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Public debt and aggregate stability with endogenous growth and a state-dependent consumption tax

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Abstract

We analyze a basic endogenous growth model with public debt and a state-dependent consumption tax rate. We show that the balanced budget rule guarantees that the long-run growth path of the economy is unique and saddle point stable unless the tax rate is strongly regressive. In case of a strongly regressive consumption tax rate over a certain range, multiple balanced growth paths and local indeterminacy can arise.

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1. Introduction

It is well known that a balanced government budget gives rise to a unique saddle point stable balanced growth path in models with a representative infinitely lived agent, when endogenous growth is allowed for and when public debt is taken into account explicitly, assuming constant tax rates (see Greiner and Fincke, 2015, and Greiner, 2014). Since tax rates may be state-dependent, the question of arises whether this result also holds when tax rates are state-dependent.

In a recent contribution, Nourry et al. (2013) present an exogenous growth model where the consumption tax rate is a function of the level of consumption that can be progressive or regressive, i.e. pro-cyclical or counter-cyclical in their terminology. The authors demonstrate that a regressive consumption tax rate may give rise to indeterminate equilibrium paths, given a balanced government budget. In this note, we extend their model by allowing for ongoing, i.e. endogenous, growth and by taking into account public debt. Then, we analyze the effects of a balanced government budget implying a constant level of public debt at all times. Note that such a situation is different to the one where public debt is not present at all since the first implies a positive debt to GDP ratio on the transition path. We can show that the balanced budget rule gives rise to a unique saddle point stable balanced growth path, unless the consumption tax rate is strongly regressive. In case of a strongly regressive consumption tax rate, multiple balanced growth paths can occur and locally indeterminate equilibrium paths can arise.[1]

The rest of the paper is organized as follows. In the next section we present the structure of our endogenous growth model allowing for public debt and a state-dependent consumption tax rate. Section 3 analyzes stability of the model and section 4, finally, concludes.

2. The model with a state-dependent consumption tax rate

The structure of our model is basically the same as in Guo and Harrison (2004) except that we allow for a state-dependent tax rate as in Nourry et al. (2013) and we take into account public debt. The economy consists of three sectors: A household sector which receives labour income and income from its saving, a productive sector and the government. First, we describe the household sector.

The household sector

The household sector is represented by one household which maximizes the discounted

[1] That also holds for a rule based policy with a permanently growing level of public debt, as shown in the working paper version (see Greiner and Bondarev, 2014).

stream of utility arising from per-capita consumption, $C(t)$, and from leisure, $L^m - L(t)$, over an infinite time horizon subject to its budget constraint, taking factor prices as given. L^m denotes the maximum available amount of time and $L(t)$ is the actual labor input. The maximization problem of the representative household can be written as

$$\max_{C(t),L(t)} \int_0^\infty e^{-\rho t} \left(\ln C(t) - L(t)^{1+\gamma}/(1+\gamma) \right) dt, \tag{1}$$

subject to

$$(w(t)L(t) + r(t)K(t) + r_p(t)B(t) + \pi(t)) = \dot{W}(t) + (1 + \tau_c(c(t)))C(t) + \delta K(t), \tag{2}$$

with $\rho \in (0,1)$ the household's rate of time preference, $\gamma \geq 0$ is the inverse of the elasticity of labour supply and $\delta \in (0,1)$ is the depreciation rate of capital. The variable $w(t)$ denotes the wage rate and $r(t)$ is the return to capital and $r_p(t)$ is the interest rate on government bonds. $W(t) \equiv B(t) + K(t)$ gives wealth which is equal to public debt, $B(t)$, and capital, $K(t)$, and $\pi(t)$ gives possible profits of the productive sector, the household takes as given in solving its optimization problem. The dot gives the derivative with respect to time. Finally, $0 < \tau_c(c(t)) < \infty$, with $c(t) := C(t)/K(t)$, is the consumption tax rate that depends on consumption relative to capital and will be discussed in detail in the section describing the government sector.[2]

Following Nourry et al. (2013) we assume that the representative household treats the consumption tax rate as a parameter, i.e. it does not take into account that it is a function of consumption relative to capital. The economic justification for that assumption is as follows. Since there is a large number of identical households, consumption of one representative household has only a negligible effect on the tax rate. Therefore, the household does not take this effect into account in solving its optimization problem but treats the tax rate as a constant.

