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Polynomial Trends, Nonstationary Volatility and the Eicker-White Asymptotic Variance Estimator

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Abstract

The problem of inference in autoregressions around polynomial trends, under nonstationary, possibly explosive, volatility is investigated. It is shown that the well-known t-statistics that incorporate the Eicker-White covariance matrix estimator are asymptotically standard normal. Simulation results show that the application of a residual-based recursive-design wild bootstrap reduces significantly the size distortions in small samples.

I dedicate this paper to my father Dr. Constantine N. Kourogenis, who passed away unexpectedly in June 2015.

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1. Introduction

An increasing interest can be observed in the econometric literature, of studying models that allow for nonstationary volatility. The effects of unconditional heteroskedasticity on the usual unit root and stationarity tests have been studied by Cavaliere (2004), Phillips and Xu (2006), Cavaliere and Taylor (2005, 2007, 2008) and Xu and Phillips (2008). Stochastic volatility models have also been studied in the linear regression framework (Hansen 1995, Boswijk 2001, Chung and Park 2007, Cavaliere and Taylor 2009).

Xu (2008) examined stable autoregressions around polynomial deterministic trends with nonstationary volatility, and proposed test statistics for the coefficients of the model. To derive these tests, Xu followed a variation of the procedures of Eicker (1963) and White (1980), which requires *a priori* knowledge of the rate, γ_n, that describes the (possibly) explosive behavior of the nonstationary volatility of the errors. As a consequence, his tests are based on a covariance matrix estimator which, in contrast to the Eicker-White estimator, involves γ_n. For the application of these tests, Xu also required that $\gamma_n \propto n^k$ with $k \in (-\infty, 1/2)$.

In this paper, I prove that even when the autoregression follows a polynomial trend and the volatility is explosive, the heteroskedasticity robust test of White (1980), which does not require any knowledge of γ_n, remains asymptotically standard normal. The result holds for any of the autoregressive coefficients or the coefficients of the polynomial trend. A small Monte Carlo study shows that an application of a residual-based recursive-design wild bootstrap procedure (RRWB) to White's test statistic leads to significant improvements on the actual type I errors in small samples.

2. Theoretical Results

The model under study is of the form

$$y_t = \sum_{j=1}^{p} \theta_j y_{t-j} + \sum_{i=0}^{m} \delta_i t^i + \varepsilon_t,\ 1 \le t \le n,\ p, m < \infty\ . \tag{1}$$

It is assumed that every root of the lag-polynomial $\theta(L) = 1 - \left(\sum_{j=1}^{p} \theta_j L^j\right)$ lies outside the unit circle, $Var(\varepsilon_t) < \infty$, $\varepsilon_t = \sigma_t \eta_t$, where η_ts are zero mean, iid, and $\sigma_t^2 := Var(\varepsilon_t|\mathcal{F}_{t-1}) = Var(y_t|\mathcal{F}_{t-1})$ is a strictly positive (stochastic) volatility process, $\mathcal{F}_{t-1}$ representing the information generated by all ε_i, σ_{i+1}, $i < t$, and, possibly, other processes occurred up to time $t-1$. Weak convergence with respect to the uniform metric on $[0,1]$ is denoted by '$\Longrightarrow$'.

Assumption 1 (Boswijk 2005, Xu 2008)**:**

(i) *There exists a non-negative process $\sigma(\cdot)$, with piecewise continuous sample paths, such that $E\left[\int \sigma^2\right] < \infty$, and*

$$\begin{bmatrix} n^{-\frac{1}{2}} \sum_{\tau=j+1}^{t} f_{\tau j}\eta_\tau \\ \gamma_n^{-1}\sigma_t \end{bmatrix} \overset{t=[ns]}{=} \begin{bmatrix} n^{-\frac{1}{2}} \sum_{\tau=j+1}^{[ns]} f_{\tau j}\eta_\tau \\ \gamma_n^{-1}\sigma_{[ns]} \end{bmatrix} \Longrightarrow \begin{bmatrix} W^{(j)}(s) \\ \sigma(s) \end{bmatrix} ,$$

as $n \to \infty$, for some non-random sequence $\{\gamma_n\}_{n\ge1}$, where $f_{\tau j} = 1$ if $j = 0$, $f_{\tau j} = \eta_{\tau-j}$ if $j \ge 1$, and for every $j \ge 0$, $W^{(j)}(\cdot)$ is a standard Brownian motion.

