Economics Bulletin

Volume 35, Issue 1

The EANSC: a weighted extension and axiomatization

Yu-hsien Liao
Department of Applied Mathematics, National Pingtung University, Taiwan

Ling-Yun Chung
Graduate School of Technological and Vocational Education, YunTech

Po-hang Wu
Department of Applied Mathematics, National Pingtung University of Education

Abstract

Here we propose a weighted extension of the equal allocation of nonseparable costs (EANSC). Further, an axiomatization is also proposed by applying consistency.

The authors are very grateful to the editor, the associate editor, the anonymous referees for helpful suggestions and comments.
Citation: Yu-hsien Liao and Ling-Yun Chung and Po-hang Wu, (2015) "The EANSC: a weighted extension and axiomatization", *Economics Bulletin*, Volume 35, Issue 1, pages 475-480
Contact: Yu-hsien Liao - twincos@ms25.hinet.net, Ling-Yun Chung - darlingyun@gmail.com, Po-hang Wu - pulo6240@hotmail.com.
Submitted: August 10, 2013. **Published:** March 11, 2015.

1 Introduction

The *equal allocation of nonseparable costs* (EANSC) is a well-known solution concept in cooperative game theory. Based on the EANSC, each player receives his or her marginal contribution plus an equal share of the remaining utility. Since all players are dissimilar, it is reasonable that weights could be assigned to the "players" to modify the discriminations among all players. In this note, we adopt the *weight functions* to propose a weighted extension of the EANSC, which we name as the *weighted allocation of nonseparable costs* (WANSC). Based on the WANSC, each player receives his or her marginal contribution plus the part of the remains that does not correspond to an equal share of the remains, but it corresponds to the given (relative) weights of each player.

Consistency is a crucial property of solutions. Consistency allows us to deduce, from the desirability of an outcome for some problem, the desirability of its restriction to each subgroup for the associated reduction the subgroup faces. It has been investigated in various classes of problems, always based on reductions. Moulin (1985) introduced a notion of consistency to characterize the EANSC. Hwang (2006) characterized the EANSC not by means of a consistency property with respect to a reduced game, but making use instead of a property that analyzes the behavior of the solution with respect to an associated game. Hwang and Hsiao (2007) showed that the EANSC can be expressed as the sum of both internal dividends and external losses. Subsequently, Liao (2008, 2010) extended the EANSC to multi-choice TU games and interval TU games respectively.

Here we study the reduction and related consistency introduced by Moulin (1985). Inspired by the axiomatization of the Shapley value (1953) due to Hart and Mas-Colell (1989), we also axiomatize the WANSC by means of the consistency and the *weighted two-person standardness*. The weighted two-person standardness asserts that all players allocate utilities by applying the weights proportionably in all two-person games.

2 The weighted allocation of nonseparable costs

Let U be a non-empty and finite set of players. A coalition is a non-empty subset of U. A coalitional game with transferable utility (TU game) is a pair (N, v) where N is a coalition and v is a mapping such that $v : 2^N \longrightarrow \mathbb{R}$ and $v(\emptyset) = 0$. Denote the class of all TU games by G. A *solution* on G is a function ψ which associates with each game

$(N,v) \in G$ an element $\psi(N,v)$ of $I\!R^N$. Let ψ be a solution on G. We say that ψ satisfies **efficiency** if for all $(N,v) \in G$, $\sum_{i \in N} \psi_i(N,v) = v(N)$.

Definition 1 *The* **equal allocation of nonseparable costs (EANSC)***, denoted by η, is the solution which associates with $(N,v) \in G$ and each player $i \in N$ the value*

$$\eta_i(N,v) = \beta_i(N,v) + \frac{1}{|N|} \cdot \big[v(N) - \sum_{k \in N} \beta_k(N,v)\big], \tag{1}$$

where $\beta_i(N,v) = \big[v(N) - v(N \setminus \{i\})\big]$.

Next, we propose a weighted extension of the EANSC. Let $w : U \to \mathbb{R}^+$ be a positive function, then w is called a **weight function**. Given $(N,v) \in G$ and a weight function w, for all $S \subseteq N$, we define $w(S) = \sum_{i \in S} w(i)$. The weighted extension of the EANSC is as follows.

