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A note on Ramsey's conjecture with AK technology

Shinya Tsukahara
Kyoto University

Abstract

We consider an endogenously growing economy with heterogeneous households, each of which prefers capital (or wealth) as well as consumption. Regarding Ramsey's conjecture on the long-run distribution of capital among households, we present some extended versions of the result that was shown by Nakamura (2014, "On Ramsey's Conjecture with AK Technology," Economics Bulletin, 34(2), pp. 875-884). One of our results is that if aggregate capital productivity is low, the most impatient household could eventually own the entire capital (not "almost all" the capital) of the economy.

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Contact: Shinya Tsukahara - tskhrs@gmail.com.
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1. Introduction

Ramsey's conjecture on long-run wealth distribution is that the most *patient* household eventually owns the entire wealth of the economy.[1] Becker (1980) and Mitra and Sorger (2013) confirmed Ramsey's conjecture in the models with neoclassical production technology. Contrary to these studies, Nakamura (2014) proved that in an endogenously growing economy with what is called AK technology, a class of the most *impatient* households could eventually own almost all the capital (but not the entire capital).

In this note, we revisit Ramsey's conjecture with a modified model of Nakamura (2014) by relaxing the common assumption in previous studies that no household feels direct utility from the private capital (or wealth) holdings. Since at least Kurz (1968), many macroeconomic models assuming that accumulated wealth *per se* yields direct utility have appeared. See, for example, Zou (1994), Hosoya (2002), Long and Sorger (2006), and Roy (2010).[2] Following these authors, we also state that *wealth effects*, accompanied by capital accumulation, exist when the instantaneous utility of households depends on privately owned capital stock as well as private consumption. By adding the presence of wealth effects to Nakamura's model, we confirm and extend his counter-result to Ramsey's conjecture. More specifically, it is proved that if aggregate capital productivity is low, the most *impatient* household could eventually own the entire capital (not "almost all" the capital) of the economy.

The remainder of this note is organized as follows. Section 2 sets up the model. Section 3 derives analytical results. Section 4 provides concluding remarks. Some technical details are explained in the Appendices.

2. Model

Here, we describe an endogenous growth model in continuous time, following Nakamura (2014).

Time is denoted by t and goes from 0 to $+\infty$. Consider a closed economy in which there are always two types of households indexed by $i \in \{H, L\}$. Households H are more *impatient* than households L in the sense that the former have a higher subjective discount rate. There is no population growth and the total population of households is normalized to unity. Let $\lambda \in (0, 1)$ denote the share of households H to the population. Of course, the share of households L is $1-\lambda$. For simplicity and tractability, λ is assumed constant over time. Government activities are ignored.

[1] Ramsey's conjecture originates from his remarks in Ramsey (1928).

[2] Zou (1994) attempted to justify the dependence of utility on not only consumption but also wealth itself by quoting from the writings of the "giants" of economic philosophy. Interested readers should refer to the literature cited therein.

At each moment, household $i \in \{H, L\}$ derives utility from capital holding, $k_i(t)$, as well as private consumption, $c_i(t)$. The flow budget constraint of household i is written as

$$\dot{k}_i(t) := \frac{\mathrm{d}k_i(t)}{\mathrm{d}t} = r(t)k_i(t) - c_i(t), \tag{1}$$

where $r(t)$ is the rental price of capital at time t, which each household takes as given. The capital stock household i initially owns, $\bar{k}_i$, is given exogenously.

In the rest of this note, we use a dot over variable z to denote the first derivative of z with respect to time, that is, $\dot{z} := \mathrm{d}z/\mathrm{d}t$. Hereafter, the time argument (t) is often suppressed for brevity.

Suppose that all households live infinitely and have perfect foresight. By choosing the time path of $c_i(t)$, household i seeks to maximize

$$U_i = \int_0^{+\infty} u_i(c_i, k_i)e^{-\rho_i t}\mathrm{d}t \tag{2}$$

subject to (1) and $k_i(0) = \bar{k}_i$. In (2), $\rho_i > 0$ denotes the subjective discount rate of household i. Throughout this note, we assume

Assumption 1: $\rho_H > \rho_L$.

In order to obtain clear-cut results, we specify the functional form of instantaneous utility as

$$u_i(c_i, k_i) := \frac{1}{1-\varepsilon_i^{-1}}(c_i \cdot k_i^{\beta_i})^{1-\varepsilon_i^{-1}} \quad 1 \neq \varepsilon_i^{-1} > 0 \tag{2'}$$

where ε_i is the elasticity of inter-temporal substitution with respect to c_i and $\beta_i \geq 0$ denotes the strength of the *spirit of capitalism* (e.g., Zou [1994]) or desire for wealth. For strict concavity of $u_i(c_i, k_i)$, it is sufficient to assume $\varepsilon_i^{-1} > 1$ (or $0 < \varepsilon_i < 1$).

