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The True Impact of Voting Rule Selection on Condorcet Efficiency

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Abstract

We provide representations giving the Condorcet Efficiency of Weighted Scoring Rules in three-alternative elections by considering the Modified Impartial Anonymous Culture condition (MIAC). This assumption only considers voting situations for which all Weighted Scoring Rules do not elect the same winner. It is concluded that the selection of a voting rule has a clear impact on the resulting Condorcet Efficiency. This makes a significant difference in what can be concluded from earlier studies.

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1. Introduction

Consider an election with n voters when each voter has a complete preference ranking on three candidates $\{A, B, C\}$. With complete preference rankings, individual voter indifference between candidates is prohibited; and intransitive, or cyclic, individual preferences are not allowed. When we restrict attention to the limiting case of a large electorate as $n \to \infty$, the six feasible candidate preference rankings for individual voters are shown in Figure 1:

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
p_1	p_2	p_3	p_4	p_5	p_6

Figure 1. Feasible voter preference rankings on candidates with three candidates.

The p_i terms denote the proportion of the n voters with the associated ith candidate preference ranking for $1 \leq i \leq 6$ in a given election as $n \to \infty$. Let $A \succ B$ denote an individual voter's preference on a pair of candidates, such that Candidate A is preferred to Candidate B. With this definition, p_1 voters have preferences with $A \succ B$, $A \succ C$ and $B \succ C$. A *voting situation* defines a specific outcome of an election for which n voters have candidate preferences with $\sum_{i=1}^{6} p_i = 1$. Any given voting situation is therefore completely specified by a six-dimensional vector $\boldsymbol{p}$ with its six associated p_i components from Figure 1. We define $\boldsymbol{P}$ as the space of all possible $\boldsymbol{p}$ with $\sum_{i=1}^{6} p_i = 1$.

A pairwise majority preference *A**M**B* exists on the pair of Candidates A and B if more voters have $A \succ B$ than those who have $B \succ A$, with $p_1 + p_2 + p_4 > p_3 + p_5 + p_6$. Candidate A is the *Condorcet Winner* (CW) in a voting situation if it defeats both remaining candidates by pairwise majority preference, with both *A**M**B* and *A**M**C*. The CW would obviously be a good candidate to select as the winner in an election, but it is well known that a CW does not always exist. That is, voting situations can exist where pairwise majority cycles are possible, such as when *A**M**B*, *B**M**C* and *C**M**A*. The existence of such cyclic majorities is known as an occurrence of Condorcet's Paradox. Most voting rules will not always elect the CW, but their overall propensity to do so is measured by Condorcet Efficiency, which is formally defined as the conditional probability that a specified voting rule will elect the CW, given that a CW does exist.

We consider voting rules in the context of Weighted Scoring Rules of the form $Rule(\lambda)$. Any $Rule(\lambda)$ is defined in terms of weights $(1, \lambda, 0)$; and every voter assigns 1 point to their most preferred candidate, λ points to their middle ranked candidate and 0 points to their least preferred candidate. The candidate that receives the greatest number of accumulated points from all voters is declared as the winner. $Rule(0)$ is equivalent to the commonly used Plurality Rule (PR), $Rule(1)$ is equivalent to Negative Plurality Rule (NPR) or Anti-Plurality Rule and $Rule(1/2)$ is the well-known Borda Rule (BR). Extensive analysis has been performed to show that BR has many excellent properties when it is compared to other $Rule(\lambda)$ [see for example Saari (1990)].

Our objective is to consider the impact that the selection of λ has on the resulting Condorcet Efficiency of the associated $Rule(\lambda)$.

2. Representations for Condorcet Efficiency with IAC

The Condorcet Efficiency of any $Rule(\lambda)$ will clearly be dependent on the probability that various voting situations will be observed. One of the most common assumptions regarding this probability is the Impartial Anonymous Culture Condition (IAC), which is equivalent to assuming that all possible $\boldsymbol{p}\epsilon\boldsymbol{P}$ are equally likely to be observed when $n \to \infty$. Probabilities of election outcomes can therefore be obtained from the consideration of volumes of subspaces in $\boldsymbol{P}$ with the assumption of IAC.