Definition 2 *Let w be a weight function. The* **weighted allocation of nonseparable costs (WANSC)***, η^w, is the solution which associates with $(N,v) \in G$ and all players $i \in N$ the value*

$$\eta_i^w(N,v) = \beta_i(N,v) + \frac{w(i)}{w(N)} \cdot \big[v(N) - \sum_{k \in N} \beta_k(N,v)\big]. \tag{2}$$

3 Reduced game and axiomatization

In this section, we introduce the reduced game introduced by Moulin (1985) to characterize the WANSC by means of related consistency.

Definition 3 (Moulin, 1985) *Given a solution ψ, $(N,v) \in G$ and $S \subseteq N$, the* **reduced game** $(S, v_{S,\psi})$ **with respect to** ψ **and** S *is defined by for all $T \subseteq S$,*

$$v_{S,\psi}(T) = \begin{cases} 0 & \text{, if } T = \emptyset, \\ v(T \cup (N \setminus S)) - \sum\limits_{i \in N \setminus S} \psi_i(N,v) & \text{, otherwise.} \end{cases}$$

The reduced game is based on the idea that, when renegotiating the solution ψ within S, it is assumed that the coalition $T \subseteq S$ cooperates with all the members of $N \setminus S$, paying off each of them at initial payoff $\psi_i(N,v)$, where $i \in N \setminus S$. For the reduced game, there is a corresponding *consistency* as follows.

Definition 4 *A solution ψ satisfies* **consistency** *if $\psi_i(S, v_{S,\psi}) = \psi_i(N,v)$ for all $(N,v) \in G$ with $|N| \geq 2$, for all $S \subseteq N$ and for all $i \in S$.*

Remark 1 *Moulin (1985) characterized the EANSC by means of the consistency property.*

Lemma 1 *The WANSC satisfies efficiency.*

Proof. Let $(N,v) \in G$. By Definition 2,

$$\begin{aligned}\sum_{i\in N} \eta_i^w(N,v) &= \sum_{i\in N} \Big[\beta_i(N,v) + \tfrac{w(i)}{w(N)} \cdot \big[v(N) - \sum_{k\in N} \beta_k(N,v)\big]\Big] \\ &= \sum_{i\in N} \beta_i(N,v) + \sum_{i\in N} \tfrac{w(i)}{w(N)} \cdot \big[v(N) - \sum_{k\in N} \beta_k(N,v)\big] \\ &= \sum_{i\in N} \beta_i(N,v) + \tfrac{w(N)}{w(N)} \cdot \big[v(N) - \sum_{k\in N} \beta_k(N,v)\big] \\ &= v(N).\end{aligned}$$

Hence, the WANSC η^w satisfies efficiency. ∎

Lemma 2 *The WANSC η^w satisfies consistency.*

Proof. Given $(N,v) \in G$ and $S \subseteq N$. If $S = \{i\}$ for some $i \in N$, then by efficiency of η^w,

$$\eta_i^w(S, v_{S,\eta^w}) = v_{S,\eta^w}(S) = v(N) - \sum_{k\neq i} \eta_k^w(N,v) = \eta_i^w(N,v).$$

Assume that $|N| \geq 2$ and $|S| \geq 2$. By the definitions of β and v_{S,η^w}, for all $i \in S$,

$$\begin{aligned}\beta_i(S, v_{S,\eta^w}) &= \big[v_{S,\eta^w}(S) - v_{S,\eta^w}(S \setminus \{i\})\big] \\ &= \big[v(N) - \sum_{k\in N\setminus S} \eta_k^w(N,v) - v(N\setminus\{i\}) + \sum_{k\in N\setminus S} \eta_k^w(N,v)\big] \\ &= \big[v(N) - v(N\setminus\{i\})\big] \\ &= \beta_i(N,v).\end{aligned} \tag{3}$$

