2 \pi}{\partial(\Delta m)^2}, \frac{\partial^2 y}{\partial(\Delta m)^2} < 0$ *for all* $\Delta m \in \mathbb{R}^*_+$*.*

By substituting (A.3) and (2.6) into (2.1), the second order condition fo the central bank's problem is given by

$$\frac{\partial^2 L}{\partial(\Delta m)^2} = \frac{\lambda a(1 - \ln \Delta m + \pi^e)}{(\Delta m)^2} + \frac{(1 - \ln \Delta m + \pi^*)}{(\Delta m)^2}, \tag{A.4}$$

which may be greater or lower than zero, depending on the sign of the two terms in parentheses.

Observe also that, for the strict convexity of L*, it is sufficient that* $\ln \Delta m < \min\{1 + \pi^e, 1 + \pi^*\}$*. But such a condition may not be satisfied for some of the economy's state. Indeed, if the economy is booming enough, such that* $\pi - \pi^* = \ln \Delta m - \pi^* > 1$ *and* $y - y^* = a(\ln \Delta m - \pi^e) > a$*, one can see in (A.4) that* $\frac{\partial^2 L}{\partial(\Delta m)^2} < 0$ *and then* L *is not convex.*

Example A.3 *Let the central bank's loss function be given by (2.1) and consider its problem of minimizing (2.1) by choosing the change of the money supply,* Δm*, and the nominal interest rate,* i*. Here we show the endogeneity of these instruments.*

Assume also that current inflation and output functions are given by

$$\pi = \pi_1 \Delta m + \pi_2 i + u \tag{A.5}$$

and

$$y = y^* + a(\pi - \pi^e) + v, \tag{A.6}$$

where $\pi_1 > 0$*,* $\pi_2 < 0$ *and* $a > 0$*. By substituting (A.5) into (A.6) we have that both* π *e* y *are linear in* Δm *and* i*, and that assumptions 2.1, 2.2, 2.3 and 2.7 are satisfied.*

If we let L *be function only of the intruments, we have the following central bank's problem:*

$$\min_{\Delta m, i} \frac{\lambda}{2} \left(a\pi_1 \Delta m + a\pi_2 i + au - a\pi^e + v\right)^2 + \frac{1}{2} \left(\pi_1 \Delta m + \pi_2 i + u - \pi^*\right)^2, \tag{A.7}$$

which has solution[7]*:*

$$\Delta m = \frac{\pi^* + \lambda a(a\pi^e - v)}{(\lambda a^2 + 1)\pi_1} - \frac{u}{\pi_1} - \frac{\pi_2}{\pi_1} i. \tag{A.8}$$

Notice that the problem has not an unique minimum: any linear combination of Δm *and* i *satisfying (A.8) is a solution of (A.7). In this sense, we say the monetary instruments are dependents (endogenous): either the central bank chooses the change in money*

[7] First order conditions are given by:

$$\begin{aligned} \frac{\partial L}{\partial \Delta m} &= \pi_1(\pi_1 \Delta m + \pi_2 i + u - \pi^e)(\lambda a^2 + 1) + \lambda a \pi_1 v = 0 \\ \frac{\partial L}{\partial i} &= \pi_2(\pi_1 \Delta m + \pi_2 i + u - \pi^e)(\lambda a^2 + 1) + \lambda a \pi_2 v = 0, \end{aligned}$$

which form a linearly dependent system.

supply and the interest rate adjusts to this choice; or it chooses the interest rate and the change in money supply must adjust. The reason of such a behavior is that now L is just a (not scrictly) convex function of the instruments. This fact may be seen through its Hessian:

$$H_2 = (\lambda a^2 + 1)\begin{bmatrix} \pi_1^2 & \pi_1\pi_2 \\ \pi_1\pi_2 & \pi_2^2 \end{bmatrix}, \tag{A.9}$$

in which we conclude that $|H_2| = 0$ and $|H_1| = \pi_1^2(\lambda a^2 + 1) > 0$. Thus, L has infinity minimum points, given by line (A.8).

