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Group contests and technologies

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Abstract

We study the role of returns of scale of the technology on the characterization of the Nash equilibrium in group contests. In the general setting of group contests, we first show that there exists only one type of equilibrium, for given each technology, when the technology is with constant or decreasing returns to scale. However, when the technology exhibits increasing returns to scale, we find that there may exist various types of equilibria according to the parameter values in the model.

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1 Introduction

In the real world, people exert a great deal of effort to put aside their rivalries and win a prize. This includes electoral competition, litigation, rent-seeking, patent races, sporting competition, competition in entrance examination, military combat, etc. These situations are referred to as *contests*, and there is a large and growing literature on the theory of contests and its application from the seminal contributions by Tullock (1980).[1]

Among the literature on contests, Pérez-Castrillo and Verdier (1992), Baye et al. (1994), and Cornes and Hartley (2005) explore the existence and uniqueness of the Nash equilibrium in the Tullock contest in which $n(\geq 2)$ individual players compete against each other to win a prize and player i's contest success function is defined as $p_i = x_i^r / \sum_{j=1}^n x_j^r$ where x_i is the player i's effort level and $r(> 0)$ characterizes the returns to scale of the technology (each player's production function) of the contest. Specifically, Pérez-Castrillo and Verdier (1992) find that, when the technology exhibits constant ($r = 1$) or decreasing returns of scale ($r < 1$) or the restricted increasing returns of scale ($r \leq \frac{n}{n-1}$), there exists a unique symmetric Nash equilibrium, while there exist multiple asymmetric equilibria when the technology is with the increasing returns of scale ($r > \frac{n}{n-1}$). In the simplified Tullock contest with $n = 2$, Shogren and Baik (1991) show that when $r = 3$, there is no Nash equilibrium. Baye et al. (1994) further find that there exists the symmetric mixed-strategy equilibrium when $r > 2$ as well as the symmetric pure-strategy equilibrium existing when $r \leq 2$. Cornes and Hartley (2005) consider the case in which the players are heterogeneous, and find that the increasing returns of the scale of the technology complicates payoff-maximization problems for the players in the contest and hence typically there exist multiple equilibria.

Note that the above studies investigate the impact of the returns to scale of the technology on the characterization of the Nash equilibrium in the contest among individual players. On the other hand, Baik (2008) and Epstein and Mealem (2009) consider the *group contest* in which each group consists of several players (group members) and the groups compete against each other to win a group-specific public-good prize, given different returns to scale of the technology, respectively. Baik (2008) assumes the linear technology for players' production function, i.e., constant returns of scale of the technology, and examines the Nash equilibrium in the group contest where the players within each group may have different valuations on the prize. He shows that full free-riding exists at equilibrium, i.e., the highest-valuation players in each group expend efforts and the others in that group do nothing. Unlike Baik (2008), Epstein and Mealem (2009) adopt decreasing returns of scale in the players' production function in the group contest, and obtain different results. In their equilibrium, all the players in each group exert efforts proportionally to their valuations and thus free-riding is decreased relative to in Baik (2008). Then, a natural question to ask should be about the characterization of the equilibrium in the group contest when the technology exhibits increasing returns to scale. However, this question has not yet been investigated. In this paper, we try to answer this.

The paper proceeds as follows. In Section 2, we develop our model and analyze it in Section 3. Finally, Section 4 concludes.

[1] For details, see Corchón (2007), Garfinkel and Skaperdas (2007), and Konrad (2009).

2 The model

We adopt the group contest model of Baik (2008). Let us consider a contest in which n groups vie for winning a group-specific public-good prize, where $n \geq 2$. Group i consists of m_i risk-neutral players who exert effort to win the prize, where $m_i \geq 2$. We denote the valuation player k in group i puts on the prize by v_{ik} and assume the following.