(ii) *For some* $a>0$, $E\left[|\eta_t|^{4+a}\right]<\infty$ *and* $\max_{1\le t\le n}E\left[\left(\gamma_n^{-1}\sigma_t\right)^{4+a}\right]<\infty$ *uniformly in* n.

Assumption 2: σ_t *and* $\eta_{t'}$ *are independent for every* t, t'.

It can be easily proved that $W^{(j)}(\cdot)$ and $W^{(k)}(\cdot)$, $j\neq k$, are independent processes.

2.1 Asymptotic Inference

Let the $(p+m+1)\times(p+m+1)$ matrix $\widehat{D}=(\widehat{d}_{i,j})_{1\le i,j\le p+m+1}$ be defined as

$$\widehat{D}=\begin{bmatrix}\widehat{D}_{11} & \widehat{D}_{12}\\ \widehat{D}_{21} & \widehat{D}_{22}\end{bmatrix}:=\left(\sum_{t=1}^{n}X_tX_t'\right)^{-1}\left(\sum_{t=1}^{n}X_tX_t'\widehat{\varepsilon}_t^2\right)\left(\sum_{t=1}^{n}X_tX_t'\right)^{-1},$$

where $X_t=[y_{t-1},\dots y_{t-p},1,t,\dots,t^m]'$ and $\widehat{D}_{11}$ is $(p\times p)$. Let $\beta=[\theta_1,\dots\theta_p,\ \delta_0,\dots,\delta_m]'$, $\widehat{\beta}_n:=\left[\widehat{\theta}_{1,n},\dots\widehat{\theta}_{p,n},\widehat{\delta}_{0,n},\ \dots,\widehat{\delta}_{m,n}\right]'$ its OLS estimate for a sample of n observations, and $\widehat{\theta}_n:=\left[\widehat{\theta}_{1,n},\dots\widehat{\theta}_{p,n}\right]'$ be the estimated vector of the slope coefficients $\theta=[\theta_1,\dots\theta_p]'$.

Under Assumptions 1 and 2, $\widehat{\theta}_n$ is consistent, while the consistency of $\widehat{\delta}_{i,n}$ requires, in addition, that $\gamma_n=o\left(n^{(1+i)/2}\right)$ (Xu 2008). If, additionally, $\gamma_n\propto n^k$, $k\in(-\infty,1/2)$, then $\sqrt{n}(\widehat{\theta}_{j,n}-\theta_j)/\sqrt{\widehat{C}_{X,j,j}}\Longrightarrow N(0,1)$, where $\widehat{C}_{X,j,j}$ is the (j,j) element of the matrix

$$\widehat{C}_X=\left(\mathcal{Y}^{-1}\sum_{t=1}^{n}X_tX_t'\right)^{-1}\left(\sum_{t=1}^{n}X_tX_t'\widehat{\varepsilon}_t^2\right)\left(\sum_{t=1}^{n}X_tX_t'\mathcal{Y}^{-1}\right)^{-1},$$

and $\mathcal{Y}=diag\{\overbrace{\sqrt{n},\sqrt{n},\dots,\sqrt{n}}^{p\text{ times}},\sqrt{n}\gamma_n^{-1},n^{3/2}\gamma_n^{-1},\dots,n^{(2m+1)/2}\gamma_n^{-1}\}$ (Xu 2008, Corollary 3.1). In order for $\widehat{C}_{Xjj}$ to be feasible, γ_n must be known. Note that $\widehat{D}\neq\widehat{C}_X/n$ when $m>0$ or $\gamma_n\nrightarrow 1$, because