Now we are able to verify that requirement (ii) of corollary 2.9 is not satisfied in this model. In fact, the MRS of inflation and output are equal:

$$\frac{\frac{\partial \pi}{\partial \Delta m}}{\frac{\partial \pi}{\partial i}} = \frac{\pi_1}{\pi_2} = \frac{a\pi_1}{a\pi_2} = \frac{\frac{\partial y}{\partial \Delta m}}{\frac{\partial y}{\partial i}}.$$

One can also see the importance of the relationship between the two MRS on the optimal choice of Δm and i through the level curves of π and y. For the sake of simplicity, assume that $u = v = 0$, $\lambda = 1$, $y^* = 1$, $\pi^* = 1$, $\pi_1 = 1$, $\pi_2 = -1$, $a = 2$, $\pi^e = 1$ and $y_2 = -1$. With these values the curves at levels $\overline{\pi}$ and $\overline{y}$ are given by

$$\Delta m = \overline{\pi} + i \tag{A.10}$$

and

$$\Delta m = \left(\frac{\overline{y} + 2}{2}\right) + i. \tag{A.11}$$

One can check that they are parallel, given that MRSs are equal. Moreover, solution of (A.8) becomes

$$\Delta m = \frac{7}{5} + i,$$

which has the same slope as level curves (A.10) and (A.11). For $\overline{\pi} = \overline{y} = 0$, graphically we have

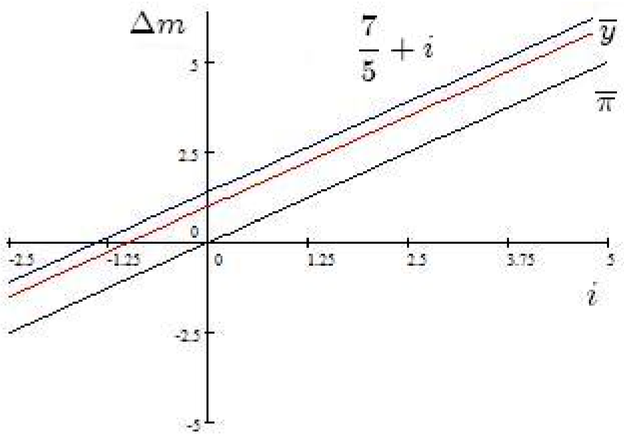

Figure 1: Dependent instruments

Because solution and level curves are parallel, there exists only one possibility to optimal choice: all the three curves coincide. Therefore, it is necessary that their intercepts are equal and this unique line provides infinity optimal points to central bank. In the *specific case above,* $\overline{\pi} = \frac{7}{5}$ *and* $\overline{y} = \frac{4}{5}$ *ensure that result.*

Example A.4 *Consider almost the same structure used in example A.3, just replacing* i *by* a_2*, an instrument that satisfies assumption 2.7 - the statutory reserve requirement, for instance - and (A.6) by*

$$y = y^* + a(\pi - \pi^e) + y_2 a_2 + v, \tag{A.12}$$

where $y_2 < 0$*, in order to satisfy the cited assumption. We can justify the inclusion of* a_2 *in (A.12) by assuming that it has a direct effect on the current output - besides the indirect effect through inflation. For instance, changes in* a_2 *can impact capital cost and then investment, what affects output as well.*

Observe that there is no equality between MRS of π *and* y*:*

$$\frac{\frac{\partial \pi}{\partial \Delta m}}{\frac{\partial \pi}{\partial a_2}} = \frac{\pi_1}{\pi_2} \neq \frac{a\pi_1}{a\pi_2 + y_2} = \frac{\frac{\partial y}{\partial \Delta m}}{\frac{\partial y}{\partial a_2}}$$