Assumption 1. $v_{i1} \geq v_{i2} \geq \cdots \geq v_{im_i} > 0 \; \forall \; i = 1, \ldots, n.$

Let e_{ik} represent the nonnegative effort level exerted by player k in group i. Through an increasing function $f(\cdot)$, the effort of player k in group i is transformed into his individual performance in the contest. As in Szidarovszky and Okuguchi (1997), we name $f(\cdot)$ players' 'production function for the contest'. Denoting the performance of player k in group i by x_{ik}, we define it as follows:

$$x_{ik} := f(e_{ik}) = e_{ik}^{\alpha} \tag{1}$$

where $\alpha > 0$. Each player's performance increases with his effort level, i.e., $\frac{dx_{ik}}{de_{ik}} > 0$. The rate of increase in the individual performance depends on the value of α. If $\alpha = 1$, the production function exhibits constant returns to scale (CRS): each player's performance increases with its effort level at a constant rate. If $\alpha < 1$, it exhibits decreasing returns to scale (DRS) and the performance increases at a decreasing rate, while, if $\alpha > 1$, it exhibits increasing returns to scale (IRS) and the performance increases at an increasing rate.

We assume that all the players have a common effort-cost function with a constant marginal cost: $c(e_{ik}) = e_{ik}$. Although we assume that the cost function is linear and the players' marginal costs for expending effort are identical, these are not a restriction.[2]

Individual performances of the players within each group are mapped onto the performance of that group through a group impact function $F(\cdot)$. Denoting the performance of group i by X_i, we define it as follows:

$$X_i := F(x_{i1}, x_{i2}, \cdots, x_{im_i}) = \sum_{l=1}^{m_i} x_{il}, \tag{2}$$

which implies that individual performances of the players within a group are perfect substitutes and the contest becomes the perfect-substitute group contest in Baik (2008).[3]

Let p_i denote the probability that group i wins the prize. The winning probability of each group depends on the performance of that group and the other groups' performances

[2]Consider an alternative model in which player k in group i has its valuation V_{ik} and effort-cost function $c_{ik} \cdot e_{ik}^{\beta}$ where c_{ik} and $\beta > 0$. If we reformulate this model such that player k in group i has the valuation $v_{ik} := \frac{V_{ik}}{c_{ik}}$ and the effort-cost function e_{ik}^{β}, then the reformulated model with $\beta = 1$ is isomorphic to our original model. Besides, if we redefine $y_{ik} := e_{ik}^{\beta}$ as the effort-cost function and accordingly $x_{ik} := y_{ik}^{\frac{\alpha}{\beta}}$ as the production function for the contest as in Epstein and Mealem (2009), then the above-reformulated model is transformed into the one in which each player exerts its effort y_{ik} and has the production function y_{ik}^{γ} and the effort-cost function y_{ik}, where $\gamma = \frac{\alpha}{\beta}$. This transformed model is also isomorphic to the original one.

[3]According to the functional form of $F(\cdot)$, there appear different types of group contests. For example, if $F(\cdot) = \min\{x_{i1}, x_{i2}, ...x_{im_i}\}$, it becomes the weakest-link group contest in Lee (2012). And, if $F(\cdot) = \max\{x_{i1}, x_{i2}, ...x_{im_i}\}$, it becomes the best-shot group contest in Chowdhury et al. (2013).

as well. That is, the contest success function for group i is defined as follows:

$$p_i(X_1, X_2, ..., X_n), \tag{3}$$

which satisfies the regularity conditions for the contest success function. Assumption 2 specifies them.

Assumption 2. $0 \leq p_i \leq 1$, $\sum_{i=1}^{n} p_i = 1$, $p_i(0, \ldots, 0) = \frac{1}{n}$, $\frac{\partial p_i}{\partial X_i} \geq 0$, $\frac{\partial^2 p_i}{\partial X_i^2} \leq 0$, $\frac{\partial p_i}{\partial X_j} \leq 0$, $\frac{\partial^2 p_i}{\partial X_j^2} \geq 0$, $\frac{\partial p_i}{\partial X_i} > 0$ and $\frac{\partial^2 p_i}{\partial X_i^2} < 0$ for some $X_j > 0$, $\frac{\partial p_i}{\partial X_j} < 0$ and $\frac{\partial^2 p_i}{\partial X_j^2} > 0$ for $X_i > 0$, where $i \neq j$.