By equations (1), (2), (3) and Definition 3, for all $i \in S$,

$$\begin{aligned}&\eta_i^w(S, v_{S,\eta^w}) \\ =\ & \beta_i(S, v_{S,\eta^w}) + \tfrac{w(i)}{w(S)} \cdot \big[v_{S,\eta^w}(S) - \sum_{k\in S} \beta_k(S, v_{S,\eta^w})\big] \text{ (by equation (2))} \\ =\ & \beta_i(N,v) + \tfrac{w(i)}{w(S)} \cdot \big[v_{S,\eta^w}(S) - \sum_{k\in S} \beta_k(N,v)\big] \text{ (by equation (3))} \\ =\ & \beta_i(N,v) + \tfrac{w(i)}{w(S)} \cdot \big[v(N) - \sum_{k\in N\setminus S} \eta_k^w(N,v) - \sum_{k\in S} \beta_k(N,v)\big] \text{ (by Definition (3))} \\ =\ & \beta_i(N,v) + \tfrac{w(i)}{w(S)} \cdot \big[\sum_{k\in S} \eta_k^w(N,v) - \sum_{k\in S} \beta_k(N,v)\big] \text{ (by efficiency of } \eta^w) \\ =\ & \beta_i(N,v) + \tfrac{w(i)}{w(S)} \cdot \Big[\tfrac{w(S)}{w(N)} \cdot \big[v(N) - \sum_{k\in N} \beta_k(N,v)\big]\Big] \text{ (by equation (2))} \\ =\ & \beta_i(N,v) + \tfrac{w(i)}{w(N)} \cdot \big[v(N) - \sum_{k\in N} \beta_k(N,v)\big] \\ =\ & \eta_i^w(N,v).\end{aligned}$$

Hence, the WANSC η^w satisfies consistency. ■

Inspired by Hart and Mas-Colell (1989), we characterize the WANSC by means of the properties of consistency and *weighted standard for two-person games.*

Definition 5 *A solution ψ satisfies* **weighted standardness for two-person games (WST)** *if for all $(N,v) \in G$ with $N = \{i,j\}$,*

$$\psi_i(N,v) = \big[v(\{i,j\}) - v(\{j\})\big] + \Big(\frac{w(i)}{w(i)+w(j)}\Big) \cdot \big[v(\{j\}) + v(\{j\}) - v(\{i,j\})\big].$$

Remark 2 *It is not difficult to derive that if a solution ψ satisfies WST and consistency, then for all $(\{i\},v) \in G$, $\psi(\{i\},v) = v(\{i\})$. The technique of the proof can be found in Hart and Mas-Colell (1989).*

Lemma 3 *Let ψ be a solution on G. If ψ satisfies WST and consistency, then it also satisfies efficiency.*

Proof. Suppose ψ satisfies WST and consistency. Let $(N,v) \in G$. If $|N| = 1$, then ψ satisfies efficiency by Remark 2. If $|N| = 2$, it is trivial that ψ satisfies efficiency by WST. Assume that $|N| > 2$ and $i \in N$. By Remark 2 and definition of $v_{\{i\},\psi}$,

$$\psi_i(\{i\}, v_{\{i\},\psi}) = v_{\{i\},\psi}(\{i\}) = v(N) - \sum_{k \neq i} \psi_k(N,v). \quad (4)$$

By consistency of ψ,

$$\psi_i(\{i\}, v_{\{i\},\psi}) = \psi_i(N,v). \quad (5)$$

By equations (4) and (5),

$$v(N) = \sum_{k \in N} \psi_k(N,v).$$

Hence, ψ satisfies efficiency. ■

Theorem 1 *A solution ψ on G satisfies WST and consistency if and only if $\psi = \eta^w$.*

Proof. By Lemma 2, η^w satisfies consistency. Clearly, η^w satisfies WST.

To prove uniqueness, suppose ψ satisfies WST and consistency. By Lemma 3, ψ satisfies efficiency. Let $(N,v) \in G$. Suppose $|N| = 1$. By