By analysing the Hessian, we confirm the strict convexity of L*:*

$$H_2^2 = \begin{bmatrix} \pi_1^2(a^2\lambda + 1) & \pi_1[a\lambda(a\pi_2 + y_2) + \pi_2] \\ \pi_1[a\lambda(a\pi_2 + y_2) + \pi_2] & \lambda(a\pi_2 + y_2)^2 + \pi_2^2 \end{bmatrix}, \tag{A.13}$$

where $|H_1^2| = \pi_1^2(a^2\lambda + 1) > 0$ *and* $|H_2^2| = \lambda\pi_1^2 y_2^2 > 0$*. Thus, in this context, different MRSs ensure the existence of an unique minimum for the central bank's problem. Further, note the role of* y_2 *in the result: if* $y_2 = 0$*, then (A.13) is equal to (A.9) and* $|H_2^2| = 0$*.*

We can also see the uniqueness of the solution using a graphic. Consider the same parameters' values used in example A.3 and $y_2 = -5$*. The level curves* $\overline{\pi}$ *and* $\overline{y}$ *are given by*

$$\Delta m = \overline{\pi} + a_2$$

and

$$\Delta m = \left(\frac{\overline{y} + 1}{2}\right) + \frac{7}{2}a_2,$$

respectively. Observe that now slopes are different, what indicates the possibility of intersection between the curves.

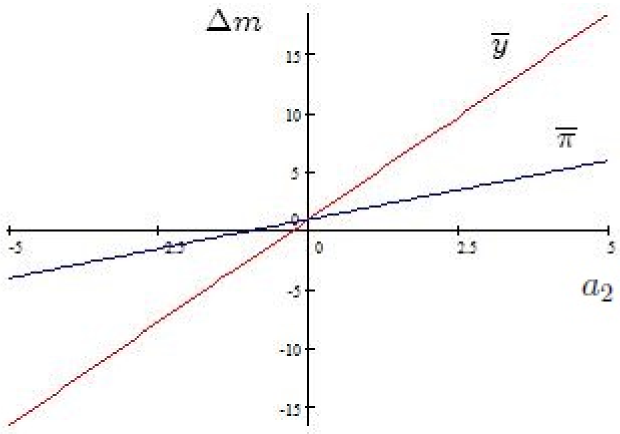

Figure 2: Independent instruments

In this context, the monetary authority can clearly choose the change in money supply and the level of a_2 in an independent way, because one variable is not function of another. Point $A = (1, 0)$ in figure 2 is the solution of central bank's problem for the proposed parameters' values. Moreover, for $\Delta m = 1$ and $a_2 = 0$, we have $\overline{\pi} = 1$ and $\overline{y} = 1$.

B Ommited proofs

Proof. Proposition 2.5. By substituting (2.3) and (2.4) into $L(\pi, y; \pi^*, y^*)$, we have $L(\pi(\Delta m; \Omega), y(\Delta m; \Omega); \pi^*, y^*)$. Now by using chain's rule,

$$\frac{\partial^2 L}{\partial(\Delta m)^2} = \frac{\partial^2 L}{\partial \pi^2}\left(\frac{\partial \pi}{\partial \Delta m}\right)^2 + \frac{\partial L}{\partial \pi}\frac{\partial^2 \pi}{\partial(\Delta m)^2} + \frac{\partial^2 L}{\partial y^2}\left(\frac{\partial y}{\partial \Delta m}\right)^2 + \frac{\partial L}{\partial y}\frac{\partial^2 y}{\partial(\Delta m)^2}. \tag{B.1}$$