Let π_{ik} represent the payoff for player k in group i. The payoff function for player k in group i is then defined as follows:

$$\pi_{ik} = v_{ik} p_i(X_1, ..., X_n) - e_{ik}. \tag{4}$$

Note that $X_i = F(x_{i1}, x_{i2}, \ldots, x_{im_i}) = F(f(e_{i1}), f(e_{i2}), \ldots, f(e_{im_i})) = F(e_{i1}^{\alpha}, e_{i2}^{\alpha}, \ldots, e_{im_i}^{\alpha}) = \sum_{l=1}^{m_i} e_{il}^{\alpha}$.

We assume that all the players in the contest choose their effort levels independently and simultaneously. All of the above is common knowledge among the players, and we employ Nash equilibrium as our solution concept.

3 Analysis of the model

3.1 $\alpha = 1$: the CRS production function

When $\alpha = 1$, group i's performance is defined as $X_i = \sum_{l=1}^{m_i} x_{il} = \sum_{l=1}^{m_i} e_{il}$. Then player k in group i chooses e_{ik} that maximizes its expected payoff

$$\pi_{ik} = v_{ik} p_i(X_i, X_{-i}) - e_{ik}, \tag{5}$$

where $X_{-i} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$.

Let e_{ik}^b denote the "imaginary" best-response of player k in group i, which means the best response of player k when he is a unique player in group i, given effort levels of all the players in the other groups. Namely, e_{ik}^b is the effort level that maximizes

$$\pi_{ik}^b = v_{ik} p_i(e_{ik}, X_{-i}) - e_{ik}. \tag{6}$$

Thus, e_{ik}^b satisfies the following first-order condition for maximizing π_{ik}^b:

$$v_{ik} \frac{\partial p_i}{\partial e_{ik}} - 1 \leq 0.[4] \tag{7}$$

[4]The second-order condition for maximizing is satisfied.

By Assumption 1 and 2, the first-order condition means that

$$e_{i1}^b(X_{-i}) \geq e_{i2}^b(X_{-i}) \geq \cdots \geq e_{im_i}^b(X_{-i}) \text{ for all } X_{-i}. \tag{8}$$

Now let e_{ik}^B denote the best-response of player k in group i to the effort levels of all the other players in the contest. By using e_{ik}^b defined above, we have the following best response of player k in group i:

$$e_{ik}^B(e_{-ik}, X_{-i}) = \begin{cases} e_{ik}^b(X_{-i}) - \sum_{l \neq k}^{m_i} e_{il} & \text{for} \quad e_{ik}^b(X_{-i}) > \sum_{l \neq k}^{m_i} e_{il} \\ 0 & \text{for} \quad e_{ik}^b(X_{-i}) \leq \sum_{l \neq k}^{m_i} e_{il}, \end{cases} \tag{9}$$

where $e_{-ik} = (e_{i1}, \ldots, e_{ik-1}, e_{ik+1}, \ldots, e_{im_i})$.

From the best responses of the players in the contest, we obtain the pure-strategy Nash equilibrium of the game. Letting a $(\sum_{j=1}^{n} m_j)-$tuple vector of effort levels, $(e_{11}^*, \ldots, e_{1m_1}^*, \ldots, e_{n1}^*, \ldots, e_{nm_n}^*)$, represent a Nash equilibrium of the game, Proposition 1 describes the equilibrium, which is found in Baik (2008).

Proposition 1 *(Baik, 2008) The following strategy profiles constitute the Nash equilibria of the game.*

(a) For group i with $v_{i1} > v_{i2}$, its players play the strategies: $e_{i1}^* = e_{i1}^b(X_{-i}^*)$ and $e_{il}^* = 0$ for $l = 2, \ldots, m_i$.

(b) For group j with $v_{j1} = v_{jt} > v_{jt+1}$ for some t, its players use strategies such that $\sum_{l=1}^{t} e_{jl}^* = e_{j1}^b(X_{-j}^*)$ and $e_{jl}^* = 0$ for $l = t+1, \ldots, m_j$, where $2 \leq t \leq m_j$.

Proposition 1 implies that, in equilibrium, the highest-valuation player(s) in each group would be active, i.e., exert positive effort, and the rest of the players in that group free ride on the active player(s).

