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Monotone comparative statics in general equilibrium

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Abstract

Under certain conditions on the excess demand function, it is shown that the set of equilibrium prices coincides with the set of maximizers of a potential function. Therefore, monotone comparative statics techniques can be employed to study how equilibrium prices change when there are shocks to the parameters of the model. As a by-product of our analysis, it turns out that the set of equilibrium prices is a convex lattice.

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1 Introduction

This paper is a contribution to the literature on comparative statics in the general equilibrium framework of exchange economies with finitely many agents and finitely many goods.

The underlying features of the exchange economy are as follows. There are N agents each having standard preferences over $L+1$ consumption goods. Agents are endowed with positive vectors of consumption goods. A Walrasian equilibrium is given by a vector of strictly positive prices and an allocation consisting of non-negative vector of consumption goods such that (a) at the given prices, every agent maximizes her preferences subject to her budget constraint; and (b) the market for each good clears.

For the comparative statics analysis, our interest is in changes of the equilibrium price vectors as a result of changes in the endowment vector of any agent. With this in mind, we keep the number of agents, number of goods and agents' preferences fixed. Given the preferences, we can focus on the aggregate excess demand function which is a function of the price vector and profile of agents' endowment vectors. Therefore, a Walrasian equilibrium price is a zero of the aggregate excess demand function. By virtue of Walras Law, it will suffice to focus on market clearing for only L goods.

It is possible to specify the exchange economy by a parameterized aggregate excess demand function, with the endowment vectors being the parameters. A very plausible conjecture in the comparative statics literature states that changes in equilibrium prices are negatively related to changes in the aggregate equilibrium consumption of goods (see Nachbar (2002) and Nachbar (2004)). It is well-known in the literature (since Hicks (1939)) that this conjecture does not hold true in general, even in the simplest exchange economy with only one agent. Therefore, additional restrictions on the fundamentals of the economy are required for the conjecture to hold. Another natural conjecture worth noting, is that the prices of goods that are in excess demand after some perturbation of the parameters go up relative to the prices of goods that are in excess supply (see Quah (2001)). These conjectures appear to be intuitive and have empirical foundations. Also, they conform to the canonical partial equilibrium comparative statics predictions that are introduced in basic economics courses. Therefore, in order for a general equilibrium model to provide a positive theory of market prices, it must be possible to prove that these standard conjectures hold under mild assumptions. In view of the above, in this paper we formalize and investigate the following conjecture: "if the endowment of any good increases and the excess demand for the same good falls, all else being equal, then the relative prices of the other goods go up".

The literature on general equilibrium comparative statics focuses on global as well as local analysis. As to the global analysis, there are two distinct approaches. The first approach establishes that an increase in the excess demand for a given commodity results in an increase in the relative price of that commodity. It relies on a type of behavior of the aggregate excess demand function which holds true if either the *weak axiom of revealed preference* (WARP henceforth) or the *gross substitution property* (GS henceforth) are assumed (see, e. g., McKenzie (2002)). The second approach is centered around lattice-theoretic techniques. Given the assumption of *strong gross substitution* (SGS henceforth), or gross substitution coupled with differentiability, equilibrium prices are shown to be fixed points of a certain increasing function. Here, equilibrium is shown to exist using Tarski's fixed point theorem, and monotone comparative statics is employed to conclude that an upward shift in the excess demand function causes both the minimal and maximal equilibrium price vectors to increase (see Mas-Colell et al. (1995, Exercises 17.F.16 and 17.G.3)). As for local analysis, WARP-like conditions or the GS property impose restrictions on the inverse of the Jacobian matrix of the excess demand function, and then the implicit function theorem is invoked (see, e.g., Mas-Colell et al. (1995)).

