Economics Bulletin

Volume 35, Issue 3

Firm Size Distribution and the Survival Bias

Antonio Palestrini
Università Politecnica delle Marche, Italy

Abstract

In this work, using the simple Kesten's process, I investigate the survival bias of the firm size distribution selecting a cohort of surviving firms. This work shows that the modified Kesten's process - in which firms exit when their size (measured as equity) cross the barrier (go bankrupt) - produces a limit distribution of the cohort more symmetric. This result provides a benchmark at comparing the distribution produced by economic models studying surviving firms.

Citation: Antonio Palestrini, (2015) "Firm Size Distribution and the Survival Bias", *Economics Bulletin*, Volume 35, Issue 3, pages 1630-1637
Contact: Antonio Palestrini - ants.pal@gmail.com.
Submitted: June 30, 2014. **Published:** July 24, 2015.

1 Introduction

Starting from the work of Gibrat (1931) asserting that the growth process of firms is independent from their size[1] (Gibrat law of proportionate effects) - and then the analysis of Simon and Bonini (1958), Steindl (1965) and Ijiri *et al.* (1977) - the literature on firm size distribution (FSD) is becoming important in economic analysis. In fact, a recent line of research claims that the FSD may be an important ingredient in explaining business cycles (Delli Gatti *et al.*, 2005; Gabaix, 2011).

A recent literature (Cabral and Mata, 2003; Hutchinson *et al.*, 2010) shows that the FSD of surviving firms seems to become more symmetric. In other terms, observing the FSD of a cohort - and thus neglecting the demography process - it seems to be best approximated by a log-normal than a Pareto distribution. Such line of research explains these phenomena with the sample selection process (exit rates are higher among smaller firm; see Jovanovic (1982)), financial constraints (financing constraints are especially relevant for young small firms) and the inter-industry diversification (firms accumulate a presence in a greater number of 4-digit industries over time).

In this work I do not want to dispute the validity of the Gibrat law. Instead, in the sections below, maintaining the Gibrat's assumption that firms' rate of growth are independent of size, I analyze the simplest possible modification of the Gibrat's process able to reproduce the Pareto distribution of firms' size. The process used in the following analysis is the Kesten's process that may be interpreted as a Gibrat's process with a lower bound on size. Using the Kesten's process, I discuss a particular condition - not analyzed in the literature - under which the distribution of surviving firms tends to become more symmetric. The explanation relies on the *survival bias* that arises when we select a cohort of firms (*i.e.*, when we follow the FSD of firms that survive from a given starting year).

This selection bias is different from the one described by Jovanovic (1982) since is not determined, in general, by selecting a different kind of firm. Instead, using the cohort, we select an *alternative multiplicative process* than the one producing a Pareto like distribution.

The purpose of the analysis is to investigate which kind of distribution we obtain following a cohort of surviving firms. The simple assumption to analyze the limiting distribution of the cohort is that a firm goes bankrupt when her size - measure as equity - falls below a given threshold. In other words, the simplifying assumptions made to analyze firms' size limit distri-

[1]The size of a firm may be measured by the number of employees, the amount of equity, total capital, sales, ecc. (Delli Gatti *et al.*, 2008).

butions of cohorts are: 1) the time t cohort is made by surviving firms from the starting period 0; 2) in order to survive firms do not have to cross a given equity threshold.

A key feature in the stochastic process - built by observing only surviving firms - is the average rate of growth. In other terms, we have to investigate what happens to the mean of the growth distribution of firms becoming old. From the work of Evans (1987) it seems that age has a slightly negative effect on firm growth and this fact seems to be more evident for developed economies. Regarding emerging economies the works of Das (1995) and Shanmugam and Bhaduri (2002) seem to show that the age of a firm positively affects its growth. The analysis of Gallegati and Palestrini (2010); Axtell *et al.* (2008) show that FSD becomes more symmetric if surviving firms have a positive average rate of growth.

Finally, Cirillo (2010) using a parametric analysis of Italian data (with the *Subbotin distribution*) finds that the mean of the firms' growth process does not appear to move significantly from zero.

This last case is the objective of analysis in this note since is not analyzed in the firm's size theoretical literature.

In section 2, we investigate the assumption of zero mean rate of growth independent from age showing the counterintuitive fact that, a stochastic multiplicative process in which surviving firms have an average rate of growth always equal to zero, also produces a selection bias.

Finally, section 3 concludes.