On the production side of the economy, many identical firms produce a homogeneous good. The total capital stock available at time t is

$$k(t) := \lambda k_H(t) + (1-\lambda)k_L(t).$$

Let $y(t)$ denote the total amount of output at time t. Following Nakamura (2014), we assume the "AK" production technology as $y(t) = Ak(t)$, where the marginal productivity of capital, $A > 0$, remains constant over time.

If the capital market is perfectly competitive, the rental price of capital is given as equal to the marginal productivity of capital:

$$r(t) = A. \tag{3}$$

3. Analytical Results

This section (except Subsection 3.3) is based on Section 2 and Section 3 of Nakamura (2014).

3.1. Balanced Growth

The competitive equilibrium paths of c_i and k_i are jointly determined by

$$\frac{\dot{c}_i}{c_i} = \{A(1+\beta_i) - \rho_i\}\varepsilon_i - A\beta_i + \frac{\beta_i c_i}{k_i}, \tag{4}$$

$$\dot{k}_i = Ak_i - c_i, \tag{5}$$

together with the appropriate transversality condition. See Appendix A for details.

Define the consumption–capital ratio of household i as $x_i := c_i/k_i$. Differentiating x_i logarithmically with respect to time, t, yields

$$\frac{\dot{x}_i}{x_i} = \frac{\dot{c}_i}{c_i} - \frac{\dot{k}_i}{k_i}. \tag{6}$$

By substituting (4) and (5) into (6), after rearrangement, we have

$$\dot{x}_i = [\{A(1+\beta_i) - \rho_i\}\varepsilon_i - A(1+\beta_i) + (1+\beta_i)x_i]x_i.$$

On the balanced growth path in which c_i and k_i grow at the same rate, $\dot{x}_i = 0$. Needless to say, $x_i = 0$ is economically meaningless and, thus, is discarded. We obtain a non-trivial steady state value of $x_i (= c_i/k_i)$ as

$$x_i^* = A(1-\varepsilon_i) + \delta_i\varepsilon_i \tag{7}$$

where $\delta_i := \rho_i/(1+\beta_i)$. There are no transitional dynamics, as in familiar AK growth models.

In the balanced growth equilibrium, k_i grows at the rate of

$$g_i := \frac{\dot{k}_i}{k_i} = A - x_i^* = (A - \delta_i)\varepsilon_i. \tag{8}$$

To ensure $g_i > 0$, we assume that the marginal productivity of capital, A, is large, such that

Assumption 2: $A > \delta_i$ for each i.

The output growth rate in this equilibrium is given by

$$g := \frac{\dot{y}}{y} = \frac{\dot{k}}{k} = A - s_H x_H^* - s_L x_L^*. \tag{9}$$

where $s_H := \lambda k_H/k$ is the capital share of household H, and $s_L := (1-\lambda)k_L/k$ is the capital share of household L. Needless to say, $s_H + s_L = 1$. See Appendix C as for the derivation of (9).

As Nakamura (2014) carefully noted, the equilibrium growth rate, g, is not constant, unlike the one that is derived from the simplest AK growth model, because the wealth distribution, s_i, can vary over time.

3.2. Dynamics of Wealth Distribution

The evolution of $s_H := \lambda k_H/k$ follows

$$\dot{s}_H = (x_H^* - x_L^*)s_H(s_H - 1). \tag{10}$$

Suppose that $s_H \in (0,1)$ initially. s_H increases to unity if and only if $x_H^* < x_L^*$.[3] In this case, as $s_H \to 1$ (or $s_L \to 0$), the equilibrium growth rate given by (9) approaches

$$g_H = (A - x_H^*)\varepsilon_H = (A - \delta_H)\varepsilon_H.$$

As stressed by Nakamura (2014), it is noteworthy that the household H will *not* own the entire capital of this economy, although $s_H \to 1$ ($s_L \to 0$), that is, household H will be much wealthier than household L in the long run. The reason is that as long as Assumption 2 is met, household L also keeps accumulating capital (i.e., $g_L > 0$).

3.3. Extending Nakamura's Result

The next proposition extends Nakamura's result.

Proposition 1: Suppose that $s_H \in (0,1)$ initially. Under Assumptions 1 and 2, the most *impatient* household eventually owns almost all the capital if and only if

$$\frac{\varepsilon_H}{\varepsilon_L} > \frac{A - \delta_L}{A - \delta_H}. \tag{11}$$

Proof. See Appendix B.