$$\frac{1}{n}\widehat{C}_X=\widehat{D}\Longleftrightarrow\left(\frac{1}{\sqrt{n}}\mathcal{Y}\right)\left(\sum_{t=1}^{n}X_tX_t'\widehat{\varepsilon}_t^2\right)\left(\frac{1}{\sqrt{n}}\mathcal{Y}\right)=\left(\sum_{t=1}^{n}X_tX_t'\widehat{\varepsilon}_t^2\right),$$

which holds only when $\mathcal{Y}=\sqrt{n}I_{p+m+1}$, I_{p+m+1} being the $(p+m+1)\times(p+m+1)$ identity matrix. Therefore, in general, $t_{\widehat{\theta}_{j,n}}:=\left(\widehat{\theta}_{j,n}-\theta_j\right)/\sqrt{\widehat{d}_{j,j}}\neq\sqrt{n}(\widehat{\theta}_{j,n}-\theta_j)/\sqrt{\widehat{C}_{X,j,j}}$, $1\le j\le p$, and $t_{\widehat{\delta}_{i,n}}:=(\widehat{\delta}_{i,n}-\delta_i)/\sqrt{\widehat{d}_{p+i+1,p+i+1}}\neq\sqrt{n}(\widehat{\delta}_{i,n}-\delta_i)/\sqrt{\widehat{C}_{X,p+i+1,p+i+1}}$, $0\le i\le m$.

Theorem 1: *If Assumptions 1 and 2 hold then*

(i) $t_{\widehat{\theta}_{j,n}}\Longrightarrow N(0,1)$, $1\le j\le p$;

(ii) *under the null hypothesis* H_0*:* $R\theta=r$, *where* R *is a* $k\times p$ *matrix of full row rank and* $r\in\mathbb{R}^k$, $(R\widehat{\theta}_n-r')(R\widehat{D}_{11}R')^{-1}(R\widehat{\theta}_n-r)\Longrightarrow\chi^2(k)$;

(iii) $t_{\widehat{\delta}_{i,n}}\Longrightarrow N(0,1)$, $0\le i\le m$.

Remarks:

(i) The test statistics in Theorem 1 are feasible and do not require any knowledge of γ_n. Theorem 1(iii), however, can be used for conducting inference on the coefficient δ_i only when $\widehat{\delta}_{i,n}$ is consistent.

(ii) In fact, $n\widehat{D}$ is the Eicker-White asymptotic covariance matrix estimator. In the context of our analysis, however, $n\widehat{D}$ does not necessarily converge.

2.2 Proof of Theorem 1

Xu (2008) showed that there exists a matrix G of the form

$$G = \begin{bmatrix} I_p & H \\ \mathbf{0} & I_{m+1} \end{bmatrix} , \tag{2}$$

such that for $\widetilde{X}_t := GX_t$, under Assumptions 1 and 2, $\gamma_n^{-2}\mathcal{Y}^{-1}\sum_{t=1}^n \widetilde{X}_t\widetilde{X}_t'\mathcal{Y}^{-1} \Longrightarrow Q = diag\{Q_1, Q_2\}$ and $\gamma_n^{-4}\mathcal{Y}^{-1}\sum_{t=1}^n \widetilde{X}_t\widetilde{X}_t'\widehat{\varepsilon}_t^2\mathcal{Y}^{-1} \Longrightarrow U = diag\{U_1, U_2\}$, where Q and U depend on $\sigma(\cdot)$, and Q_1, U_1 are $p \times p$ (proofs of Lemma 3.1 and Corollary 3.1, respectively). Let $A = diag\{A_{11}, A_{22}\} := Q^{-1}UQ^{-1}$, where A_{11} is $p \times p$. Lemma 3.1(ii) in Xu (2008) proves that under Assumptions 1 and 2,

$$\mathcal{Y}(G')^{-1}\left(\widehat{\beta}_n - \beta\right) \Longrightarrow MN_{p+m+1}(\mathbf{0}, A) \ . \tag{3}$$