By assumption 2.1, $\frac{\partial^2 L}{\partial \pi^2} > 0$ and $\frac{\partial^2 L}{\partial y^2} > 0$. Hence, by using assumption 2.2, we have:

i. if economy is booming, $\frac{\partial L}{\partial \pi} > 0$ and $\frac{\partial L}{\partial y} > 0$, then $\frac{\partial^2 \pi}{\partial(\Delta m)^2} > 0$ and $\frac{\partial^2 y}{\partial(\Delta m)^2} > 0$ imply $\frac{\partial^2 L}{\partial(\Delta m)^2} > 0$;

ii. if economy is in recession, $\frac{\partial L}{\partial \pi} < 0$ and $\frac{\partial L}{\partial y} < 0$, then $\frac{\partial^2 \pi}{\partial(\Delta m)^2} < 0$ and $\frac{\partial^2 y}{\partial(\Delta m)^2} < 0$ imply $\frac{\partial^2 L}{\partial(\Delta m)^2} > 0$;

iii. if economy is with high inflation, $\frac{\partial L}{\partial \pi} > 0$ and $\frac{\partial L}{\partial y} < 0$, then $\frac{\partial^2 \pi}{\partial(\Delta m)^2} > 0$ and $\frac{\partial^2 y}{\partial(\Delta m)^2} < 0$ imply $\frac{\partial^2 L}{\partial(\Delta m)^2} > 0$;

iv. if economy is with high output, $\frac{\partial L}{\partial \pi} < 0$ and $\frac{\partial L}{\partial y} > 0$, then $\frac{\partial^2 \pi}{\partial(\Delta m)^2} < 0$ and $\frac{\partial^2 y}{\partial(\Delta m)^2} > 0$ imply $\frac{\partial^2 L}{\partial(\Delta m)^2} > 0$;

This concludes the proof. ■

Proof. Corollary 2.6. If $\pi(\Delta m; \Omega)$ e $y(\Delta m; \Omega)$ are linear functions of Δm, then $\frac{\partial^2 \pi}{\partial(\Delta m)^2} = \frac{\partial^2 y}{\partial(\Delta m)^2} = 0$. By (B.1), we have $\frac{\partial^2 L}{\partial(\Delta m)^2} > 0$. ■

Proof. Proposition 2.8. Consider the Hessian of the central bank's problem with two monetary policy's intruments

$$H_2 = \begin{bmatrix} \frac{\partial^2 L}{\partial(a_1)^2} & \frac{\partial^2 L}{\partial a_1 \partial a_2} \\ \frac{\partial^2 L}{\partial a_1 \partial a_2} & \frac{\partial^2 L}{\partial(a_2)^2} \end{bmatrix},$$

where the partial derivatives are given by:

$$\begin{aligned}
\frac{\partial^2 L}{\partial(a_1)^2} &= \frac{\partial^2 L}{\partial \pi^2}\left(\frac{\partial \pi}{\partial a_1}\right)^2 + \frac{\partial L}{\partial \pi}\frac{\partial^2 \pi}{\partial(a_1)^2} + \frac{\partial^2 L}{\partial y^2}\left(\frac{\partial y}{\partial a_1}\right)^2 + \frac{\partial L}{\partial y}\frac{\partial^2 y}{\partial(a_1)^2} \\
\frac{\partial^2 L}{\partial(a_2)^2} &= \frac{\partial^2 L}{\partial \pi^2}\left(\frac{\partial \pi}{\partial a_2}\right)^2 + \frac{\partial L}{\partial \pi}\frac{\partial^2 \pi}{\partial(a_2)^2} + \frac{\partial^2 L}{\partial y^2}\left(\frac{\partial y}{\partial a_2}\right)^2 + \frac{\partial L}{\partial y}\frac{\partial^2 y}{\partial(a_2)^2} \\
\frac{\partial^2 L}{\partial a_1 \partial a_2} &= \frac{\partial^2 L}{\partial \pi^2}\left(\frac{\partial \pi}{\partial a_1}\frac{\partial \pi}{\partial a_2}\right) + \frac{\partial L}{\partial \pi}\frac{\partial^2 \pi}{\partial a_1 \partial a_2} + \frac{\partial^2 L}{\partial y^2}\left(\frac{\partial y}{\partial a_1}\frac{\partial y}{\partial a_2}\right) + \frac{\partial L}{\partial y}\frac{\partial^2 y}{\partial a_1 \partial a_2}.
\end{aligned}$$