3.2 $\alpha < 1$: the DRS production function

When $\alpha < 1$, the performance of group i is defined as $X_i = \sum_{l=1}^{m_i} x_{il} = \sum_{l=1}^{m_i} e_{il}^{\alpha}$, and player k in group i seeks to maximize its expected payoff

$$\pi_{ik} = v_{ik} p_i(X_i, X_{-i}) - e_{ik} \tag{10}$$

with respect to e_{ik}. Then e_{ik}^B satisfies the first-order condition for maximizing π_{ik} is:

$$v_{ik} \frac{\partial p_i}{\partial X_i} \alpha e_{ik}^{\alpha-1} - 1 = 0 \Leftrightarrow \frac{\partial p_i}{\partial X_i} \alpha = \frac{e_{ik}^{1-\alpha}}{v_{ik}}. \tag{11}$$

The second-order condition for maximizing π_{ik} is:

$$v_{ik} \alpha \left(\frac{\partial^2 p_i}{\partial X_i^2} \alpha e_{ik}^{2(\alpha-1)} + (\alpha - 1) \frac{\partial p_i}{\partial X_i} e_{ik}^{\alpha-2} \right) < 0 \tag{12}$$

, which is satisfied for any $e_{ik} > 0$ since the two terms within the bracket have negative signs by Assumption 2 and $\alpha < 1$. This means that the interior maximizer e_{ik}^B satisfying the first-order condition (11) is the global maximizer.

Let a $(\sum_{j=1}^{n} m_j)-$tuple vector of positive effort levels, $(e_{11}^*, \ldots, e_{1m_1}^*, \ldots, e_{n1}^*, \ldots, e_{nm_n}^*)$, represent a Nash equilibrium. Then, at the equilibrium, the first-order condition (11) must be satisfied for all $i = 1, \ldots, n$ and $k = 1, \ldots, m_i$, which implies that

$$\frac{e_{i1}^{*\ 1-\alpha}}{v_{i1}} = \frac{e_{i2}^{*\ 1-\alpha}}{v_{i2}} = \cdots = \frac{e_{im_i}^{*\ \ 1-\alpha}}{v_{im_i}} > 0 \ \forall\ i = 1, 2, \ldots, n. \tag{13}$$

By Assumption 1, we then have the following:

$$e_{i1}^* \geq e_{i2}^* \geq \cdots \geq e_{im_i}^* > 0 \ \forall\ i = 1, 2, \ldots, n. \tag{14}$$

Proposition 2 summarize the equilibrium of the game, which are specified in Epstein and Mealem (2009).

Proposition 2 *(Epstein and Mealem, 2009) The following strategy profile constitutes the Nash equilibria of the game.*

(a) Each player plays the strategy: $e_{ik}^* = e_{ik}^B(e_{-ik}^*, X_{-i}^*) > 0$.

(b) $e_{ik}^* \geq e_{ik+1}^*$ *where* $1 \leq k \leq m_i - 1$.

Proposition 2 means that all the players exert positive efforts in equilibrium and within each group, the higher valuation the player has, the greater effort he exerts. There doesn't exist the full free-ride problem shown in the previous case where $\alpha = 1$.

3.3 $\alpha > 1$: the IRS production function

The performance of group i is defined as $X_i = \sum_{l=1}^{m_i} x_{il} = \sum_{l=1}^{m_i} e_{il}^{\alpha}$, and player k in group i seeks to maximize its expected payoff

$$\pi_{ik} = v_{ik} p_i(X_i, X_{-i}) - e_{ik} \tag{15}$$

with respect to e_{ik}. Then, the first-order condition for maximizing π_{ik} is given as follows and e_{ik}^B satisfies this:

$$v_{ik} \frac{\partial p_i}{\partial X_i} \alpha e_{ik}^{\alpha-1} - 1 = 0 \Leftrightarrow \frac{\partial p_i}{\partial X_i} \alpha = \frac{1}{v_{ik} e_{ik}^{\alpha-1}}. \tag{16}$$

The second-order condition for the maximization is as follows:

$$v_{ik} \alpha \left(\frac{\partial^2 p_i}{\partial X_i^2} \alpha e_{ik}^{2(\alpha-1)} + (\alpha - 1) \frac{\partial p_i}{\partial X_i} e_{ik}^{\alpha-2} \right) < 0 \tag{17}$$