Diss and Gehrlein (2012) use results from Sommerville (1958, pgs. 125-126) to show that the entire volume of $\boldsymbol{P}$ is given by $Volume(IAC)$ with

$$Volume(IAC) = \frac{\sqrt{6}}{120}. \tag{1}$$

Cervone et al (2005) considers the Condorcet Efficiency of Weighted Scoring Rules by determining the volume of two subspaces of $\boldsymbol{P}$. The first is the volume of the subspace for which Candidate *A* is the CW with IAC, which we denote as $Volume(IAC, A = CW)$, with

$$Volume(IAC, A = CW) = \frac{\sqrt{6}}{384}. \tag{2}$$

The second subspace considers voting situations for which Candidate *A* is both the CW and the winner by $Rule(\lambda)$, with a volume denoted by $Volume\big(IAC, A = CW = Rule(\lambda)\big)$:

$$Volume\big(IAC, A = CW = Rule(\lambda)\big) = \frac{\sqrt{6}\left(8\lambda^7+28\lambda^6+65\lambda^5-1036\lambda^4+1534\lambda^3+335\lambda^2-1647\lambda+714\right)}{155520(\lambda+1)(\lambda-2)(\lambda-1)^3}, \quad \text{for } 0 \leq \lambda \leq 1/2. \tag{3}$$

$$= \frac{\sqrt{6}\left(192\lambda^8+5984\lambda^7-16764\lambda^6+4496\lambda^5+17522\lambda^4-8395\lambda^3+1163\lambda^2-126\lambda+8\right)}{622080\lambda^3(\lambda+1)(\lambda-2)(1-3\lambda)}, \quad \text{for } 1/2 \leq \lambda \leq 1.$$

The IAC assumption is symmetric with regards to Candidates *A*, *B* and *C*, so the Condorcet Efficiency $CE\big(IAC, Rule(\lambda)\big)$ of $Rule(\lambda)$ with IAC as $n \to \infty$ is obtained from

$$CE\big(IAC, Rule(\lambda)\big) = \frac{Volume\big(IAC, A = CW = Rule(\lambda)\big)}{Volume(IAC, A = CW)}. \tag{4}$$

The representations from (2), (3) and (4) are used to obtain values of $CE\big(IAC, Rule(\lambda)\big)$ for each $\lambda = 0(.05)1$ and the results are listed in Table 1.

λ	$CE(IAC, Rule(\lambda))$	$CE(MIAC, Rule(\lambda))$	λ	$CE(IAC, Rule(\lambda))$	$CE(MIAC, Rule(\lambda))$
0.00	0.8815	0.7668	0.50	0.9111	0.8315
0.05	0.8899	0.7851	0.55	0.8943	0.7947
0.10	0.8979	0.8027	0.60	0.8720	0.7461
0.15	0.9055	0.8192	0.65	0.8461	0.6896
0.20	0.9123	0.8342	0.70	0.8176	0.6273
0.25	0.9182	0.8470	0.75	0.7874	0.5612
0.30	0.9227	0.8567	0.80	0.7560	0.4926
0.35	0.9252	0.8622	0.85	0.7240	0.4228
0.40	0.9249	0.8617	0.90	0.6919	0.3528
0.45	0.9208	0.8527	0.95	0.6603	0.2838
0.50	0.9111	0.8315	1.00	0.6296	0.2167

Table 1. Condorcet Efficiency of $Rule(\lambda)$ with IAC and MIAC.

The results from Table 1 show that $CE(IAC, PR) = .8815$, $CE(IAC, BR) = .9111$ and $CE(IAC, NPR) = .6296$. The value of $CE(IAC, Rule(\lambda))$ is maximized at $\lambda \approx .37228$, so BR does not maximize Condorcet Efficiency with IAC. Table 1 shows that $CE(IAC, Rule(\lambda))$ is quite stable for a wide range of values around the efficiency maximizing λ. This indicates that the selection of λ for a voting rule with IAC is not very critical in regions that are not relatively far removed from the most efficient λ, particulary for λ that are closer to PR.

Given this background, Gehrlein et al (2011) found that there were identifiable subsets of voting situations that are based on the proximity of the voting situations to having perfectly single-peaked or perfectly single-dipped preferences, such that these voting situations resulted in extremely poor expected performance for PR and NPR on the basis of Condorcet Efficiency with IAC. The same outcome was never observed for BR, which always performed quite well. The results in Table 1 do not seem to reflect nearly as strongly just how poorly PR and NPR can perform in some scenarios relative to BR with IAC.