It suffices to show that $|H_2| > 0$ for each one of the economy's state, because we can use proposition 2.5 in order to ensure $|H_1| > 0$. Note that the determinant of H_2 is given by

$$
\begin{aligned}
|H_2| &= \frac{\partial^2 L}{\partial(a_1)^2}\frac{\partial^2 L}{\partial(a_2)^2} - \left(\frac{\partial^2 L}{\partial a_1 \partial a_2}\right)^2 \\
&= \frac{\partial^2 L}{\partial \pi^2}\frac{\partial \pi}{\partial a_1}\frac{\partial L}{\partial \pi}\left(\frac{\partial \pi}{\partial a_1}\frac{\partial^2 \pi}{\partial(a_2)^2} - \frac{\partial \pi}{\partial a_2}\frac{\partial^2 \pi}{\partial a_1 \partial a_2}\right) + \frac{\partial^2 L}{\partial \pi^2}\frac{\partial^2 L}{\partial y^2}\left(\frac{\partial \pi}{\partial a_1}\frac{\partial y}{\partial a_2} - \frac{\partial \pi}{\partial a_2}\frac{\partial y}{\partial a_1}\right)^2 \\
&\quad + \frac{\partial^2 L}{\partial \pi^2}\frac{\partial \pi}{\partial a_1}\frac{\partial L}{\partial y}\left(\frac{\partial \pi}{\partial a_1}\frac{\partial^2 y}{\partial(a_2)^2} - \frac{\partial^2 y}{\partial a_1 \partial a_2}\frac{\partial \pi}{\partial a_2}\right) \\
&\quad + \frac{\partial^2 L}{\partial \pi^2}\frac{\partial L}{\partial \pi}\frac{\partial \pi}{\partial a_2}\left(\frac{\partial \pi}{\partial a_2}\frac{\partial^2 \pi}{\partial(a_1)^2} - \frac{\partial \pi}{\partial a_1}\frac{\partial^2 \pi}{\partial a_1 \partial a_2}\right) \\
&\quad + \left(\frac{\partial L}{\partial \pi}\right)^2\left[\frac{\partial^2 \pi}{\partial(a_2)^2}\frac{\partial^2 \pi}{\partial(a_1)^2} - \left(\frac{\partial^2 \pi}{\partial a_1 \partial a_2}\right)^2\right] \\
&\quad + \frac{\partial L}{\partial \pi}\frac{\partial^2 L}{\partial y^2}\frac{\partial y}{\partial a_2}\left(\frac{\partial y}{\partial a_2}\frac{\partial^2 \pi}{\partial(a_1)^2} - \frac{\partial y}{\partial a_1}\frac{\partial^2 \pi}{\partial a_1 \partial a_2}\right) \\
&\quad + \frac{\partial L}{\partial \pi}\frac{\partial L}{\partial y}\left(\frac{\partial^2 \pi}{\partial(a_1)^2}\frac{\partial^2 y}{\partial(a_2)^2} - \frac{\partial^2 \pi}{\partial a_1 \partial a_2}\frac{\partial^2 y}{\partial a_1 \partial a_2}\right) \\
&\quad + \frac{\partial^2 L}{\partial y^2}\frac{\partial L}{\partial \pi}\frac{\partial y}{\partial a_1}\left(\frac{\partial y}{\partial a_1}\frac{\partial^2 \pi}{\partial(a_2)^2} - \frac{\partial y}{\partial a_2}\frac{\partial^2 \pi}{\partial a_1 \partial a_2}\right) \\
&\quad + \frac{\partial^2 L}{\partial y^2}\frac{\partial L}{\partial y}\frac{\partial y}{\partial a_1}\left(\frac{\partial y}{\partial a_1}\frac{\partial^2 y}{\partial(a_2)^2} - \frac{\partial y}{\partial a_2}\frac{\partial^2 y}{\partial a_1 \partial a_2}\right) \\
&\quad + \frac{\partial L}{\partial y}\frac{\partial^2 L}{\partial \pi^2}\frac{\partial \pi}{\partial a_2}\left(\frac{\partial \pi}{\partial a_2}\frac{\partial^2 y}{\partial(a_1)^2} - \frac{\partial \pi}{\partial a_1}\frac{\partial^2 y}{\partial a_1 \partial a_2}\right) \\
&\quad + \frac{\partial L}{\partial y}\frac{\partial L}{\partial \pi}\left(\frac{\partial^2 y}{\partial(a_1)^2}\frac{\partial^2 \pi}{\partial(a_2)^2} - \frac{\partial^2 \pi}{\partial a_1 \partial a_2}\frac{\partial^2 y}{\partial a_1 \partial a_2}\right) \\
&\quad + \frac{\partial L}{\partial y}\frac{\partial^2 L}{\partial y^2}\frac{\partial y}{\partial a_2}\left(\frac{\partial y}{\partial a_1}\frac{\partial^2 y}{\partial(a_1)^2} - \frac{\partial y}{\partial a_1}\frac{\partial^2 y}{\partial a_1 \partial a_2}\right) \\
&\quad + \left(\frac{\partial L}{\partial y}\right)^2\left[\frac{\partial^2 y}{\partial(a_1)^2}\frac{\partial^2 y}{\partial(a_2)^2} - \left(\frac{\partial^2 y}{\partial a_1 \partial a_2}\right)^2\right] \\
&= A + B + C + D + E + F + G + H + I + J + K + L + M
\end{aligned}
\tag{B.2}
$$