, which is not necessarily satisfied for any $e_{ik} > 0$ since the first term within the bracket has a negative sign but the second one has a positive sign. The second-order condition is satisfied only under the following condition:

$$\frac{\alpha - 1}{\alpha} \frac{\frac{\partial p_i}{\partial X_i}}{-\frac{\partial^2 p_i}{\partial X_i^2}} < e_{ik}^{\alpha}, \tag{18}$$

which means that the interior maximizer e_{ik}^B, satisfying the first-order condition (16) and the second-order condition (17), is the local maximizer for sure, but not necessarily global maximizer.

Let a $(\sum_{j=1}^{n} m_j)-$tuple vector of positive effort levels, $(e_{11}^*, \ldots, e_{1m_1}^*, \ldots, e_{n1}^*, \ldots, e_{nm_n}^*)$, represent a Nash equilibrium in which the first-order and second-order conditions are satisfied for all $i = 1, ..., n$ and $k = 1, ..., m_i$. Then, at this equilibrium, the first-order condition (16) implies that

$$v_{i1} {e_{i1}^*}^{\alpha-1} = v_{i2} {e_{i2}^*}^{\alpha-1} = \cdots = v_{im_i} {e_{im_i}^*}^{\alpha-1} \ \forall \ i = 1, 2, \ldots, n. \tag{19}$$

By Assumption 1, we then have

$$0 < e_{i1}^* \leq e_{i2}^* \leq \cdots \leq e_{im_i}^* \forall \ i = 1, 2, \ldots, n. \tag{20}$$

Inequalities (20) mean that all the players exert positive efforts in equilibrium and within each group, **the lower valuation** the player has, **the more effort** he exerts. This is opposite to the result in the previous case where $\alpha < 1$. Then, dose this type of equilibrium really exist?

Here we have to note that the second-order condition (17) is satisfied only under the certain condition (18), not for any arbitrary $e_{ik} > 0$. This implies that a strategy profile, satisfying the first-order condition (16) and the second-order condition (18) for all $i = 1, \ldots, n$ and $k = 1, \ldots, m_i$, may not constitute a Nash equilibrium, because the second-order condition (18) does not ensure that each player's strategy, i.e., its interior maximizer in that strategy profile, is the global maximizer. Namely, for a player, its interior solution satisfying the first-order condition may be maximizing its payoff locally, not globally. And, in this case, there may exist an incentive for the player to change its strategy from its interior solution to the corner solution, given the other players' strategies. For this reason, the set of interior solutions satisfying all the first-order conditions may not be a Nash equilibrium, even though they meet all the second-order conditions for the (local) maximum. Hence, when $\alpha > 1$, we must be careful in determining whether a strategy profile obtained from the first-order and second-order conditions constitutes a Nash equilibrium or not. In order to understand this more specifically, we consider the following simple example in which two groups compete against each other, each group consists of two members, and the members within each group have the valuations on the prize, $k \geq 1$ and 1, respectively.[5] We use the Tullock-form contest success function for our specific analysis, i.e., $p_i(X_1, X_2) = \frac{X_i}{X_1 + X_2}$.

[5] I thank the anonymous referee and the associate editor for the insightful suggestion on this example.

3.3.1 The symmetric two-group-two-member case

Using the parameters $n = 2$, $m_1 = m_2 = 2$, $v_{11} = v_{21} = k$, $v_{12} = v_{22} = 1$, and the contest success function $p_i(X_1, X_2) = \frac{X_i}{X_1+X_2}$ for $i = 1, 2$, we obtain the following symmetric interior solutions which satisfy the first-order condition (16) for all players:

$$e_{11}^* = e_{21}^* = \frac{\alpha k}{4(1+k^{\frac{\alpha}{\alpha-1}})} \quad \text{and} \quad e_{12}^* = e_{22}^* = \frac{\alpha k^{\frac{\alpha}{\alpha-1}}}{4(1+k^{\frac{\alpha}{\alpha-1}})}. \tag{21}$$