We have

$$\begin{aligned} & \mathcal{Y}(G')^{-1}\widehat{D}G^{-1}\mathcal{Y} \\ = \ & \mathcal{Y}(G')^{-1}\left(\sum_{t=1}^n X_tX_t'\right)^{-1} G^{-1}G\left(\sum_{t=1}^n X_tX_t'\widehat{\varepsilon}_t^2\right)G'(G')^{-1}\left(\sum_{t=1}^n X_tX_t'\right)^{-1}G^{-1}\mathcal{Y} \\ = \ & \left(\gamma_n^{-2}\mathcal{Y}^{-1}\sum_{t=1}^n \widetilde{X}_t\widetilde{X}_t'\mathcal{Y}^{-1}\right)^{-1}\left(\gamma_n^{-4}\mathcal{Y}^{-1}\sum_{t=1}^n \widetilde{X}_t\widetilde{X}_t'\widehat{\varepsilon}_t^2\mathcal{Y}^{-1}\right)\left(\gamma_n^{-2}\mathcal{Y}^{-1}\sum_{t=1}^n \widetilde{X}_t\widetilde{X}_t'\mathcal{Y}^{-1}\right)^{-1} , \end{aligned}$$

hence,

$$\mathcal{Y}(G')^{-1}\widehat{D}G^{-1}\mathcal{Y} \Longrightarrow A \ . \tag{4}$$

Let $\Gamma_n := diag\{\sqrt{n}\gamma_n^{-1}, n^{3/2}\gamma_n^{-1}, \ldots, n^{(2m+1)/2}\gamma_n^{-1}\}$. Then,

$$\mathcal{Y} = \begin{bmatrix} \sqrt{n}I_p & \mathbf{0} \\ \mathbf{0} & \Gamma_n \end{bmatrix} \Rightarrow \mathcal{Y}(G')^{-1} = \begin{bmatrix} \sqrt{n}I_p & \mathbf{0} \\ -\Gamma_nH' & \Gamma_n \end{bmatrix} . \tag{5}$$

Applying (5) in (3) we obtain $\sqrt{n}\left(\widehat{\theta}_n - \theta\right) \Longrightarrow MN_p(\mathbf{0}, A_{11})$. Applying now (5) in (4) we have that $n\widehat{D}_{11} \Longrightarrow A_{11}$, and Theorems 1(i) and (ii) follow directly.

Let, now, $P(1,j)$ be the $(p+m+1) \times (p+m+1)$ permutation matrix for rows 1 and j. Let also $\widehat{F}_{1,j,L}$ be the lower triangular invertible matrix with strictly positive diagonal elements, of the Cholesky decomposition of $\widehat{F}_{1,j} := P(1,j)\widehat{D}P(1,j)$, i.e. $\widehat{F}_{1,j} = \widehat{F}_{1,j,L}\widehat{F}_{1,j,L}'$. Because $P(1,j) = P(1,j)' = P(1,j)^{-1}$, by virtue of (4) we have that

$$\mathcal{Y}(G')^{-1}\widehat{D}G^{-1}\mathcal{Y} = \mathcal{Y}(G')^{-1}P(1,j)\widehat{F}_{1,j,L}\widehat{F}_{1,j,L}'P(1,j)G^{-1}\mathcal{Y} \Longrightarrow A \ . \tag{6}$$

Combining (6) with (3) we conclude that

$$\widehat{F}_{1,j,L}^{-1}P(1,j)G'\mathcal{Y}^{-1}\mathcal{Y}(G')^{-1}\left(\widehat{\beta}_n - \beta\right) = \widehat{F}_{1,j,L}^{-1}P(1,j)\left(\widehat{\beta}_n - \beta\right) \Longrightarrow N(0, I_{p+m+1}). \tag{7}$$

For $j = p+1+i$, $0 \leq i \leq m$, the first element of the vector $P(1,j)(\widehat{\beta}_n - \beta)$ is $\widehat{\delta}_{i,n}$. Theorem 1(iii) follows from (7), because $\widehat{F}_{1,j,L}$ is a lower triangular matrix and its upper left element is indeed $\sqrt{d_{j,j}}$.