Lets analyse each one of the economy's state individually. We must show that every term in (B.2) is non-negative when we apply conditions of proposition 2.8. First, observe that term B is non-negative in any economy's state, given that $\frac{\partial^2 L}{\partial \pi^2}, \frac{\partial^2 L}{\partial y^2} > 0$ by assumption 2.1.

I. **Booming economy**. In this case, by assumption 2.2 we have $\frac{\partial L}{\partial \pi}, \frac{\partial L}{\partial y} > 0$. By using assumptions 2.2, 2.3 and 2.7 we have that all terms outside of parentheses in A, C, E, G, H, I, K and M are positive. Conditions (i), (ii) and (iii) still imply that all terms inside of parentheses are positive for those letters. Further, for D, F, J and L, terms outside of parentheses are negative. By applying (i), (ii) and (iii), we have that terms inside of parentheses are also negative for those letters. Therefore, $A, C, D, E, F, G, H, I, J, K, L, M > 0$, what implies $|H_2| > 0$.

II. **Economy is in recession**. Now we have $\frac{\partial L}{\partial \pi}, \frac{\partial L}{\partial y} < 0$. Terms outside of parentheses in D, E, F, G, J, K, L and M are positive. By using (i), (ii) and (iii), we have that terms inside of parentheses of those letters are positive as well. In turn, for A, C and H, terms outside of parentheses are negative, but conditions (i), (ii) and (iii) ensure that terms inside of parentheses have the same sign. Thus, $A, C, D, E, F, G, H, I, J, K, L, M > 0$, what implies $|H_2| > 0$.

III. **Economy with high inflation**. We have $\frac{\partial L}{\partial \pi} > 0$ and $\frac{\partial L}{\partial y} < 0$. Now, terms outside of parenthses in A, E, H, J, L and M are greater than zero, while the correspondent terms in C, D, F, G, I and K are lower than zero. By applying (i), (ii) and (iii), terms inside of parentheses of the first group are positive and of the second one are negative. Therefore, $A, C, D, E, F, G, H, I, J, K, L, M > 0$ and then $|H_2| > 0$.

IV. **Economy with high output**. In this case, $\frac{\partial L}{\partial \pi} < 0$ and $\frac{\partial L}{\partial y} > 0$. Note now that, terms outside of parentheses in C, E, I and M are positive, while the correspondent terms of A, D, F, G, H, J, K and L are negative. However, terms inside of parentheses in both groups of letters are negative as we use conditions (i), (ii) and (iii). Thus, we have $A, C, D, E, F, G, H, I, J, K, L, M > 0$, what guarantees $|H_2| > 0$.