Thus, the main features of the canonical approach to general equilibrium comparative statics can be summarized as follows: the analysis based on the implicit function theorem yields local results, not global, and requires the equilibrium price vectors to be regular. On the other hand, both global and local analysis rely on WARP, the SGS or GS properties, which are strong assumptions. Indeed, in view of the Debreu-Mantel-Sonnenschein indeterminacy results (see Debreu (1974), Mantel (1974), Mantel (1976), and Sonnenschein (1973)), it is not surprising that general equilibrium comparative statics requires very specific assumptions on the aggregate excess demand function.

Our approach to comparative statics is as follows. First, we normalize the price vector by setting the price of one of the goods to be equal to one and call it the numéraire good. Additional structure is placed on the partial derivatives of the excess demand function with respect to prices of the non - numéraire goods. Indeed, we assume that the Jacobian matrix of the aggregate excess demand function is symmetric (undoubtedly a strong assumption) and negative semidefinite everywhere on the domain. The assumption of symmetry is required for the existence of a potential function. Given our assumptions, the aggregate excess demand function admits a concave and supermodular potential function. Therefore, it turns out that the set of (normalized) market-clearing prices coincides with set of maximizers of the potential function. Hence, on the one hand, comparative statics analysis of equilibrium prices boils down to an application of monotone comparative statics methods to the maximization problem whose objective function is a potential function; on the other hand, it turns out that the set of equilibrium prices is a *convex lattice* (see Corollary 1). Notice that we can perform comparative statics for the set of equilibrium prices as a whole. In fact, to compare two different sets of equilibrium prices we use the notion of *strong set order*. The notion of strong set order reveals more information on the change of equilibrium prices than the information we could obtain by restricting attention only to changes in the minimal and maximal equilibrium price selections.

Also, our approach enables us to perform global comparative statics analysis even in those settings where the implicit function theorem cannot be used. This is so because our assumptions can accommodate singular matrices and we do not require the aggregate excess demand function to be differentiable with respect to the endowment vectors. Moreover, WARP neither implies nor is implied by our assumptions. Besides, it is worth pointing out that we slightly relax the assumption of gross substitution.

This is not the first attempt to use monotone comparative statics techniques in general equilibrium theory. In fact, Milgrom and Shannon (1994) apply these techniques to a general equilibrium model with gross substitutes. They consider a game derived from the underlying economy where gross substitution property implies that such a game exhibits strategic complementarities. The gross substitution property allows them to apply monotone comparative statics results for pure strategies Nash equilibria (equilibrium prices). While they assume gross substitution (GS), we assume weak gross substitution, which is a milder condition than

GS.

It is worth clarifying and expanding on our choice of price normalization and the partial order on the set of prices. As Nachbar (2002) clarifies, a "good" price normalization should possess the following characteristics. Firstly, it must be possible to prove the desired comparative statics conjectures / outcomes under minimal conditions. Secondly, the normalization must give an easy and sensible way of interpreting the equilibrium price changes. It should be such that changes in normalized equilibrium prices have a natural interpretation in terms of changes in the relative prices (i.e., if the normalized price of a good falls, then its relative prices should also fall).

As for the price normalization chosen in this paper, we simply pick a commodity and set its price equal to one (numéraire good). We would like to argue that our normalization exhibits the desired features outlined above. Moreover, our analysis by no means depends on the choice of numéraire. To substantiate our claim, consider the following remarks. Our price normalization implies that (a) the set of prices is a lattice with respect to the canonical component-wise order; and (b) the price domain of the excess demand function is an open and convex set. The first property enables us to use lattice theoretic methods for comparative statics. The second property ensures the existence of a potential function for the aggregate excess demand. Using the potential function, we can carry out monotone comparative statics of equilibrium prices in a straightforward manner. By construction, with our normalization, normalized prices are relative prices; hence, changes in normalized equilibrium prices coincide with changes in relative prices. This implies that our normalization yields a very easy interpretation of equilibrium price changes. Furthermore, the main result of our paper (Proposition 1) is robust against the choice of the numéraire good. In fact, change in the numéraire would be equivalent to relabeling the consumption goods and applying the same assumptions of Proposition 1 to the relabeled goods. Then, it would still be possible to prove, verbatim, that the set of normalized equilibrium prices is monotone non-decreasing in the endowment of the numéraire good.