A no-arbitrage condition requires that the return to capital equals the return to government bonds yielding $r_p(t) = r(t) - \delta$. Thus, the budget constraint of the household can be written as

$$\dot{W}(t) = w(t)L(t) + r(t)W(t) + \pi(t) - C(t)(1 + \tau_c(c(t))) - \delta W(t). \tag{3}$$

The current-value Hamiltonian for this optimization problem is written as

$$\mathcal{H} = \ln C(t) - \frac{L(t)^{1+\gamma}}{1+\gamma} + \lambda(t) \left[w(t)L(t) + r(t)W(t) + \pi(t) - \delta W(t) - C(t)(1 + \tau_c(\cdot)) \right], \tag{4}$$

where $\lambda(t)$ is the co-state variable or the shadow price of wealth.

[2]The function $\tau_c(\cdot)$ is assumed to be continuous and at least piecewise C^1.

Necessary optimality conditions are given by

$$
\begin{aligned}
C(t)(1+\tau_c(c(t))) &= w(t)\,L(t)^{-\gamma} && (5)\\
\dot{C}(t) &= C(t)\,r(t) - C(t)\,(\rho+\delta) && (6)
\end{aligned}
$$

If the transversality condition $\lim_{t\to\infty} e^{-\rho t} W(t)/C(t) = 0$ holds, which is fulfilled for a time path on which assets grow at the same rate as consumption, the necessary conditions are also sufficient.

The productive sector

The productive sector is represented by one firm which behaves competitively and which maximizes static profits. The production function of the firm is given by,

$$
Y(t) = K(t)^{1-\alpha}\overline{K(t)}^{\xi} L(t)^{\beta}, \tag{7}
$$

with $(1-\alpha) \in (0,1)$ the capital share, $\beta \in (0,1)$ the labour share and $(1-\alpha)+\beta \leq 1$. The variable $Y(t)$ denotes output and $\overline{K(t)}$ represents the average economy-wide level of capital and we assume constant returns to capital in the economy, i.e. $(1-\alpha)+\xi = 1$, thus following the approach by Romer (1986).

Using $(1-\alpha)+\xi = 1$ and that $K(t) = \overline{K(t)}$ in equilibrium, profit maximization gives

$$
\begin{aligned}
r(t) &= (1-\alpha)L(t)^{\beta} && (8)\\
w(t) &= \beta L(t)^{\beta-1}K(t) && (9)
\end{aligned}
$$

The Government

The government in our economy receives tax revenues from consumption taxation and has revenues from issuing government bonds. As concerns public spending, $C_p(t)$, we again follow Guo and Harrison (2004) and Nourry et al. (2013) and assume that it is a mere waste of resources that is neither productive nor yields utility for the household.

The accounting identity describing the accumulation of public debt is given by:

$$
\dot{B}(t) = r_p(t)B(t) - S(t), \tag{10}
$$

where $S(t)$ is the government surplus exclusive of net interest payments that is given by $S(t) = \tau_c C(t) - C_p(t)$. As mentioned above, the consumption tax rate is a function that depends on consumption relative to capital. For growing economies it is more appropriate to define the tax base relative to capital or output. Otherwise, the tax rate would be permanently growing which would cause analytical problems and that would not be realistic either.

When the consumption tax rate is exogenously determined, it is a constant parameter, $\tau_c' = 0$, and we have the usual case typically analyzed in the economics literature. When the consumption tax rate rises as consumption relative to capital increases, i.e. if $\tau_c' > 0$ holds, we say that the consumption tax rate is progressive and it is regressive in case the tax rate declines as consumption relative to capital increases, i.e. for $\tau_c' < 0$.[3] Further, we distinguish between a weakly regressive consumption tax and a strongly regressive one. The consumption tax is weakly regressive if the marginal tax revenue relative to capital, $T_c(t) = C(t)\tau_c/K(t)$, declines by less than one as consumption relative to capital rises, i.e. if $-1 < (\partial T_c/\partial c) < 0$ holds, and the consumption tax is strongly regressive if the marginal tax revenue declines by more than one, i.e. when $(\partial T_c/\partial c) < -1$. Or, defining $\eta := (\partial T_c/\partial c)(c/T_c)$ as the elasticity of the total consumption tax revenue relative to capital, the consumption tax rate is weakly (strongly) regressive if the elasticity of the total tax revenue relative to capital is larger (smaller) than the negative of the inverse of the consumption tax rate, that is when $\eta > (<) - 1/\tau_c$ holds. This implies that the elasticity of consumption multiplied by one plus the tax rate, relative to capital, is positive (negative) in case of a weakly (strongly) regressive consumption tax rate. It will turn out that this distinction is decisive as regards the number and stability of balanced growth paths.