■

Proof. Corollary 2.9. Observe that (i) implies that all second order partial derivatives of π and y are null. Therefore, (B.2) becomes

$$|H_2| = B = \frac{\partial^2 L}{\partial \pi^2}\frac{\partial^2 L}{\partial y^2}\left(\frac{\partial \pi}{\partial a_1}\frac{\partial y}{\partial a_2} - \frac{\partial \pi}{\partial a_2}\frac{\partial y}{\partial a_1}\right)^2,$$

which is not negative, given the assumption 2.1. Observe also that $|H_2| > 0$ if, and only if, $\frac{\partial \pi}{\partial a_1}\frac{\partial y}{\partial a_2} \neq \frac{\partial \pi}{\partial a_2}\frac{\partial y}{\partial a_1}$. By rewriting this last expression we have condition (ii):

$$\frac{\frac{\partial \pi}{\partial a_1}}{\frac{\partial \pi}{\partial a_2}} \neq \frac{\frac{\partial y}{\partial a_1}}{\frac{\partial y}{\partial a_2}}.$$

■

Proof. Proposition 2.10. Consider the Hessian of the problem with n instruments,

$$H_n = \begin{bmatrix} \frac{\partial^2 L}{\partial a_1^2} & \frac{\partial^2 L}{\partial a_1 \partial a_2} & \cdots & \frac{\partial^2 L}{\partial a_1 \partial a_n} \\ \frac{\partial^2 L}{\partial a_2 \partial a_1} & \frac{\partial^2 L}{\partial a_2^2} & \cdots & \frac{\partial^2 L}{\partial a_1 \partial a_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 L}{\partial a_n \partial a_1} & \frac{\partial^2 L}{\partial a_n \partial a_2} & \cdots & \frac{\partial^2 L}{\partial a_n^2} \end{bmatrix}.$$

We must show that $|H_i| \geq 0$ for all $i = 1, ..., n$ and $|H_j| > 0$ for some j, such that L is a (not strictly) convex function. First, consider the derivatives that compose the two first rows of H_n:

$$\begin{aligned}
\frac{\partial^2 L}{\partial a_1^2} &= \frac{\partial^2 L}{\partial \pi^2}\left(\frac{\partial \pi}{\partial a_1}\right)^2 + \frac{\partial^2 L}{\partial y^2}\left(\frac{\partial y}{\partial a_1}\right)^2 \\
\frac{\partial^2 L}{\partial a_1 \partial a_2} &= \frac{\partial^2 L}{\partial \pi^2}\frac{\partial \pi}{\partial a_1}\frac{\partial \pi}{\partial a_2} + \frac{\partial^2 L}{\partial y^2}\frac{\partial y}{\partial a_1}\frac{\partial y}{\partial a_2} \\
&\vdots \\
\frac{\partial^2 L}{\partial a_1 \partial a_n} &= \frac{\partial^2 L}{\partial \pi^2}\frac{\partial \pi}{\partial a_1}\frac{\partial \pi}{\partial a_n} + \frac{\partial^2 L}{\partial y^2}\frac{\partial y}{\partial a_1}\frac{\partial y}{\partial a_n} \\
\frac{\partial^2 L}{\partial a_2 \partial a_1} &= \frac{\partial^2 L}{\partial \pi^2}\frac{\partial \pi}{\partial a_2}\frac{\partial \pi}{\partial a_1} + \frac{\partial^2 L}{\partial y^2}\frac{\partial y}{\partial a_2}\frac{\partial y}{\partial a_1} \\
\frac{\partial^2 L}{\partial a_2^2} &= \frac{\partial^2 L}{\partial \pi^2}\left(\frac{\partial \pi}{\partial a_2}\right)^2 + \frac{\partial^2 L}{\partial y^2}\left(\frac{\partial y}{\partial a_2}\right)^2 \\
&\vdots \\
\frac{\partial^2 L}{\partial a_2 \partial a_n} &= \frac{\partial^2 L}{\partial \pi^2}\frac{\partial \pi}{\partial a_2}\frac{\partial \pi}{\partial a_n} + \frac{\partial^2 L}{\partial y^2}\frac{\partial y}{\partial a_2}\frac{\partial y}{\partial a_n},
\end{aligned}$$

where the second order partial derivatives of π and y are absence because they are null by linearity assumption.

