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Condorcet's Paradox and the Median Voter Theorem for Randomized Social Choice

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Abstract

Condorcet's paradox is one of the most prominent results in social choice theory. It says that there may not exist any alternative that a net majority prefers over every other alternative. When outcomes need not be deterministic alternatives, we show that a similar paradox still exists even if preferences are dichotomous. Thus relaxing the requirement of discrete alternatives does not help in circumventing Condorcet's paradox. On the other hand, we show that a fractional/randomized version of Black's Median Voter Theorem still holds for single-peaked preferences.

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1. Introduction

Condorcet's paradox is one of the most fundamental and central results in social choice theory [3]. It shows that even if individual agents (voters) are rational, society as a whole may not be rational. In other words, even though preferences of individual voters are transitive, collective preferences can be cyclic intransitive. A devastating consequence of the paradox is that a Condorcet winner — an alternative that is preferred by a majority over all other alternatives — may not exist. The paradox has attracted a lot of research [5] with various approaches proposed to circumvent it. For example Black [2] showed that when preferences are single-peaked, a Condorcet winner does exist. In its simplest form, the paradox has remained restricted to the setting in which agent express preferences over discrete alternatives and a single alternative is to be selected. In this note, we extend the Condorcet's paradox to the more general context of *randomized* social choice in which each outcome is lottery over the set of discrete alternatives. The setting is a strict generalization of the classic social choice setting since a degenerate lottery precisely corresponds to the selection of a single discrete alternative. Although the proof is straightforward, we have not (to the best of our knowledge) seen Condorcet's paradox expressed in the general domain of randomized social choice [1, 6]. We then complement this result by showing that for strict and single-peaked preferences, a weak majority lottery with respect to *SD* exists. The result is an extension of Black's Median Voter Theorem.

2. Setup

Consider the social choice setting in which there is a set of agents $N = \{1, \ldots, n\}$, a set of alternatives $A = \{a_1, \ldots, a_m\}$ and a preference profile $\succsim = (\succsim_1, \ldots, \succsim_n)$ such that each $\succsim_i$ is a complete and transitive relation over A. We write $a \succsim_i b$ to denote that agent i values alternative a at least as much as alternative b and use $\succ_i$ for the strict part of $\succsim_i$, i.e., $a \succ_i b$ iff $a \succsim_i b$ but not $b \succsim_i a$. Finally, $\sim_i$ denotes i's indifference relation, i.e., $a \sim_i b$ iff both $a \succsim_i b$ and $b \succsim_i a$. The relation $\succsim_i$ results in equivalence classes $E_i^1, E_i^2, \ldots, E_i^{k_i}$ for some k_i such that $a \succ_i a'$ if $a \in E_i^l$ and $a' \in E_i^{l'}$ for some $l < l'$. We will use these equivalence classes to represent the preference relation of an agent as a preference list $i : E_i^1, E_i^2, \ldots, E_i^{k_i}$. For example, we will denote the preferences $a \sim_i b \succ_i c$ by the list $i : \{a, b\}, \{c\}$. An agent i's preferences are *dichotomous* if he partitions the alternatives into just two equivalence classes, i.e., $k_i = 2$.

Let $\Delta(A)$ denote the set of all *lotteries* (or *probability distributions*) over A. The support of a lottery $p \in \Delta(A)$, denoted by $\text{supp}(p)$, is the set of all alternatives to which p assigns a positive probability, i.e., $\mathrm{supp}(p) = \{x \in A \mid p(x) > 0\}$. For $A' \subseteq A$, we will (slightly abusing notation) denote $\sum_{a \in A'} p(a)$ by $p(A')$. We say that a lottery is *degenerate* if it assigns probability one to an alternative and probability zero to the other alternatives. We will denote by $[a]$ the degenerate lottery that assigns probability/fraction one to alternative $a \in A$.

A *lottery extension* extends preferences over alternatives to (possibly incomplete) preferences over lotteries. Given $\succsim_i$ over A, a *lottery extension* $\mathscr{X}$ extends $\succsim_i$ to $\succsim_i^{\mathscr{X}}$ over the set of lotteries $\Delta(A)$. We now define some particular lottery extensions that we will later refer to. Under *stochastic dominance (SD)*, an agent prefers a lottery that, for each alternative $x \in A$, has a higher probability of selecting an alternative that is at least as good as x. Formally,

$$p \succsim_i^{SD} q \text{ iff for all } y \in A\colon \sum_{x \in A : x \succsim_i y} p(x) \geq \sum_{x \in A : x \succsim_i y} q(x).$$

A lottery extension $\mathscr{X}$ is a *refinement* of SD if $p \succsim_i^{SD} q$ implies $p \succsim_i^{\mathscr{X}} q$.

For dichotomous preferences, the SD relation is complete and equivalent to any refinement of SD. Similarly, when lotteries are degenerate, the SD relation is complete and equivalent to any refinement of SD. For trichotomous preferences, the SD relation is not complete. For example if agent i has preferences $a \succ_i b \succ_i c$, then lotteries $[a : 1/3, b : 1/3, c : 1/3]$ and $[a : 0, b : 1, c : 0]$ are incomparable.