2 FSD of surviving firms with mean zero distribution

The starting point of the FSD literature is the famous Gibrat's stochastic model leading to a lognormal distribution of firm size

$$S_{it} = (1 + g_{it})S_{it-1} \tag{1}$$

where S_{it} is the equity measure of firm-i size and g_{it} is the IID rate of growth from $t-1$ to t. Gibrat assumes that g_{it} is independent of S_{it-1} (*law of proportional effect*).

Modifying it by introducing a lower bound (or reflective barrier), the process generates a power law distribution. To be precise, the *multiplicative process with lower bound or reflecting barrier*, also called *Kesten process* (Axtell, 2001; Gallegati and Palestrini, 2010) is defined by the following stochastic difference equation

$$S_{it} = \max[(1 + g_{it})S_{it-1}, \bar{S}]$$

where $\bar{S}$ is the minimum size.

This process may be interpreted as a Gibrat process with the addition of a very simple demography[2]. In other terms, when a firm goes to the barrier it is replaced by another firm with a size equal to the barrier.

It can be shown that with the condition

$$E[\log(1+g_{it})] < 0, \tag{2}$$

the process generates an invariant distribution (a clear analysis is in Bhattacharya and Majumdar (2007). The FSD tends to a Pareto distribution.

When the multiplicative shock has a positive mean $\mu > 0$, firm size evolves according to the equation

$$S_{it} = \max[(1+\mu+g_{it})S_{it-1}, \bar{S}] \tag{3}$$

with g_{it} an IID mean zero shock.

In particular, when μ is such that

$$\mu : \quad E[\log(1+\mu+g_{it})] > 0 \tag{4}$$

firm size is no longer convergent to an invariant distribution. Firms tend to growth and the lower bound (minimum size) does not bind (except for events that have measure zero). The non-stationary process (3) is similar to the standard Gibrat one (1) with a shape approximating the lognormal distribution (Champernowne, 1953; Kesten, 1973; Mitzenmacher, 2004). The work of Gallegati and Palestrini (2010) shows that in this case the selection/survivor bias produces more symmetric distributions.

In the following it is shown that even if the distribution of the rate of growth remains the same (invariant) and equal to zero for surviving firms - *with a mean equal to zero independent of a firm's age and size* - we are observing a completely different stochastic process compared to the Kesten/Gibrat processes.

This process is difficult to analyze since it converges asymptotically to a degenerate situation. The number of firms goes to zero with exponential velocity. To show this consider the following process

$$S_{it} = (1+g_{it})S_{it-1} \quad if \quad S_{it-1} \geq 20$$

[2] The more general case with a non-constant reflecting barrier is studied in Blank and Solomon (2000).

otherwise the firm exits. The initial size is equal to 1000 for all 100,000 firms[3], and the rate of growth g_{it} is an *i.i.d.* Gaussian distribution with zero mean and standard deviation $\sigma = 0.1$. Now consider 2000 iterations of this stochastic process[4]. Figure 1 shows - in a semi-log plane - that from iteration 1000 and 2000 the number of firms goes down with a slope equal to -0.0018.

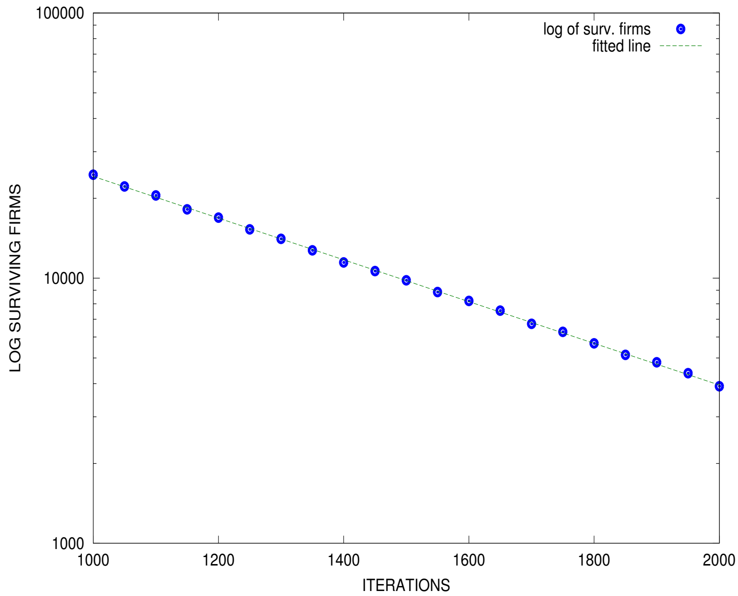

Figure 1: *Exponential decay for surviving firms.* The iteration step is 50.