3 Simulation Results

This section presents the results of Monte Carlo simulations, assessing the small-sample performance of the tests provided by Theorem 1 and by the following RRWB applied on these tests (see also Gonçalves and Kilian 2004, and Xu 2008):

Estimate model (1) by OLS and set $\widehat{\varepsilon}_t = y_t - X_t'\widehat{\beta}$, $1 \leq t \leq n$. Let $\{v_t\}_{1 \leq t \leq n} \sim NIID(0,1)$, $\widehat{\varepsilon}_t^* = v_t \widehat{\varepsilon}_t$, $1 \leq t \leq n$, be the series of bootstrap errors, and $\{y_t^*\}$ be defined by setting for $-p+1 \leq t \leq 0$, $y_t^* = y_t$, and for $1 \leq t \leq n$, $y_t^* = X_t^{*\prime}\widehat{\beta} + \widehat{\varepsilon}_t^*$, where $X_t^* = \left[y_{t-1}^*, \ldots, y_{t-p}^*, 1, t, \ldots, t^m\right]'$. Finally, let $\widehat{\beta}^*$ be the OLS estimate of $\widehat{\beta}$ for model $y_t^* = X_t^{*\prime}\widehat{\beta} + \widehat{\varepsilon}_t^*$, and $\widetilde{\varepsilon}_t^* = y_t^* - X_t^{*\prime}\widehat{\beta}^*$ be the bootstrap residuals. The bootstrap analog of $\widehat{D}$ is $\widehat{D}^* := \left(\sum_{t=1}^n X_t^* X_t^{*\prime}\right)^{-1} \left(\sum_{t=1}^n X_t^* X_t^{*\prime} \left(\widetilde{\varepsilon}_t^*\right)^2\right) \left(\sum_{t=1}^n X_t^* X_t^{*\prime}\right)^{-1}$.

The data generating process is the following: $y_t = \delta_0 + \delta_1 t + \delta_2 t^2 + \theta_1 y_{t-1} + \theta_2 y_{t-2} + \varepsilon_t$, $1 \leq t \leq n$, $y_{-1} = y_0 = 0$,where $\varepsilon_t = \sigma_t \eta_t$, $n \in \{50, 100, 250\}$ and $(\theta_1, \theta_2) \in \{(0.4, 0.2), (0.6, 0.3)\}$. We also set $\delta_0 = 2$, $\delta_1 = -0.04$ and $\delta_2 = 0.0002$. Because $m > 0$, $t_{\widehat{\theta}_{j,n}}$ and $t_{\widehat{\delta}_{i,n}}$ do not coincide with the corresponding statistics of Xu (2008). The number of replications is 4000, and the number of bootstrap replications is 999. The test statistics of Theorem 1(i) (asymptotic) and of the RRWB are used in order to obtain the 90% equal tailed percentile-t confidence intervals. Concerning $\{\sigma_t\}$, let $[u_t, \eta_t]' \sim NIID\left(\mathbf{0}, diag\{0.16, 1\}\right)$, $1 \leq t \leq n$, $z_0 = 0$ and for $t \geq 1$, $z_t = z_{t-1} + 0.5u_t$. We consider cases where $\sigma_t = |z_{t-1}|^k$, $k \in \{0.5, 1, 2, 3\}$, and $\sigma_t = e^{z_{t-1}}$. When $\sigma_t = |z_{t-1}|^k$, γ_n increases at a rate similar to $n^{k/2}$. Therefore, $\widehat{\delta}_{i,n}$ is consistent only if $k < 1 + i$. Chung and Park (2007) examined the case $\sigma_t = e^{z_{t-1}}$, but not in an autoregressive framework. When $\sigma_t = e^{z_{t-1}}$, none of the estimators $\widehat{\delta}_{i,n}$ is consistent. Nevertheless, Theorem 1 implies that $\widehat{\theta}_n$ remains consistent and inference on the autoregressive coefficients can be conducted.

Table I: Actual type I errors for the 90% confidence intervals based on the asymptotic distribution and the RRWB for the autoregressive coefficients.