3. Another Perspective on Probability Representations with MIAC

We begin the new focus of the current study by considering a modification of IAC to account for this observation. This modification is based on the idea that there are many voting situations in $\boldsymbol{P}$ for which every $Rule(\lambda)$ for $0 \leq \lambda \leq 1$ will elect the same winner. It is well known that the same winner will be elected for all $Rule(\lambda)$ in a three-candidate voting situation if that candidate is the winner by both PR and NPR [see for example, Moulin (1988)], and the limiting probability that PR and NPR both elect the same winner is found to be .5231 with IAC in Gehrlein (2002). What if PR and NPR only have their greater than anticipated values of Condorcet Efficiency with IAC in Table 1 as a result of the fact that there is a relatively large probability that every $Rule(\lambda)$ will elect the same winner?

For example, NPR already has a relatively weak Condorcet Efficiency value of 63% with IAC. If the 52% of voting situations are excluded from consideration when NPR and every other

$Rule(\lambda)$ elect the same winner, to make the selection of λ irrelevant, the true performance of NPR in relevant scenarios will be measured. NPR might therefore be shown actually to have abysmal expected performance in the relevant scenarios, rather than just having weak performance overall. The same general argument can be made for all $Rule(\lambda)$ when we wish to measure their relative performance on the basis of Condorcet Efficiency.

It is important to note that the removal from consideration of the subset of voting situations for which all $Rule(\lambda)$ elect the same winner does not have the same impact on the Condorcet Efficiency of all voting rules, so it is not a trivial exercise to consider the impact of this extension of IAC. In order to illustrate this point, we temporarily leave aside the case of $n \to \infty$ and suppose instead that the number of voters is $n = 3$. A given voting situation is now defined by $(n_1, n_2, n_3, n_4, n_5, n_6)$. The n_i *t*erms denote the number of voters with the associated *i*th candidate preference ranking for $1 \leq i \leq 6$ in a given election with three voters from Figure 1. The total number of possible voting situations is $\frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{120} = 56$. Two of these voting situations (1,0,0,1,1,0) and (0,1,1,0,0,1) do not have a CW, so they are excluded from any further consideration. Without any loss of generality, we suppose that the CW is Candidate *A*, which occurs in 18 of these 54 remaining voting situations. Candidate *A* is found to be the strict winner by PR, NPR, and BR respectively in 16, 8, and 16 of these 18 voting situations. A strict winner is elected without it having any ties with other candidates. These values are used with the symmetry that IAC has with respect to candidates to obtain the Condorcet Efficiency of PR $\left(\frac{16}{18} = .889\right)$, NPR $\left(\frac{8}{18} = .444\right)$ and BR $\left(\frac{16}{18} = .889\right)$ with IAC for $n = 3$.

Among these 18 voting situations, there are six[1] that will lead to the election of the same strict winner for every *Rule* (λ) for $0 \leq \lambda \leq 1$, based on the fact that the same candidate is elected by PR and NPR. This is the CW Candidate *A* in each case, but it is possible that PR and NPR could both select the same winner that is not the CW. In other words, the selection of $Rule(\lambda)$ has no impact on the ultimate election result (Candidate *A* wins) in one-third of these possible voting situations since six out of 18 voting situations are found among these cases.

However, if the preference structure of voters leads to a voting situation such as the one given by (2,0,0,0,0,1), the analysis is quite different. The CW is still the same Candidate *A* and the selection of a *Rule* (λ) has a clear impact on the selection (or not) of the CW since the winner of PR and BR both select *A* to have a positive impact on their Condorcet Efficiencies. But, NPR selects Candidate *B* to have a negative impact on its Condorcet Efficiency.