In order to investigate what happens to a Pareto like firm size distribution we can implement a *two-stage simulation.* In the first stage there is a 1000 iterations simulation of the Kesten process

$$S_{it} = \max[(1 + g_{it})S_{it-1}, 20] \quad t = 1, 2, ..., 1000$$

The initial size is equal to 1000 for all 20,000 firms, and the rate of growth g_{it} is an *i.i.d.* Gaussian distribution with zero mean and standard deviation $\sigma = 0.1$. This process - at iteration 1000 - converges approximately to a Pareto distribution. The log of the size has excess kurtosis equal to 4.3577 and skewness equal to 1.8850.

[3]This setting of the stochastic process parameters has the following simple interpretation: Initial equity (size) is normalized to 1000 and the firm goes bankrupt if her equity falls below 2% of the initial value.

[4]The simulations were implemented in *GNU Octave* version 3. See Eaton *et al.* (2008).

In the second stage the process starts with the distribution at iteration 1000 of the first stage and evolve according to the process (same as above)

$$S_{it} = (1 + g_{it})S_{it-1} \quad if \quad S_{it-1} \geq 20; \quad t = 1000, 1001, ..., 2000.$$

Otherwise ($S_{it-1} < 20$) the firm exits. After other 1000 iterations the firm log-size distribution - of the 1261 survived firms - shows an excess kurtosis equal to 1.4168 and skewness equal to 1.1058 and so a more symmetric distribution with less fat tails. The distributions of the two steps - estimated with a Gaussian kernel - are compared in figure 2 where we can also see the rightward shift of the distribution. The intuition is simple: we are observing only firms that never go to the barrier after the second step of 1000 iterations.

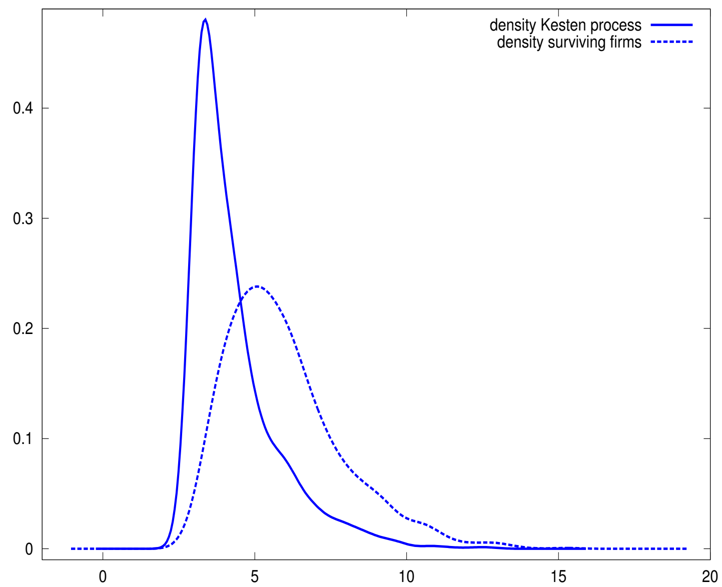

Figure 2: *Density estimation (Gaussian kernel) of unconditional distribution and conditional to survival.*

Thanks to the exponential decay discovery described above, we can go even further and increment the initial number of firms from 20,000 to 120,000 and implement a second step with 2000 iteration. In this way we can have an insight of statistical properties of the evolving survival firms distribution.

After the second stage of 2000 iterations, the firm's log-size distribution (of the 1257 survived firms[5]) shows an excess kurtosis equal to 1.2917 and skewness equal to 1.0701. The surviving firms' distribution continues to become more symmetric with less fat tails even though at at slower rate.

The firm size evolution is summarized in table 1

date	*kurtosis*	*skewness*
log-size step 1	4.3577	1.8850
log-size step 2 (1000 iter.)	1.4168	1.1058
log-size step 2 (2000 iter.)	1.2917	1.0701

Table 1: skewness and excess kurtosis (*i.e.*, kurtosis - 3) of the cohort.

where we can evaluate the bias toward a more symmetric distribution in the case in which the mean of a firm's growth is zero and does not change with the firm's age.