By multiplying the first row of H_n by $\frac{\partial y}{\partial a_2}\left(\frac{\partial y}{\partial a_1}\right)^{-1}$ and using the assumption of $\frac{\partial \pi}{\partial a_i}\left(\frac{\partial \pi}{\partial a_j}\right)^{-1} = \frac{\partial y}{\partial a_i}\left(\frac{\partial y}{\partial a_j}\right)^{-1}$ for $i = 1$ and $j = 2$, we have

$$\begin{aligned}
\frac{\partial y}{\partial a_2}\left(\frac{\partial y}{\partial a_1}\right)^{-1}\left[\frac{\partial^2 L}{\partial \pi^2}\left(\frac{\partial \pi}{\partial a_1}\right)^2 + \frac{\partial^2 L}{\partial y^2}\left(\frac{\partial y}{\partial a_1}\right)^2\right] &= \frac{\partial^2 L}{\partial \pi^2}\frac{\partial \pi}{\partial a_2}\frac{\partial \pi}{\partial a_1} + \frac{\partial^2 L}{\partial y^2}\frac{\partial y}{\partial a_2}\frac{\partial y}{\partial a_1} \\
&= \frac{\partial^2 L}{\partial a_2 \partial a_1} \\
\frac{\partial y}{\partial a_2}\left(\frac{\partial y}{\partial a_1}\right)^{-1}\left[\frac{\partial^2 L}{\partial \pi^2}\frac{\partial \pi}{\partial a_1}\frac{\partial \pi}{\partial a_2} + \frac{\partial^2 L}{\partial y^2}\frac{\partial y}{\partial a_1}\frac{\partial y}{\partial a_2}\right] &= \frac{\partial^2 L}{\partial \pi^2}\left(\frac{\partial \pi}{\partial a_2}\right)^2 + \frac{\partial^2 L}{\partial y^2}\left(\frac{\partial y}{\partial a_2}\right)^2 \\
&= \frac{\partial^2 L}{\partial a_2^2} \\
&\vdots \\
\frac{\partial y}{\partial a_2}\left(\frac{\partial y}{\partial a_1}\right)^{-1}\left[\frac{\partial^2 L}{\partial \pi^2}\frac{\partial \pi}{\partial a_1}\frac{\partial \pi}{\partial a_n} + \frac{\partial^2 L}{\partial y^2}\frac{\partial y}{\partial a_1}\frac{\partial y}{\partial a_n}\right] &= \frac{\partial^2 L}{\partial \pi^2}\frac{\partial \pi}{\partial a_2}\frac{\partial \pi}{\partial a_n} + \frac{\partial^2 L}{\partial y^2}\frac{\partial y}{\partial a_2}\frac{\partial y}{\partial a_n} \\
&= \frac{\partial^2 L}{\partial a_2 \partial a_n},
\end{aligned}$$

what shows that first row is a multiple of the second one. Then, $|H_n| = 0$. It is possible to apply the same reasoning for any principal minor of dimention $n > 1$, such that $|H_n| = 0$.

In order to complete the proof, observe the terms of main diagonal of H_n,

$$\frac{\partial^2 L}{\partial a_i^2} = \frac{\partial^2 L}{\partial \pi^2}\left(\frac{\partial \pi}{\partial a_i}\right)^2 + \frac{\partial^2 L}{\partial y^2}\left(\frac{\partial y}{\partial a_i}\right)^2 > 0,$$

what is ensured for $i = 1, ..., n$ by assumption 2.1. ∎