Corollary (Nakamura [2014]): Suppose that $\beta_i = 0$ for each i and that $A > \rho_H > \rho_L$. (a) For any $s_H \in (0,1)$ given initially, the most *impatient* household eventually owns almost all the capital if and only if

$$\frac{\varepsilon_H}{\varepsilon_L} > \frac{A - \rho_L}{A - \rho_H}. \tag{12}$$

[3] Conversely if and only if $x_H^* > x_L^*$, s_H decreases to zero, irrespective of $s_H \in (0,1)$ given initially.

(b) $\varepsilon_H > \varepsilon_L$ is necessary for (12) to hold because $\rho_H > \rho_L$.

The next two remarks on Proposition 1 are noteworthy.

Remark 1: If $\beta_H > \beta_L$, (11) can hold even when $\varepsilon_H \le \varepsilon_L$.

If $\beta_H > \beta_L$, $\delta_H := \rho_H/(1+\beta_H)$ can be smaller than $\delta_L := \rho_L/(1+\beta_L)$ even though $\rho_H > \rho_L$. If so, $A - \delta_H > A - \delta_L (> 0)$ and, thus, (11) can hold even when $\varepsilon_H \le \varepsilon_L$.

Remark 1 implies that even if $\varepsilon_H \le \varepsilon_L$, the most *impatient* household whose desire for wealth is assumed the *strongest* (i.e., $\beta_H > \beta_L$) can own almost all the capital. In the case of $\varepsilon_H \le \varepsilon_L$, the difference in households' desire for wealth (rather than the difference in households' patience) could be a main determinant of wealth distribution. With regard to this, Futagami and Shibata (1998) derived a similar result in a model that differs from the present one. See Section 3 of their paper.

Remark 2: As long as $\varepsilon_H > \varepsilon_L$, $\beta_H < \beta_L$ is compatible with (11).

Suppose $\beta_H < \beta_L$. Then, it follows that $\delta_H > \delta_L$ or $(0 <) A - \delta_H < A - \delta_L$. Thus, as far as $\varepsilon_H > \varepsilon_L$, (11) can hold even if $\beta_H < \beta_L$.

Remark 2 implies that as long as $\varepsilon_H > \varepsilon_L$, the most *impatient* household whose desire for wealth is assumed the *weakest* (i.e., $\beta_H < \beta_L$) can own almost all the capital in the long run, although this seems counter-intuitive.

Finally, suppose that A is low, such that $\rho_H > \rho_L > A$ and that β_H is sufficiently larger than β_L. It should be noted that even though $\rho_H > \rho_L$,

$$\underbrace{\frac{\rho_L}{1+\beta_L}}_{=\delta_L} > A > \underbrace{\frac{\rho_H}{1+\beta_H}}_{=\delta_H} \tag{13}$$

can hold only in the presence of wealth effects (i.e., $\beta_H > \beta_L \ge 0$). When (13) instead of Assumption 2 is assumed, $g_H > 0$ while $g_L < 0$. See (8) again. Thus, as $t \to +\infty$, k_H grows infinitely while k_L approaches zero. In other words, the most *impatient* household eventually owns the *entire* capital (not "almost all" the capital). This is a stronger version of Nakamura's result.

As a numerical example illustrating the last result, we assume the values of parameters as $(\rho_H, \rho_L, \beta_H, \beta_L) = (0.2, 0.15, 1, 0)$. Under these values, (13) is satisfied as long as

$$0.15 > A > 0.1.$$

Suppose, for example, $A = 0.12$ and $\varepsilon_H = \varepsilon_L = 0.5$. From (8), we obtain $g_H = 0.01 > 0$, while $g_L = -0.015 < 0$. This example implies that the impatient households (labeled H), having the desire for wealth (i.e., $\beta_H > 0$), become increasingly richer by accumulating private capital forever, while the patient households (labeled L), having no desire for wealth (i.e., $\beta_L = 0$), finally become trapped in the zero-capital state.

4. Concluding Remarks

Nakamura (2014) proved that Ramsey's conjecture on long-run wealth distribution may fail to hold in a perpetually growing economy, that is, the most *impatient* households can own almost all the capital (not the entire capital) of the economy with sustained growth.

As an extension of his analysis, we considered an endogenous growth model in which each heterogeneous household derives utility from capital itself as well as consumption. Although the model is simple, it yields some interesting results as follows.