3 Conclusions

In this paper, we aim to shed some light on the surviving-firms' size distribution. We find that observing a cohort of firms may produce a *sample selection (survival) bias* depending on the fact that - observing only surviving firms - we select a different stochastic process from the one that may have produced the Pareto like shape. We made our analysis modifying the Kesten process. As shown in the paper, if we do not reintroduce firms when they go below the barrier, then the modified process produces an exponential decay of the number of firms.

The main result is that when the growth distribution of surviving firms has a mean equal to zero and independent from age, the selection of the cohort produces a bias towards a more symmetric distribution. This work shows that the modified Kesten's process - in which firms exit when their size (measured as equity) cross the barrier - produces a limit distribution of the cohort more symmetric.

This statistical result may be useful in studying the limit distribution of cohorts since it provides a benchmark to compare the distribution produced by economic models studying surviving firms.

[5]Note that increasing the number of firms from 20,000 to 120,000 we have approximately the same number of surviving firms so that a comparison with the previous simulation is possible.

References

Axtell, R., Gallegati, M., and Palestrini, A. (2008). Common components in firms' growth and the sectors scaling puzzle. *Economics Bulletin*, **12**(35), 1–8.

Axtell, R. L. (2001). Zipf distribution of us firm sizes. *Science*, **293**(5536), 1818–1820.

Bhattacharya, R. and Majumdar, M. (2007). Random dynamical systems. *Cambridge Books*.

Blank, A. and Solomon, S. (2000). Power laws in cities population, financial markets and internet sites (scaling in systems with a variable number of components). *Physica A: Statistical Mechanics and its Applications*, **287**(1), 279–288.

Cabral, L. M. and Mata, J. (2003). On the evolution of the firm size distribution: Facts and theory. *American economic review*, pages 1075–1090.

Champernowne, D. G. (1953). A model of income distribution. *The Economic Journal*, pages 318–351.

Cirillo, P. (2010). An analysis of the size distribution of italian firms by age. *Physica A: Statistical Mechanics and its Applications*, **389**(3), 459–466.

Das, S. (1995). Size, age and firm growth in an infant industry: The computer hardware industry in india. *International Journal of Industrial Organization*, **13**(1), 111–126.

Delli Gatti, D., Di Guilmi, C., Gaffeo, E., Giulioni, G., Gallegati, M., and Palestrini, A. (2005). A new approach to business fluctuations: heterogeneous interacting agents, scaling laws and financial fragility. *Journal of Economic behavior & organization*, **56**(4), 489–512.

Delli Gatti, D., Palestrini, A., Gaffeo, E., Giulioni, G., and Gallegati, M. (2008). *Emergent macroeconomics*. Springer.

Eaton, J., Bateman, D., and Hauberg, S. (2008). Gnu octave manual: Version 3, network theory limited.

Evans, D. S. (1987). The relationship between firm growth, size, and age: Estimates for 100 manufacturing industries. *The journal of industrial economics*, pages 567–581.

Gabaix, X. (2011). The granular origins of aggregate fluctuations. *Econometrica*, **79**(3), 733–772.

Gallegati, M. and Palestrini, A. (2010). The complex behavior of firms size dynamics. *Journal of Economic Behavior & Organization*, **75**(1), 69–76.

Gibrat, R. (1931). *Les inégalités économiques*. Recueil Sirey.

Hutchinson, J., Konings, J., and Walsh, P. P. (2010). The firm size distribution and inter-industry diversification. *Review of Industrial Organization*, **37**(2), 65–82.

Ijiri, Y., Simon, H. A., Bonini, C. P., and van Wormer, T. A. (1977). *Skew distributions and the sizes of business firms*. North-Holland Publishing Company New York.

Jovanovic, B. (1982). Selection and the evolution of industry. *Econometrica: Journal of the Econometric Society*, pages 649–670.

Kesten, H. (1973). Random difference equations and renewal theory for products of random matrices. *Acta Mathematica*, **131**(1), 207–248.

Mitzenmacher, M. (2004). A brief history of generative models for power law and lognormal distributions. *Internet mathematics*, **1**(2), 226–251.

Shanmugam, K. and Bhaduri, S. N. (2002). Size, age and firm growth in the indian manufacturing sector. *Applied Economics Letters*, **9**(9), 607–613.

Simon, H. A. and Bonini, C. P. (1958). The size distribution of business firms. *The American Economic Review*, pages 607–617.

Steindl, J. (1965). *Random processes and the growth of firms: A study of the Pareto law*. Griffin London.