Note that $e_{i2}^* \geq e_{i1}^* > 0$ because $\alpha > 1$ and $k \geq 1$. The second-order condition (18), required for the above solutions to be interior maximizers, is given:

$$k < \left(\frac{1}{\alpha-1}\right)^{\frac{\alpha-1}{\alpha}}, \tag{22}$$

which requires the sufficient condition $\alpha < 2$ because $k \geq 1$. At the strategy profile $(e_{11}^*, e_{12}^*, e_{21}^*, e_{22}^*)$, the players' expected payoffs are

$$\pi_{11}^* = \pi_{21}^* = \frac{k}{4}\left(2 - \frac{\alpha}{1+k^{\frac{\alpha}{\alpha-1}}}\right) \quad \text{and} \quad \pi_{12}^* = \pi_{22}^* = \frac{1}{4}\left(2 - \frac{\alpha k^{\frac{\alpha}{\alpha-1}}}{1+k^{\frac{\alpha}{\alpha-1}}}\right). \tag{23}$$

We now examine whether the strategy profile $(e_{11}^*, e_{12}^*, e_{21}^*, e_{22}^*)$ constitutes a Nash equilibrium. For that strategy profile to be the Nash equilibrium, there shouldn't be any incentive for any player to deviate from it. In other words, each player shouldn't have any incentive to change its strategy from its interior maximizer in (21) to its boundaries, 0 or ∞, given the other players' strategies. First, we consider the non-deviation condition for player 1 (the high-valuation player) in each group. Denoting $\pi_{i1}^d(e_{i1})$ by the expected payoff for player 1 in group i obtained when he changes its strategy from e_{i1}^* to $e_{i1} = 0$ or ∞, we have

$$\pi_{i1}^d(e_{i1} = 0) = \frac{k^{\frac{2\alpha-1}{\alpha-1}}}{2k^{\frac{\alpha}{\alpha-1}} + 1} \quad \text{and} \quad \pi_{i1}^d(e_{i1} = \infty) = -\infty. \tag{24}$$

Comparing $\pi_{i1}^d(e_{i1} = 0)$ in (24) and π_{i1}^* in (23), we obtain the following non-deviation condition for player 1 in each group:

$$\pi_{i1}^* - \pi_{i1}^d(e_{i1} = 0) \geq 0 \Leftrightarrow k \leq \left(\frac{2-\alpha}{2(\alpha-1)}\right)^{\frac{\alpha-1}{\alpha}}, \tag{25}$$

which requires the sufficient condition $\alpha < \frac{4}{3}$ because $k \geq 1$.

By the same way, we derive the non-deviation condition for player 2 in each group. This is given as follows:

$$\pi_{i2}^* - \pi_{i2}^d(e_{i1} = 0) \geq 0 \Leftrightarrow k \geq \left(\frac{2(\alpha-1)}{2-\alpha}\right)^{\frac{\alpha-1}{\alpha}}, \tag{26}$$

which are trivially satisfied from the sufficient condition $\alpha < \frac{4}{3}$ above.

Combining the second-order condition (22) and the non-deviation conditions (24) and (25), we obtain the following equilibrium condition for the strategy profile $(e^*_{11}, e^*_{12}, e^*_{21}, e^*_{22})$ to constitute a Nash equilibrium:

$$\alpha < \frac{4}{3} \text{ and } k \leq \left(\frac{2-\alpha}{2(\alpha-1)}\right)^{\frac{\alpha-1}{\alpha}}. \tag{27}$$

Lemma 3 summarizes the results for the two-group-two-member case.

Lemma 3 *(The symmetric two-group-two-member case) When* $\alpha < \frac{4}{3}$ *and* $k \leq \left(\frac{2-\alpha}{2(\alpha-1)}\right)^{\frac{\alpha-1}{\alpha}}$*, the following strategy profile constitutes the Nash equilibria of the game.*

(a) $\left(e^*_{11} = \frac{\alpha k}{4(1+k^{\frac{\alpha}{\alpha-1}})}, e^*_{12} = \frac{\alpha k^{\frac{\alpha}{\alpha-1}}}{4(1+k^{\frac{\alpha}{\alpha-1}})}, e^*_{21} = \frac{\alpha k}{4(1+k^{\frac{\alpha}{\alpha-1}})}, e^*_{22} = \frac{\alpha k^{\frac{\alpha}{\alpha-1}}}{4(1+k^{\frac{\alpha}{\alpha-1}})}\right)$.