If the six voting situations that are irrelevant to the selection of *Rule* (λ) are removed from consideration for the example with $n = 3$, the modified voting rule efficiencies become: PR $\left(\frac{10}{12} = .833\right)$, NPR $\left(\frac{2}{12} = .167\right)$ and BR $\left(\frac{10}{12} = .833\right)$. All of the efficiencies are reduced, but the

[1] These voting situations are (2,1,0,0,0,0), (1,2,0,0,0,0), (0,2,1,0,0,0), (2,0,0,1,0,0), (1,1,1,0,0,0), and (1,1,0,1,0,0)

impact is particularly dramatic for NPR, which shows only a 16% chance of selecting the CW for voting situations where the selection of *Rule* (λ) might actually have an impact on the election outcome. This strongly suggests that there is absolutely no reason to ever consider the possibility of using NPR, or anything like it, in three-voter elections. It does perform well when everything works, but it performs very poorly when the voting rule selection actually makes a difference. The obvious question is to wonder if this dramatic outcome is simply an aberration that results from the fact that we are only considering the special case with $n = 3$.

The Modified Impartial Anonymous Culture Condition (MIAC) follows what was done in the example with $n = 3$ and only considers voting situations in $\boldsymbol{P}' \subset \boldsymbol{P}$ for which $Rule(\lambda)$ does not elect the same winner for all $0 \leq \lambda \leq 1$. As in the case of IAC, it is assumed with MIAC that all possible voting situations in $\boldsymbol{P}'$ are equally likely to be observed, and we now start to focus again on the limiting case as $n \to \infty$.

So, we begin our analysis with MIAC by finding the volume $Volume(IAC, A = PR = NPR)$ of the subspace of IAC voting situations in $\boldsymbol{P}$ for which Candidate *A* is the winner by both PR and NPR. This is done with the same general procedure that was used in Cervone et al (2005) to obtain the volumes that are given above in (1), (2) and (3). Precise details of how this procedure is implemented are not included in the current study since they can be found at the original source. However, we do present an outline of how the results were obtained for the simplest case that is considered in this study in order to provide a basic illustration of how the procedure works.

The volume of the possible IAC voting situations in $\boldsymbol{P}$ is defined in six-dimensional space with six vertices υ_i^0 at:

$$\upsilon_1^0 = [1,0,0,0,0,0] \qquad \upsilon_3^0 = [0,0,1,0,0,0] \qquad \upsilon_5^0 = [0,0,0,0,1,0]$$
$$\upsilon_2^0 = [0,1,0,0,0,0] \qquad \upsilon_4^0 = [0,0,0,1,0,0] \qquad \upsilon_6^0 = [0,0,0,0,0,1].$$

This space corresponds to a five-dimensional simplex Δ^5 and its volume is given above in (1).

Suppose that Candidate *A* is the winner by both PR and NPR. This requires that:

$$p_1 + p_2 - p_3 - p_5 > 0 \qquad \text{(}A \text{ beats } B \text{ by PR)} \tag{5}$$

$$p_1 + p_2 - p_4 - p_6 > 0 \qquad \text{(}A \text{ beats } C \text{ by PR)} \tag{6}$$

$$p_2 + p_4 - p_5 - p_6 > 0 \qquad \text{(}A \text{ beats } B \text{ by NPR)} \tag{7}$$

$$p_1 + p_3 - p_5 - p_6 > 0 \qquad \text{(}A \text{ beats } C \text{ by NPR).} \tag{8}$$

Hyperplane H1 is then defined from (5) with

$$\text{H1: } p_1 + p_2 - p_3 - p_5 = 0. \tag{9}$$

This hyperplane identifies voting situations for which there is a PR tie between Candidates A and B, and it is used to partition Δ^5 into two subspace regions for which $p_1 + p_2 - p_3 - p_5 > 0$ (with A beats B by PR) and $p_1 + p_2 - p_3 - p_5 < 0$ (with B beat A by PR). The partition subspace for which B beat A by PR is discarded along with all vertices that are included in it, and we then use a procedure in Cervone et al (2005) to determine all of the new vertices that are created when H1 cuts some of the edges of Δ^5 to form new faces in the remaining subspace in which A beats B by PR.

Hyperplane H2 is then defined from (6) in the same manner to determine the voting situations for which there is a PR tie between A and C, with

$$\text{H2}: p_1 + p_2 - p_4 - p_6 = 0. \tag{10}$$

Then, H2 is used to partition the simplex partition component with A beats B by PR into the subspace in which both A beats B by PR and A beats C by PR (with $p_1 + p_2 - p_4 - p_6 > 0$) and the subspace in which both A beats B by PR and C beats A by PR (with $p_1 + p_2 - p_4 - p_6 < 0$) This second subspace is discarded along with all vertices that are included in it, and we then determine all of the new vertices that are created when H2 cuts some edges of the simplex partition component with A beats B by PR to form new faces in the remaining subspace in which A beats both B and C by PR.