	$\theta_1 = .4$, $\theta_2 = .2$					$\theta_1 = .6$, $\theta_2 = .3$			
	ASY		WB			ASY		WB	
σ_t	θ_1	θ_2	θ_1	θ_2	$n = 50$	θ_1	θ_2	θ_1	θ_2
$\sqrt{\lvert z_{t-1}\rvert}$	.176	.153	.113	.121		.221	.169	.136	.125
$\lvert z_{t-1}\rvert$	.181	.158	.113	.125		.217	.177	.126	.132
$\lvert z_{t-1}\rvert^2$	.198	.176	.116	.133		.218	.190	.127	.137
$\lvert z_{t-1}\rvert^3$	.211	.192	.111	.134		.227	.207	.123	.141
$e^{z_{t-1}}$	.226	.196	.128	.140		.271	.211	.153	.143
	θ_1	θ_2	θ_1	θ_2	$n = 100$	θ_1	θ_2	θ_1	θ_2
$\sqrt{\lvert z_{t-1}\rvert}$	.147	.128	.100	.096		.141	.124	.102	.103
$\lvert z_{t-1}\rvert$	.150	.133	.115	.118		.147	.134	.119	.121
$\lvert z_{t-1}\rvert^2$	.170	.158	.115	.119		.174	.150	.122	.120
$\lvert z_{t-1}\rvert^3$	.184	.180	.130	.138		.190	.173	.134	.140
$e^{z_{t-1}}$	.181	.170	.120	.126		.177	.156	.121	.123
	θ_1	θ_2	θ_1	θ_2	$n = 250$	θ_1	θ_2	θ_1	θ_2
$\sqrt{\lvert z_{t-1}\rvert}$	.129	.123	.094	.099		.119	.113	.091	.093
$\lvert z_{t-1}\rvert$	.136	.127	.099	.101		.132	.127	.102	.105
$\lvert z_{t-1}\rvert^2$	.148	.140	.113	.109		.145	.140	.111	.113
$\lvert z_{t-1}\rvert^3$	.163	.151	.110	.117		.164	.153	.114	.115
$e^{z_{t-1}}$	.167	.160	.114	.114		.158	.147	.109	.108

Tables I and II report the results that correspond to the autoregressive coefficients and the (consistently estimated) coefficients of the deterministic trend, respectively. When $n = 50$ the test based on the asymptotic confidence intervals (ASY) suffers from severe size distortions. When symmetric bootstrap confidence intervals (WB) are employed, size distortions are significantly reduced. As n increases, the type I errors of ASY decrease, albeit at a slow rate.[1] On the other hand, the rejection rates of WB for $n = 250$ are fairly close to 10%.

Recall that $\widehat{D}$ and $\widehat{C}_X/n$ do not coincide when $m > 0$ or $\gamma_n \nrightarrow 1$. A small simulation study provides some evidence about the performance of the tests proposed in Xu (2008). Note that these tests require knowledge of γ_n. We set $n = 1000$, the number of replications equal to 6000 and $(\theta_1, \theta_2) \in \{(0.4, 0.2), (0.6, 0.3)\}$. When $\{\sigma_t\} \in \left\{\left\{\sqrt{|z_{t-1}|}\right\}, \{|z_{t-1}|\}\right\}$, the rejection rates of the asymptotic procedure for all the regression coefficients are practically 0%. When $\sigma_t = |z_{t-1}|^2$, the rejection rates that correspond to the autoregressive coefficients range from 11.58% to 12.35%. However, the rejection rates that correspond to δ_2, which is the only consistently estimated coefficient of the deterministic trend, are again practically 0%.

Table II: Actual type I errors for the 90% confidence intervals based on the asymptotic distribution and the RRWB for the deterministic trend coefficients.