(b) $e^*_{11} = e^*_{21} \leq e^*_{12} = e^*_{22}$*: the low-valuation players exert more efforts.*

Note that, at this equilibrium, all the players exert positive efforts and thus there is no free-riding issue within each group. Along with this equilibrium, we guess that there exists another type of equilibrium in which only a player in each group puts positive effort and the other in that group puts nothing, i.e., the full free-riding problem occurs. Since, in our model, each group's performance is defined as the sum of individual performances of the players within that group, which implies that the individual performances are perfect substitutes within the group, the low-valuation player in each group is likely to free ride on the high-valuation player in that group. Thus, we predict the existence of the Nash equilibrium in which only the high-valuation player in each group puts some positive effort and the low-valuation player does nothing. Furthermore, we also expect the existence of other possible equilibria in which the active player in each group is not necessarily the high-valuation player, depending on the values of the parameters. To investigate the existence of these equilibria, we consider the following numerical examples.

3.3.2 Numerical examples and another type of equilibria

Example 1. $n = 2$, $m_1 = m_2 = 2$, $k = 1.1$, $\alpha = 1.2$
Given these parameters, the equilibrium conditions in (27) are met and, therefore, there exists the Nash equilibrium presented in Lemma 3. Putting the parameters into (a) in Lemma 3, we get the following symmetric numerical solutions satisfying the first-order and the second-order conditions for maximizing each players' payoffs:

$$(e^*_{11} = 0.119066476, e^*_{12} = 0.19175775, e^*_{21} = 0.119066476, e^*_{22} = 0.19175775).$$

The numerical plots of each player's expected payoff for given the other players' effort levels above, e.g., the graph $\pi_{11}(e_{11})$ for $e_{12} = 0.19175775$, $e_{21} = 0.119066476$, and $e_{22} = 0.19175775$, reveal that $e_{i1} = 0.119066476$ and $e_{i2} = 0.19175775$ give each players the highest expected payoffs, respectively, i.e., they are genuinely the global maximizers. Therefore, the above player's effort levels obtained from the first-order conditions constitutes a Nash equilibrium. Again, at this equilibrium, the low-valuation player in each group exerts more than the high-valuation player in that group.

In addition to this equilibrium, we have checked the existence of another type of equilibria in which only a player in each group puts positive effort and the other in that group puts nothing. Actually, in this example, those equilibria exist as well:

$$(e^*_{11} = 0.33, e^*_{12} = 0, e^*_{21} = 0.33, e^*_{22} = 0),$$

$$(e^*_{11} = 0, e^*_{12} = 0.30, e^*_{21} = 0, e^*_{22} = 0.30),$$

$$(e^*_{11} = 0.328923166, e^*_{12} = 0, e^*_{21} = 0, e^*_{22} = 0.29902106),$$

$$(e^*_{11} = 0, e^*_{12} = 0.29902106, e^*_{21} = 0.328923166, e^*_{22} = 0).$$

Example 2. $n = 2$, $m_1 = m_2 = 2$, $k = 1.25$, $\alpha = 1.2$
Given these parameters, the equilibrium conditions in (27) are not satisfied and, thus, the equilibrium in Lemma 3 does not exist. Specifically, putting these parameters into (a) in Lemma 3, we have the following symmetric numerical solutions satisfying the first-order and the second-order conditions for maximizing each players' payoffs:

$$(e^*_{11} = 0.077886517, e^*_{12} = 0.237690786, e^*_{21} = 0.077886517, e^*_{22} = 0.237690786).$$