The inter-temporal budget constraint of the government is fulfilled if

$$\lim_{t\to\infty} e^{-\int_0^t r_p(\mu)d\mu} B(t) = 0 \tag{11}$$

holds, which is the no-Ponzi game condition. Of course, the inter-temporal budget constraint of the government is fulfilled for a balanced government budget that is given for $S(t) = r_p(t)B(t)$ which implies $\dot{B}(t) = 0$.

3. Analysis of the model

Before we analyze our model we give the definition of an equilibrium and of a balanced growth path. An equilibrium allocation for our economy is defined as follows.

Definition 1 *An equilibrium is a sequence of variables $\{C(t), K(t), B(t)\}_{t=0}^{\infty}$ and a sequence of prices $\{w(t), r(t)\}_{t=0}^{\infty}$ such that, given prices and the balanced budget rule, the firm maximizes profits, the household solves (1) subject to (2) and the budget constraint of the government (10) is fulfilled, with $K(0) = K_0$, $B(0) = B_0$ given, $C(0)$ free and* $\lim_{t\to\infty} e^{-\rho t}(K(t) + B(t))/C(t) = 0$, $\lim_{t\to\infty} e^{-rt}B(t) = 0$.

In definition 2 we define a balanced growth path.

Definition 2 *A balanced growth path (BGP) is a path such that the economy is in equilibrium and such that consumption and capital grow at the same strictly positive constant growth rate, that is $\dot{C}(t)/C(t) = \dot{K}(t)/K(t) = \dot{C}_p(t)/C_p(t) = g$, $g > 0$, $g = constant$, and $\dot{B}(t) = 0$.*

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[3] A progressive (regressive) tax implies that the tax rate declines (rises) with a higher investment share and, thus, with a higher growth rate.

To study our model, we note that it is completely described by the following differential equations,

$$\begin{aligned}
\frac{\dot{C}(t)}{C(t)} &= (1-\alpha)\,\omega\left(\frac{C(t)}{K(t)}\right)^{-\beta/(1-\beta+\gamma)} h(c(t))^{-\beta/(1-\beta+\gamma)} - (\rho+\delta),\ C(0)>0, && (12)\\
\frac{\dot{K}(t)}{K(t)} &= \omega\left(\frac{C(t)}{K(t)}\right)^{-\beta/(1-\beta+\gamma)} h(c(t))^{-\beta/(1-\beta+\gamma)} - \left(\frac{C(t)+C_p(t)}{K(t)}\right) - \delta,\ K_0>0, && (13)\\
\frac{\dot{B}(t)}{B(t)} &= (1-\alpha)\omega\left(\frac{C(t)}{K(t)}\right)^{-\beta/(1-\beta+\gamma)} h(c(t))^{-\beta/(1-\beta+\gamma)} - \frac{S(t)}{B(t)} - \delta,\ B_0>0, && (14)
\end{aligned}$$

with $\omega := \beta^{\beta/(1-\beta+\gamma)}$ and $h(c(t)) := 1+\tau(c(t))$. The initial conditions with respect to capital and public debt are assumed to be given while consumption can be chosen by the household at time $t=0$. The growth rate of $C(t)$ has been obtained from (6) with $r(t)$ determined by (8), $\dot{K}(t)/K(t)$ results by combining (3) with (10) and $\dot{B}(t)/B(t)$, finally, results from (10), with $r(t)$ determined by (8) and $r_p(t) = r(t)-\delta$.

To analyze our economy around a BGP we use $c(t) = C(t)/K(t)$ and define the variable $b(t) := B(t)/K(t)$. Differentiating these variables with respect to time leads to a two dimensional system of differential equations given by

$$\begin{aligned}
\dot{c}(t) &= c(t)\left(c(t)h(c(t)) - S(t)/K(t) - \alpha\,\omega\,c(t)^{-\beta/(1-\beta+\gamma)}h(c(t))^{-\beta/(1-\beta+\gamma)} - \rho\right), && (15)\\
\dot{b}(t) &= b(t)\left(c(t)h(c(t)) - S(t)/K(t) - \alpha\omega\,(c(t)h(c(t)))^{-\beta/(1-\beta+\gamma)} + \frac{S(t)}{B(t)}\right), && (16)
\end{aligned}$$

where we used $C_p(t)/K(t) = \tau_c c(t) - S(t)/K(t)$ and where $S(t)/K(t)$ has to be set to $S(t)/K(t) = r_p(t)B(t)/K(t)$ so that the government budget is balanced. A solution of $\dot{c}(t) = \dot{b}(t) = 0$ with respect to $c(t), b(t)$ gives a BGP for our model and the corresponding ratios $c^\star$, $b^\star = 0$ on the BGP, where the * denotes BGP values.