The partitioning process continues in the same fashion by using hyperplanes H3 from (7) and H4 from (8), to find the polyhedron that remains for the subset of voting situations from $\boldsymbol{P}$ for which Candidate A beats both B and C by both PR and NPR. The final polyhedron that remains from this partitioning has 18 vertices v_i^1, with:

$v_1^1 = \left[\frac{1}{3}, 0, 0, \frac{1}{3}, \frac{1}{3}, 0\right]$	$v_7^1 = \left[\frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}, 0\right]$	$v_{13}^1 = \left[\frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0\right]$
$v_2^1 = [0,1,0,0,0,0]$	$v_8^1 = \left[0, \frac{1}{2}, \frac{1}{4}, 0, \frac{1}{4}, 0\right]$	$v_{14}^1 = \left[0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0\right]$
$v_3^1 = [1,0,0,0,0,0]$	$v_9^1 = \left[\frac{1}{3}, \frac{1}{6}, 0, \frac{1}{6}, 0, \frac{1}{3}\right]$	$v_{15}^1 = \left[\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{6}, 0, \frac{1}{6}\right]$
$v_4^1 = \left[0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right]$	$v_{10}^1 = \left[0, \frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}\right]$	$v_{16}^1 = \left[\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, 0, \frac{1}{3}, 0\right]$
$v_5^1 = \left[\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}, 0, 0\right]$	$v_{11}^1 = \left[0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0\right]$	$v_{17}^1 = \left[\frac{1}{3}, \frac{1}{3}, 0, 0, 0, \frac{1}{3}\right]$
$v_6^1 = \left[\frac{1}{2}, 0, 0, \frac{1}{4}, 0, \frac{1}{4}\right]$	$v_{12}^1 = \left[\frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0\right]$	$v_{18}^1 = \left[0, \frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, 0\right]$

The volume of the polyhedron that is bounded by these vertices is:

$$Volume(IAC, A = PR = NPR) = \frac{113\sqrt{6}}{77760}. \tag{11}$$

The symmetry of IAC with respect to the candidates along with (1) and (11) leads to a representation for the probability that all $Rule(\lambda)$ elect the same winner that is denoted by $Probability(IAC, PR = NPR)$ with

$$Probability(IAC, PR = NPR) = \frac{3Volume(IAC, A=PR=NPR)}{Volume(IAC)} = \frac{113}{216} \approx .523. \quad (12)$$

The probability in (12) clearly shows that there is a significant likelihood that all $Rule(\lambda)$ elect the same winner.[2] As mentioned above, the value in (12) verifies a known result from Gehrlein (2002). We chose not to use that source as a starting point here in order to illustrate how the procedure from Cervone et al (2005) works on the simplest possible problem that we are considering, and because it would still be necessary to work from that starting point back to the volumes that are used in the remainder of the study. It then follows from definitions that the MIAC subspace volume is given by

$$Volume(MIAC) = Volume(IAC) - 3Volume(IAC, A = PR = NPR) = \frac{103\sqrt{6}}{25920}. \quad (13)$$

4. Condorcet Efficiency with MIAC

In order to develop a representation for the Condorcet Efficiency of $Rule(\lambda)$ with MIAC, we first need to determine the volume of the IAC subspace for which the same winner is elected by both PR and NPR when a CW exists. Two subspace volumes are required to obtain that. The first volume is $Volume(IAC, A = CW = PR = NPR)$ in which Candidate A is the CW that is also elected by every $Rule(\lambda)$. The partitioning process that was described above is used to find the 29 v_i^2 vertices of the resulting polyhedron:

$$v_1^2 = \left[0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0\right] \quad v_{11}^2 = \left[\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, 0, 0, \frac{1}{3}\right] \quad v_{21}^2 = [0, 1, 0, 0, 0, 0]$$
$$v_2^2 = \left[\frac{1}{2}, 0, 0, \frac{1}{4}, 0, \frac{1}{4}\right] \quad v_{12}^2 = \left[0, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}, \frac{1}{6}\right] \quad v_{22}^2 = \left[\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}, 0, 0\right]$$
$$v_3^2 = \left[0, \frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, 0\right] \quad v_{13}^2 = \left[\frac{1}{3}, 0, \frac{1}{6}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}\right] \quad v_{23}^2 = \left[\frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}, 0\right]$$
$$v_4^2 = \left[\frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0\right] \quad v_{14}^2 = \left[\frac{1}{2}, 0, 0, \frac{1}{4}, \frac{1}{4}, 0\right] \quad v_{24}^2 = \left[0, \frac{1}{2}, \frac{1}{4}, 0, 0, \frac{1}{4}\right]$$
$$v_5^2 = \left[\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{6}, 0, \frac{1}{6}\right] \quad v_{15}^2 = \left[\frac{1}{3}, \frac{1}{6}, 0, \frac{1}{6}, \frac{1}{3}, 0\right] \quad v_{25}^2 = \left[0, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}, 0, \frac{1}{6}\right]$$
$$v_6^2 = \left[0, \frac{1}{2}, \frac{1}{4}, 0, \frac{1}{4}, 0\right] \quad v_{16}^2 = \left[\frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0\right] \quad v_{26}^2 = \left[\frac{1}{6}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}, 0, \frac{1}{3}\right]$$
$$v_7^2 = [1, 0, 0, 0, 0, 0] \quad v_{17}^2 = \left[\frac{1}{3}, \frac{1}{6}, 0, \frac{1}{6}, 0, \frac{1}{3}\right] \quad v_{27}^2 = \left[\frac{3}{8}, 0, \frac{1}{8}, \frac{1}{4}, \frac{1}{4}, 0\right]$$
$$v_8^2 = \left[0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0\right] \quad v_{18}^2 = \left[\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right] \quad v_{28}^2 = \left[\frac{1}{4}, \frac{1}{6}, \frac{1}{12}, \frac{1}{6}, \frac{1}{3}, 0\right]$$
$$v_9^2 = \left[\frac{1}{3}, \frac{1}{3}, 0, 0, 0, \frac{1}{3}\right] \quad v_{19}^2 = \left[0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right] \quad v_{29}^2 = \left[\frac{1}{3}, 0, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, 0\right]$$
$$v_{10}^2 = \left[0, \frac{3}{8}, \frac{1}{4}, \frac{1}{8}, 0, \frac{1}{4}\right] \quad v_{20}^2 = \left[\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, 0, \frac{1}{3}, 0\right]$$

The volume of the polyhedron defined by these vertices is given by:

[2] It is shown in Gehrlein and Fishburn (1983) that this limiting probability with the well-known assumption of Impartial Culture (IC) with complete independence among voters' preferences has $Probability(IC, PR = NPR) \approx .535$, so the degree of dependence that IAC introduces among voters' preferences has a remarkably small impact on this probability.

$$Volume(IAC, A = CW = PR = NPR) = \frac{3437\sqrt{6}}{2488320}. \tag{14}$$

The second additional volume that is required is $Volume(IAC, A = CW, B = PR = NPR)$ in which Candidate A is the CW while B is elected by all $Rule(\lambda)$. The same partitioning process that was described above is used to find the 13 v_i^3 vertices of the resulting polyhedron:

$$v_1^3 = \left[0, \frac{1}{2}, \frac{1}{2}, 0,0,0\right] \quad v_6^3 = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0,0, \frac{1}{4}\right] \quad v_{11}^3 = \left[\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{12}, 0, \frac{1}{4}\right]$$

$$v_2^3 = \left[\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right] \quad v_7^3 = \left[\frac{1}{4}, \frac{1}{8}, \frac{1}{4}, \frac{1}{8}, 0, \frac{1}{4}\right] \quad v_{12}^3 = \left[\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, 0, \frac{1}{6}\right]$$

$$v_3^3 = \left[\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{6}, 0, \frac{1}{6}\right] \quad v_8^3 = \left[0, \frac{1}{2}, \frac{1}{4}, 0, \frac{1}{4}, 0\right] \quad v_{13}^3 = \left[\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, 0, \frac{1}{6}, \frac{1}{6}\right]$$

$$v_4^3 = \left[\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, 0,0, \frac{1}{4}\right] \quad v_9^3 = \left[\frac{1}{2}, 0, \frac{1}{2}, 0,0,0\right]$$

$$v_5^3 = \left[\frac{1}{12}, \frac{1}{3}, \frac{1}{4}, 0, \frac{1}{6}, \frac{1}{6}\right] \quad v_{10}^3 = \left[\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, 0, \frac{1}{3}, 0\right]$$