At this strategy profile, the high-valuation player in each group obtains its expected payoff $\pi_{i1} = 0.5471134831$, while he gets higher expected payoff $\pi^d_{i1} = 0.5525731340$ when he reduces her effort level from $e^*_{i1} = 0.077886517$ to $e_{i1} = 0$. This means that $e^*_{i1} = 0.077886517$ results in the local maximum for given the other players' effort levels above. In fact, the numerical plot of his expected payoff reveals that $e^*_{i1} = 0.077886517$ is only a local maximizer and the global maximum is obtained at $e_{i1} = 0$. Therefore, the above players' effort levels computed from the first-order and the second conditions is not a Nash equilibrium. However, as in Example 1, we find other equilibria in which only a player within each group exerts effort and the other does noting:

$$(e^*_{11} = 0.375, e^*_{12} = 0, e^*_{21} = 0.375, e^*_{22} = 0),$$

$$(e^*_{11} = 0, e^*_{12} = 0.3, e^*_{21} = 0, e^*_{22} = 0.3),$$

$$(e^*_{11} = 0.368357461, e^*_{12} = 0, e^*_{21} = 0, e^*_{22} = 0.294685969),$$

$$(e^*_{11} = 0, e^*_{12} = 0.294685969, e^*_{21} = 0.368357461, e^*_{22} = 0).$$

Example 3. $n = 2$, $m_1 = m_2 = 2$, $k = 1.75$, $\alpha = 1.2$
Given these parameters, the equilibrium conditions in (27) are not satisfied. Thus, the equilibrium in Lemma 3 does not exist. In this example, we find that there exists only a unique symmetric Nash equilibrium where the high-valuation players in each group exert

efforts and the low-valuation players free ride:

$$(e_{11}^* = 0.525, e_{12}^* = 0, e_{21}^* = 0.525, e_{22}^* = 0).$$

So far we have analyzed the symmetric two-group-two-member case and considered its several numerical examples. Through this exercise, we have gained a basic understanding of the existence and the structure of the Nash equilibrium in the group contest with the IRS production function, although our analysis does not provide complete information about the equilibrium in the general setting with $n(\geq 2)$ asymmetric groups. Proposition 4 summarizes it.

Proposition 4 *When $\alpha > 1$, the following strategy profile may constitute the Nash equilibria of the game.*

- *The equilibrium derived from the first-order conditions for maximizing each players' payoffs*
  - *(a) Each player plays the strategy: $e_{ik}^* = e_{ik}^B(e_{-ik}^*, X_{-i}^*) > 0$.*
  - *(b) $e_{ik}^* \leq e_{ik+1}^*$ where $1 \leq k \leq m_i - 1$.*
- *The equilibrium in which the highest-valuation player in each group exerts positive effort and the others in that group free ride on the active player*
  - *(a) Each players play the strategies: $e_{i1}^* = e_{i1}^b(X_{-i}^*)$ and $e_{il}^* = 0$ for $l = 2, \ldots, m_i$.*
  - *(b) $(e_{11}^*, 0, \ldots, 0, e_{21}^*, 0, \ldots, 0, \cdots, e_{n1}^*, 0, \ldots, 0)$.*
- *Equilibrium in which the k_ith-highest-valuation player in each group i exerts positive effort and the others in that group free ride on the active player, other than $k_1 = k_2 = \cdots = k_n = 1$*
  - *(a) Each players play the strategies: $e_{ik_i}^* = e_{ik_i}^b(X_{-i}^*)$ and $e_{il}^* = 0$ for $l = 1, 2, \ldots, k_i - 1, k_i + 1, \ldots, m_i$.*
  - *(b) $(0, \ldots, 0, e_{1k_1}^*, 0, \ldots, 0, \cdots, 0, \ldots, 0, e_{nk_n}^*, 0, \ldots, 0)$.*

4 Conclusion

In the group contest, the production function for the contest has an important role in shaping the nature of the equilibrium. The constant returns to scale in the production function brings us the equilibrium in which the full free-ride problem occurs, i.e., only the highest-valuation players in each group are active. On the other hand, the full free-ride problem does not occur in case of the decreasing returns to scale in the production function: the higher valuation the player has, the more effort he exerts. Lastly, when the production function exhibits increasing returns to scale, there exist possibly different types of multiple equilibria in which: 1) the lower valuation the player has, the more effort he exerts, 2) only the highest-valuation players in each group are active, and 3) in each group, there is a unique active player whose valuation is not the highest within that group.

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