Remark 2 and Lemma 3, $\psi(N,v) = \eta^w(N,v)$. If $|N| = 2$, it is trivial that $\psi(N,v) = \eta^w(N,v)$ by WST. The case $|N| > 2$: Let $i \in N$ and $S = \{i,j\}$ for some $j \in N \setminus \{i\}$. Then

$$\begin{aligned}
\psi_i(N,v) - \eta_i^w(N,v) &= \psi_i(S, v_{S,\psi}) - \eta_i^w(S, v_{S,\eta^w}) \quad \text{(by consistency of } \psi, \eta^w) \\
&= \eta_i^w(S, v_{S,\psi}) - \eta_i^w(S, v_{S,\eta^w}) \quad \text{(by WST of } \psi, \eta^w) \\
&= \tfrac{w(i)}{w(i)+w(j)} \cdot \Big[v_{S,\psi}(S) + v_{S,\psi}(\{i\}) - v_{S,\psi}(\{j\})\Big] \\
&\quad - \tfrac{w(i)}{w(i)+w(j)} \cdot \Big[v_{S,\eta^w}(S) + v_{S,\eta^w}(\{i\}) - v_{S,\eta^w}(\{j\})\Big].
\end{aligned} \tag{6}$$

By the definitions of $v_{S,\psi}$ and v_{S,η^w},

$$\begin{aligned}
v_{S,\psi}(\{i\}) - v_{S,\psi}(\{j\}) &= \Big[v(N \setminus \{j\}) - v(N \setminus \{i\})\Big] \\
&= v_{S,\eta^w}(\{i\}) - v_{S,\eta^w}(\{j\}).
\end{aligned} \tag{7}$$

By equations (6), (7), the definition of $v_{S,\psi}$ and the efficiency of ψ and η^w,

$$\begin{aligned}
\psi_i(N,v) - \eta_i^w(N,v) &= \tfrac{w(i)}{w(i)+w(j)} \cdot \Big[v_{S,\psi}(S) - v_{S,\eta^w}(S)\Big] \\
&= \tfrac{w(i)}{w(i)+w(j)} \cdot \Big[\psi_i(N,v) + \psi_j(N,v) - \eta_i^w(N,v) - \eta_j^w(N,v)\Big].
\end{aligned}$$

That is, for all $i,j \in N$,

$$\frac{w(j)}{w(i)+w(j)} \cdot \Big[\psi_i(N,v) - \eta_i^w(N,v)\Big] = \frac{w(i)}{w(i)+w(j)} \cdot \Big[\psi_j(N,v) - \eta_j^w(N,v)\Big].$$

By efficiency of ψ and η^w,

$$0 = v(N) - v(N) = \sum_{j \in N} \big[\psi_j(N,v) - \eta_j^w(N,v)\big] = \frac{w(N)}{w(i)} \cdot \big[\psi_i(N,v) - \eta_i^w(N,v)\big].$$

Hence, for all $i \in N$, $\psi_i(N,v) = \eta_i^w(N,v)$. ■

The following examples are to show that each of the axioms used in Theorem 1 is logically independent of the remaining axioms.

Example 1 *Define a solution ψ by for all $(N,v) \in G$ and for all $i \in N$,*

$$\psi_i(N,v) = \frac{v(N)}{|N|}.$$

Clearly, ψ satisfies consistency, but it violates WST.

Example 2 *Define a solution ψ by for all $(N,v) \in G$ and for all $i \in N$,*

$$\psi_i(N,v) = \begin{cases} \eta_i^w(N,v) & , \text{ if } |N| \leq 2, \\ \eta_i(N,v) & , \text{ otherwise.} \end{cases}$$

Clearly, ψ satisfies WST, but it violates consistency.

References

[1] Hart S, Mas-Colell A (1989) Potential, value, and consistency. Econometrica 57:589-614

[2] Hwang YA (2006) Associated consistency and equal allocation of nonseparable costs. Economic Theory 28:709-719

[3] Hwang YA, Hsiao YH (2007) Internal dividend, external loss and value. Economics Bulletin 3:1-5

[4] Liao YH (2008) The maximal equal allocation of nonseparable costs on multi-choice games. Economics Bulletin 3:1-8

[5] Liao YH (2010) Value, complement reduction and interval TU games. Economics Bulletin 30:2671-2679

[6] Moulin H (1985) The separability axiom and equal-sharing methods. Journal of Economic Theory 36:120-148

[7] Shapley LS (1953) A value for n-person game. In: Kuhn HW, Tucker AW(eds.) Contributions to the Theory of Games II, Annals of Mathematics Studies 28, Princeton University Press, Princeton, 307-317