The volume of the polyhedron defined by these vertices is given by:

$$Volume(IAC, A = CW, B = PR = NPR) = \frac{19\sqrt{6}}{1244160}. \tag{15}$$

Let $Volume(IAC, A = CW, PR \neq NPR)$ denote the volume of the space of MIAC that has Candidate A as the CW. By definition and MIAC symmetry with respect to candidates, with (2), (14) and (15)

$$Volume(IAC, A = CW, PR \neq NPR) = Volume(IAC, A = CW) - Volume(IAC, A = CW = PR = NPR) - 2Volume(IAC, A = CW, B = PR = NPR) = \frac{989\sqrt{6}}{829440}. \tag{16}$$

The subspace of $Volume(IAC, A = CW, PR \neq NPR)$ for which Candidate A is both the CW and the winner by $Rule(\lambda)$ has $Volume(IAC, A = CW = Rule(\lambda), PR \neq NPR)$, and

$$Volume(IAC, A = CW = Rule(\lambda), PR \neq NPR) = Volume\big(IAC, A = CW = Rule(\lambda)\big) - Volume(IAC, A = CW = PR = NPR). \tag{17}$$

We note for clarity that no adjustment is made to $Volume\big(IAC, A = CW = Rule(\lambda)\big)$ in (17) to account for $Volume(IAC, A = CW, B = PR = NPR)$ as we did in (16). This follows from the fact that if B wins by both PR and NPR then B must be the winner for all $Rule(\lambda)$, not A; so no such voting situations are included in $Volume\big(IAC, A = CW = Rule(\lambda)\big)$.

The Condorcet Efficiency of $Rule(\lambda)$ with MIAC is obtained from

$$CE\big(MIAC, Rule(\lambda)\big) = \frac{Volume(IAC, A=CW=Rule(\lambda), PR \neq NPR)}{Volume(IAC, A=CW, PR \neq NPR)}. \tag{18}$$

Using (18) with (3), (14), (16) and (17):

$$CE\big(MIAC, Rule(\lambda)\big) = \frac{128\lambda^7 + 448\lambda^6 - 2397\lambda^5 - 2828\lambda^4 + 10796\lambda^3 - 1514\lambda^2 - 9167\lambda + 4550}{2967(\lambda+1)(\lambda-2)(\lambda-1)^3}, \tag{19}$$

for $0 \leq \lambda \leq 1/2$.

$$= \frac{768\lambda^8+23936\lambda^7-56745\lambda^6+4236\lambda^5+52903\lambda^4-26706\lambda^3+4652\lambda^2-504\lambda+32}{2967\lambda^3(\lambda+1)(\lambda-2)(1-3\lambda)},$$

for $1/2 \leq \lambda \leq 1$.

The representation in (19) is used to obtain values of $CE(MIAC, Rule(\lambda))$ for each $\lambda = 0(.05)1$ and the results are listed in Table 1.[3]

5. Conclusion

A comparison of $CE(IAC, Rule(\lambda))$ and $CE(MIAC, Rule(\lambda))$ shows that the same value of λ maximizes both of the cases of IAC and MIAC. However, the results in Table 1 clearly indicate that the selection of λ for $Rule(\lambda)$ has a much greater impact on the election outcome on voting situations for which the selection of λ makes a difference than might have been concluded from earlier studies. This is particularly true for values of λ that are larger than the most efficient λ.

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[3] It is not practically possible to make comparisons of $CE(MIAC, Rule(\lambda))$ to similar results with IC, as we did above for $Probability(IC, PR = NPR)$, since this now would for example require computations on six-dimensions to simultaneously make *A* the CW, PR Winner and NPR Winner over both *B* and *C* with IC. While it is possible to perform computations on associated IC positive-orthant probabilities for up to five dimensions, any extension to six dimensions is quite intractable.