3. Generalizing Condorcet's Paradox

We say that $p \succ_{maj}^{\mathscr{X}} q$ if $|\{i \in N : p \succ_i^{\mathscr{X}} q\}| - |\{i \in N : q \succ_i^{\mathscr{X}} p\}| > 0$. We say that a lottery p is a *majority winner with respect to* $\mathscr{X}$ if $p \succsim_{maj}^{\mathscr{X}} q$ for all $q \in \Delta(A)$. A lottery p is a *weak majority winner with respect to* $\mathscr{X}$ if $q \succ_{maj}^{\mathscr{X}} p$ for no $q \in \Delta(A)$. Observe that for any lottery extension $\mathscr{X}$ that is SD or a refinement of SD, $a \succsim_i b$ iff $[a] \succsim_i^{\mathscr{X}} [b]$ where $[a]$ and $[b]$ are degenerate lotteries. In view of the this, Condorcet's paradox can be phrased as follows.

> Condorcet's Paradox: When outcomes are restricted to degenerate lotteries, a majority winner with respect to *SD* may not exist.

We now show that Condorcet's paradox even occurs when we do not restrict ourselves to degenerate lotteries.

Theorem 1. *For $|N| \geq 3$ and $|A| \geq 3$, even when any lottery can be an outcome, a weak majority winner with respect to any refinement of SD may not exist.*

Proof. Consider the following preference profile where $n \geq 3$ and $m \geq n$.

$$\begin{aligned} 1: &\quad a_1, A \setminus \{a_1\} \\ &\vdots \\ i: &\quad a_i, A \setminus \{a_i\} \\ &\vdots \\ n: &\quad a_n, A \setminus \{a_n\} \end{aligned}$$

Assume for contradiction that there exists a lottery p that is a weak majority winner with respect to *SD*. Then there exists at least one alternative $x \in A$ such that $p(x) > 0$. Without loss of generality, let us say that x is a_1. Then consider the lottery q which is defined as follows.

For all a in A,

$$q(a) = \begin{cases} p(a) - \epsilon & \text{if } a = a_1, \\ p(a) + \dfrac{\epsilon}{|A| - 1} & \text{otherwise.} \end{cases}$$

Note that

$$q \succ_i^{SD} p \text{ for } i \in N \setminus \{1\}.$$

Hence, $q \succ_{maj}^{SD} p$. Thus, p is not a majority winner with respect to *SD* which is a contradiction. □

Note that Theorem 1 does not imply nor is it implied by the original Condorcet's paradox. On one hand, the allowance of randomized outcomes makes it easier to obtain an objection by a majority because of there simply being more possible outcomes. On the other hand, there are in principle more candidates to be majority winners.

Remark 1. *Note that in the definition of majority winners, comparison by a majority of agents is critically based on the majority margin between the two lotteries where agents compare lotteries with respect to SD. If a majority comparison is based on considering the probabilities of different instantiations of the lotteries and then checking the* expected *majority margin of one lottery over the other, then an undominated outcome ("winner") with respect to expected majority margins exists in the form of a* maximal lottery *[1, 4, 7].*

The remark above shows that weak majority winner with respect SD may be an unreasonably strong generalization of deterministic weak Condorcet winners. Or the statement should be seen in the context of fractional social choice (time sharing) rather than randomized social choice. However, we next show that even this unreasonably strong generalization lends itself to a positive result in a restricted domain.

4. Generalizing Black's Median Voter Theorem

Although, a Condorcet winner may not exist in general, it does for a particularly appealing domain restriction of single-peaked preferences. In this restricition, the alternatives are linearly ordered along an axis, each agent has a most preferred alternative and alternatives further away (along the axis) from this peak are less preferred. Black [2] proved the following:

> Black's Median Voter Theorem: For strict and single-peaked preferences, the median peak alternative is the Condorcet winner.

We see that even in the context of randomized social choice there is a 'randomized' version of Black's Median Voter Theorem.

Theorem 2. *For strict and single-peaked preferences, the degenerate lottery corresponding to the median peak alternative is a weak majority winner with respect to SD.*

Proof. Suppose the axis is $a_1 < a_2 < \cdots < a_m$. Consider the peaks along the axis and let one of the (maximum of two median peaks) be a_i corresponding to agent i's peak. We claim that the lottery p that gives probability one to a_i is a weak majority winner with respect to *SD*. Assume that there is a lottery q such that $q \succ_{maj}^{\mathscr{X}} p$. Since $q \neq p$, in order to obtain q from p, there must be some alternative a on the left (right) such that $q(a) > 0$. Then for $a_j = a_i$ as well as for each a_j on the right (left) of a_i, $p(\{a \in A : a \succsim_j a_i\}) = 1$ but $q(\{a \in A : a \succsim_j a_i\}) < 1$. Hence $q \not\succsim_j^{SD} p$ for all such $j \in N$. Thus $q \not\succ_{maj}^{SD} p$. □

Note that the argument does not extend to prove the existence of a majority winner rather than weak majority winner. The reason is that the median degenerate lottery is SD-preferred by *no* agent (who does not have a median peak) over the lottery that divides the probability over the alternatives immediately left and right of the median alternative.

5. Conclusion

What we have highlighted is that although dichotomous preferences constitute such a restricted domain, even for such a domain, a weak majority winner with respect to any refinement of *SD* may not exist. On the other hand, when preferences are single-peaked, a weak majority winner lottery with respect to *SD* is guaranteed to exist, which is in line with Black's celebrated Median Voter Theorem.

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