		$\theta_1 = .4,\ \theta_2 = .2$						$\theta_1 = .6,\ \theta_2 = .3$					
		ASY			WB			ASY			WB		
n	σ_t	δ_0	δ_1	δ_2	δ_0	δ_1	δ_2	δ_0	δ_1	δ_2	δ_0	δ_1	δ_2
	$\sqrt{\|z_{t-1}\|}$	.186	.216	.227	.126	.136	.148	.263	.439	.420	.138	.237	.229
50	$\|z_{t-1}\|$		.221	.245		.143	.151		.423	.404		.220	.214
	$\|z_{t-1}\|^2$			.248			.152			.404			.199
	$\sqrt{\|z_{t-1}\|}$	.148	.144	.152	.099	.104	.107	.235	.221	.231	.120	.120	.126
100	$\|z_{t-1}\|$		.157	.162		.103	.108		.260	.272		.132	.135
	$\|z_{t-1}\|^2$			.180			.115			.302			.154
	$\sqrt{\|z_{t-1}\|}$	.130	.135	.134	.110	.112	.110	.204	.198	.200	.120	.118	.121
250	$\|z_{t-1}\|$		.137	.138		.102	.102		.203	.207		.122	.122
	$\|z_{t-1}\|^2$			.148			.110			.223			.134

4. Conclusions

This paper examined the problem of inference in stable autoregressions, around polynomial deterministic trends, under nonstationary volatility. It was shown that the Eicker-White heteroskedasticity robust $t-$statistics are asymptotically standard normal. The corresponding tests are always feasible and do not require any knowledge of the asymptotic rate of the nonstationary volatility, γ_n. These tests can be directly applied to conduct inference on the autoregressive coefficients and on any coefficient of the deterministic trend given that the corresponding OLS estimator is consistent. A small Monte Carlo study demonstrated the good performance of the residual-based recursive-design wild bootstrap procedure in terms of accuracy of the actual type I errors, even in small

[1] For example, using 6000 replications with $n = 5000$, the actual type I error was 12.6% for H_0: $\theta_1 = 0.6$, when $(\theta_1, \theta_2) = (0.6, 0.3)$ and $\sigma_t = e^{z_{t-1}}$.

samples. As the persistence of the autoregressive process increases, larger samples are required for the convergence of the rejection rates to their asymptotic value.

References

Boswijk, H.P. (2001) "Testing for a unit root with near-integrated volatility" Tinbergen Institute Discussion Paper 01-077/4.

Boswijk, H.P. (2005) "Adaptive testing for a unit root with nonstationary volatility" UvA Econometrics Discussion Paper 2005/07.

Cavaliere, G. (2004) "Unit root tests under time-varying variances" *Econometric Reviews* **6**, 259-292.

Cavaliere, G. and A.M.R. Taylor (2005) "Stationarity Tests Under Time-Varying Second Moments" *Econometric Theory* **21**, 1112-1129.

Cavaliere, G. and A.M.R. Taylor (2007) "Testing for unit roots in time series models with non-stationary volatility" *Journal of Econometrics* **140**, 919-947.

Cavaliere, G. and A.M.R. Taylor (2008) "Bootstrap unit root tests for time series with non-stationary volatility" *Econometric Theory* **24**, 43-71.

Cavaliere, G. and A.M.R. Taylor (2009) "Heteroskedastic time series with a unit root" *Econometric Theory* **25**, 1228-1276.

Chung, H. and J.Y. Park (2007) "Nonstationary nonlinear heteroskedasticity in regression" *Journal of Econometrics* **137**, 230-259.

Eicker, F. (1963) "Asymptotic Normality and Consistency of the Least Squares Estimators for Families of Linear Regressions" *Annals of Mathematical Statistics* **34**, 447-456.

Gonçalves, S. and L. Kilian (2004) "Bootstrapping autoregression with conditional heteroskedasticity of unknown form" *Journal of Econometrics* **123**, 89-120.

Hansen, B.E. (1995) "Regression with nonstationary volatility" *Econometrica* **63**, 1113-1132.

Phillips, P.C.B. and K.-L. Xu (2006) "Inference in autoregression under heteroskedasticity" *Journal of Time Series Analysis* **27**, 289-308.

White, H. (1980) "A heteroscedasticity-Consistent Covariance Matrix Estimator and a Direct Test for Heteroscedasticity" *Econometrica* **48**, 817-838.

Xu, K.-L. (2008) "Bootstrapping autoregression under nonstationary volatility" *Econometrics Journal* **11**, 1-26.

Xu, K.-L. and P.C.B. Phillips (2008) "Adaptive estimation of autoregressive models with time-varying variances" *Journal of Econometrics* **142**, 265-280.