- If $\varepsilon_H \leq \varepsilon_L$, the most *impatient* household whose desire for wealth is assumed the *strongest* (i.e., $\beta_H > \beta_L$) could eventually own almost all the capital of the economy.
- If $\varepsilon_H > \varepsilon_L$, the most *impatient* household whose desire for wealth is assumed the *weakest* (i.e., $\beta_H < \beta_L$) could eventually own almost all the capital of the economy.

where ε_H (ε_L) denotes the elasticity of inter-temporal substitution of the more (less) impatient households.

Furthermore, we proved the following.

- Suppose that aggregate capital productivity is low, such that (13) is satisfied. If the *impatient* household has the *strongest* desire for capital holdings, it could eventually own the *entire* capital (not "almost all" capital) of the economy.

These results confirm and extend Nakamura's argument that at least in the AK model with perpetual growth, the patience of households is less important as a determinant of wealth distribution than Ramsey (1928) originally conjectured.

Appendix A

Here, we derive equations (4) and (5) in the main text.

At first, equation (5) is trivially obtained by the substitution of (3) into (1).

Household i seeks to maximize its lifetime utility (2) (with (2′)) subject to (1) together with the initial condition, $k_i(0) = \bar{k}_i$. For deriving (4), we define the current-value Hamiltonian for this problem as

$$\mathcal{H}_i := \frac{1}{1-\varepsilon_i^{-1}}(c_i \cdot k_i^{\beta_i})^{1-\varepsilon_i^{-1}} + \nu_i(Ak_i - c_i),$$

where ν_i is a co-state variable associated with the dynamic equation of k_i. The relevant conditions for interior optimum are

$$\frac{\partial \mathcal{H}_i}{\partial c_i} = 0 \Leftrightarrow \nu_i = k_i^{\beta_i}(c_i k^{\beta_i})^{-\varepsilon_i^{-1}}, \tag{14}$$

$$\dot{\nu}_i = \rho_i \nu_i - \frac{\partial \mathcal{H}_i}{\partial k_i} \Leftrightarrow \dot{\nu}_i = (\rho_i - A)\nu_i - \frac{\beta_i}{k_i}(c_i k_i^{\beta_i})^{1-\varepsilon_i^{-1}}. \tag{15}$$

The following transversality condition is imposed:

$$\lim_{t \to +\infty} \nu_i k_i e^{-\rho_i t} = 0.$$

By the logarithmic differentiation of (14) with respect to time, t, we obtain

$$\frac{\dot{\nu}_i}{\nu_i} = -\frac{1}{\varepsilon_i} \cdot \frac{\dot{c}_i}{c_i} + \beta_i \left(1 - \frac{1}{\varepsilon_i}\right) \frac{\dot{k}_i}{k_i}. \tag{16}$$

Substituting (1), (14), and (15) into (16) yields (4).

Appendix B

Suppose that $s_H \in (0, 1)$ initially. Then it follows from (10) that $s_H \to 1$ as $t \to +\infty$ if and only if $x_H^* < x_L^*$. Hence, our task here is to show that $x_H^* < x_L^*$ is equivalent to (11).

Subtracting x_L^* from x_H^* gives

$$\begin{aligned} x_H^* - x_L^* &= (\varepsilon_L - \varepsilon_H)A + \delta_H \varepsilon_H - \delta_L \varepsilon_L \\ &= \varepsilon_L \Bigg\{ \underbrace{\left(1 - \frac{\varepsilon_H}{\varepsilon_L}\right) A + \delta_H \frac{\varepsilon_H}{\varepsilon_L} - \delta_L}_{=:\Omega} \Bigg\}. \end{aligned}$$

where x_i^* has been obtained by (7). As $\varepsilon_L > 0$, it is clear that under Assumption 2,

$$x_H^* < x_L^* \Leftrightarrow \Omega < 0 \Leftrightarrow \frac{\varepsilon_H}{\varepsilon_L} > \frac{A - \delta_L}{A - \delta_H}.$$

Appendix C

Equation (9) is derived here. Differentiating

$$k(t) = \lambda k_H(t) + (1 - \lambda) k_L(t)$$

with respect to t yields $\dot{k} = \lambda \dot{k}_H + (1-\lambda)\dot{k}_L$.

Recall the definitions of x_i and s_i in Section 3. Then, it is not difficult to verify

$$\begin{aligned} g := \frac{\dot{k}}{k} &= \lambda \frac{\dot{k}_H}{k} + (1-\lambda)\frac{\dot{k}_L}{k} \\ &= \frac{\dot{k}_H}{k_H} \cdot \frac{\lambda k_H}{k} + \frac{\dot{k}_L}{k_L} \cdot \frac{(1-\lambda)k_L}{k} \\ &= (A - x_H^*)s_H + (A - x_L^*)s_L \\ &= A - s_H x_H^* - s_L x_L^*. \end{aligned}$$

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