Proposition 1 gives results with respect to the existence and stability of a BGP for our economy.

Proposition 1 *There exists a unique saddle point stable BGP if the government runs a balanced budget and the consumption tax rate is constant ($\tau_c' = 0$), progressive ($\tau_c' > 0$) or weakly regressive ($-1 < \partial T_c/\partial c < 0$).*

Proof: See the appendix.

This proposition demonstrates that the economy is characterized by a unique saddle point stable balanced growth path when the government runs a balanced budget and with a progressive consumption tax rate.[4] Proposition 1 also holds for a regressive consumption

[4]Strictly speaking, we can only show that there exists a rest point for (15)-(16) but not that it implies a positive g. However, for a sufficiently small ρ and δ the balanced growth rate will always be positive.

tax rate as long as the elasticity of the total tax revenue relative to capital is not lower than the negative of the inverse of the consumption tax rate, i.e. as long as $\eta > -1/\tau_c$ holds. For a strongly regressive tax scheme over a certain interval $c(t) \in [\underline{c}, \overline{c}]$, multiple BGPs may arise. Proposition 2 gives the result.

Proposition 2 *Assume that the government runs a balanced budget and the consumption tax rate is constant ($\tau_c' = 0$), progressive ($\tau_c' > 0$) or weakly regressive ($-1 < \partial T_c/\partial c < 0$) for $0 < c(t) < \underline{c}$ and $\overline{c} < c(t) < \infty$ and strongly regressive ($\partial T_c/\partial c < -1$) for $\underline{c} \leq c(t) \leq \overline{c}$. Then, multiple (three) BGPs can exist. The first and the third BGP are locally saddle point stable and the second BGP is locally indeterminate.*

Proof: See the appendix.

Proposition 2 demonstrates that in the case of a strongly regressive consumption tax rate over a certain range of the consumption share, such that the total tax revenue declines as consumption rises, multiple BGPs may arise.The BGP that is associated with the lowest consumption share, that implies the highest balanced growth rate, and the BGP that goes along with the highest consumption share, that yields the lowest long-run growth rate, are locally saddle point stable. The medium BGP, that gives a balanced growth rate lower than the first BGP but higher than the third BGP, is locally indeterminate since the Jacobian evaluated at that rest point has two negative eigenvalues, implying that this rest point is a stable node.

Implicitly, proposition 2 also shows that there exists no BGP if the consumption tax rate is strongly regressive everywhere. There must be at least some range for which the consumption tax rate is constant, progressive or weakly regressive. The reason for that outcome is that for small values of the consumption-capital ratio, the growth rate of consumption is smaller than the growth rate of capital. Hence, consumption must grow faster than capital until the two growth rates become identical on a BGP. However, with a strongly regressive tax rate the reverse holds: capital grows faster than consumption so that the two growth rates diverge instead of converging to each other.

The emergence of indeterminacy can be explained as follows. When agents expect that the return to labour input rises as they supply more labour, the higher labour supply will lead to higher consumption relative to capital and to a lower consumption tax rate if the consumption tax rate is strongly regressive. That holds because with a strongly regressive consumption tax rate, labour supply and consumption relative to capital are positively correlated in optimum, which is seen from equations (5) and (9). Thus, there exist self-fulfilling expectations in the case of a strongly regressive consumption tax. When the consumption tax is weakly regressive, labour supply and consumption relative to capital are negatively correlated, just as for a constant and for a progressive tax rate, so that the expectation of a higher labour supply leading to higher after-tax consumption will not be fulfilled.

An example for a regressive tax scheme is the German social insurance, where the amount to be paid remains constant once a certain level of income is reached, implying a regressive system of contributions.

In order to illustrate that local indeterminacy can arise for plausible parameter values, we resort to a numerical example. We set the capital share equal to 25 percent, the labour share is assumed to be 75 percent and the rate of time preference is 5 percent. The inverse of the elasticity of labour supply, γ, is allowed to take values between 0.05 and 0.4 which are realistic values (see e.g. the discussion in Benhabib and Farmer, 1994). The consumption tax rate is given by $\tau_c = c(t)^{-2} - c(t) - 1.5$ for $c(t) \in [\underline{c} \leq c(t) \leq \overline{c}]$. With that specification, the second BGP is locally indeterminate and table 1 shows that the consumption tax rate on the BGP takes plausible values for different values of γ.

γ	0.05	0.1	0.25	0.3	0.4
τ_c	16.5%	16.8%	17.6%	17.9%	18.3%

Table 1: The consumption tax rate at the indeterminate BGP for different values of γ.

4. Conclusion

In this paper we have shown that a state-dependent consumption tax rate can lead to multiple balanced growth paths and local indeterminacy with a balanced government budget in a basic endogenous growth model. This is remarkable because in endogenous growth models with constant tax rates, a balanced government budget usually gives rise to a unique saddle point stable balanced growth path.

The emergence of multiple balanced growth paths crucially depends on the way how the consumption tax rate depends on the consumption share. Only in the case of a strongly regressive consumption tax rate over a certain range, multiple balanced growth paths may occur. Then, indeterminate equilibrium paths are feasible. The economic mechanism behind that outcome is that a strongly regressive consumption tax rate implies a positive correlation between optimal labour supply and consumption, that can give rise to self-fulfilling expectations and locally indeterminate equilibrium paths. If the consumption tax rate is progressive or exogenously given, the long-run balanced growth path is always unique and saddle point stable.

Appendix

Proof of proposition 1

To prove proposition 1, we first note that the balanced budget rule implies $b^\star = 0$ since public debt is constant while the capital stock grows over time. Thus, the equation $\dot{c}(t)/c(t)$ can be written as

$$\dot{c}(t)/c(t) = \left(f_1(c(t)) - \alpha\omega f_1(c(t))^{-\beta/(1-\beta+\gamma)} - \rho\right) \qquad (A.1)$$

with $f_1(c(t)) := c(t)h(c(t)) = c(t)(1+\tau_c(c(t)))$. It is easily seen that for $\tau_c'(\cdot) \geq 0$ we get,

$$\lim_{c(t)\to 0} (\dot{c}(t)/c(t)) = -\infty, \quad \lim_{c(t)\to\infty} (\dot{c}(t)/c(t)) = +\infty, \quad \partial(\dot{c}(t)/c(t))/\partial c(t) > 0.$$

This proves the existence of a unique $c^\star$ which solves $\dot{c}(t)/c(t) = 0$.

The Jacobian matrix for (15)-(16) is given by

$$\mathrm{J} = \begin{bmatrix} a_{11} & \partial\dot{c}(t)/\partial b(t) \\ 0 & -g \end{bmatrix},$$

with $c(t)$ and $b(t)$ evaluated at the rest point $\{c^\star, 0\}$ and $\partial\dot{K}(t)/\partial K(t) = g$. The term a_{11} is given by

$$a_{11} = c(t)\left(1 + \alpha\,\omega\,(\beta/(1-\beta+\gamma))\, f_1(c(t))^{-1-\beta/(1-\beta+\gamma)}\right) f_1'(c(t)).$$

The eigenvalues of the Jacobian are given by $\{a_{11}, -g\}$. A progressive tax rate implies $f_1'(c(t)) > 0$ and, therefore, $a_{11} > 0$ so that one eigenvalue is positive and one eigenvalue is negative. $\square$

Proof of proposition 2

A BGP is again obtained for a rest point of $\dot{c}(t)/c(t)$ (equation $(A.1)$ in the proof of proposition 1). Note that η is given by $1 + c(t)\tau_c'/\tau_c$. Further, we have $f_1' = 1 + \tau_c + c(t)\tau_c'$ showing that $f_1' < 0 \leftrightarrow \eta < -1/\tau_c$ holds. For $f_1' < 0 \leftrightarrow \eta < -1/\tau_c$, the curve $\dot{c}(t)/c(t)$ is a decreasing function in $c(t)$ for $c(t) \in [\underline{c}, \overline{c}]$, whereas it is increasing in $c(t)$ for $f_1' > 0 \leftrightarrow \eta > -1/\tau_c$. Therefore, $\dot{c}(t)/c(t)$ first rises, then declines for $c(t) \in [\underline{c}, \overline{c}]$ and monotonically rises for $c(t) > \overline{c}$ such that three BGPs can exist. The eigenvalues of the Jacobian are given by $\{a_{11}, -g\}$ (see the proof of proposition 1). Since $f_1' > 0$ holds for the first and for the third BGP, a_{11} is positive for those two BGPs, whereas a_{11} is negative at the second BGP. $\square$

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