In the special case of uniqueness of equilibria, our main result (Proposition 1) yields the following corollary: under the assumptions therein stated, every relative price is non-decreasing in the endowment of the numéraire good. Thus, under the assumption that the non-numéraire goods are normal, we not only show that our conjecture under investigation holds true, but we also obtain a natural interpretation of comparative statics changes in equilibrium prices (see Remark 3). Incidentally, notice that normal demand is a fairly standard assumption in the literature on general equilibrium comparative statics. [1]

The paper is organized as follows. In section 2, we gather lattice-theoretic concepts. In section 3, we introduce the excess demand function, the assumptions of the model, and show the equivalence between general equilibrium comparative statics and comparative statics for a properly defined maximization problem. Also, we establish a result about the structure of the set of equilibrium prices and we prove our main comparative statics theorem about equilibrium prices. Finally, in section 4, we point out open questions and we outline avenues for further research.

[1] For example, Nachbar (2002) argues that if all goods are normal, then under his normalization the sign pattern of the changes in normalized prices has the intended interpretation in terms of changes of relative prices. Also, for the role played by the assumption that goods are normal, see for example, Proposition 17.G.3 and the subsequent discussion in Mas-Colell et al. (1995).

2 Setting and preliminaries

We shall remind the reader few fundamental lattice-theoretic notions and theorems from the theory of monotone comparative statics which will be used in this paper. For further details, see Milgrom and Shannon (1994).

Let $(X, \geq)$ be a partially ordered set. The set X could be considered as a choice set, or the set of endogenous variables (prices). Given x and y both in X, let $x \vee y$ denote the least upper bound of x and y in X, if it exists. Similarly, let $x \wedge y$ denote the greatest lower bound of x and y in X, if it exists. $(X, \geq)$ is a lattice if for every pair of elements x and y in X, $x \vee y$ and $x \wedge y$ exist as elements of X. Monotone comparative statics requires an order both on the parameter space and on the set of endogenous variables. The following definitions are useful in our analysis.

Definition 1. *Let $(X, \geq)$ be a lattice, and let Z and Y be subset of X. One says that Y is greater than Z in the sense of the strong set order, denoted by $Y \geq_s Z$, if for every $z \in Z$ and $y \in Y$, $z \wedge y \in Z$ and $z \vee y \in Y$.*

Remark 1. *If Z and Y are singletons, then notice that $\{y\} \geq_s \{z\}$ if and only if $y \geq z$.*

Now consider another partially ordered set, $(\Omega, \geq)$, where Ω could be thought of as the parameter space (or space of economies).

Definition 2. *Let X be a lattice. One says that a correspondence $M : \Omega \twoheadrightarrow X$ is monotone non-decreasing if, given ω and $\tilde{\omega}$ in Ω,*

$$\tilde{\omega} \geq \omega \Rightarrow M(\tilde{\omega}) \geq_s M(\omega).$$

Definition 3. *Given a lattice X and a partially ordered set Ω, $f : X \times \Omega \to \mathbb{R}$ is said to have increasing differences in (x, ω) if for $\hat{x} \geq x$, $f(\hat{x}, \omega) - f(x, \omega)$ is monotone non-decreasing in ω.*

Definition 4. *Given a lattice X, $f : X \to \mathbb{R}$ is supermodular if*

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$$

for all $x, y \in X$.

Definition 5. *Given a lattice X, $f : X \to \mathbb{R}$ is quasi- supermodular if*

$$\begin{aligned} (i) \quad & f(x) \geq f(x \wedge y) \Rightarrow f(x \vee y) \geq f(y) \\ (ii) \quad & f(x) > f(x \wedge y) \Rightarrow f(x \vee y) > f(y) \end{aligned}$$

for all $x, y \in X$.

Definition 6. *Given a lattice X, and a partially ordered set Ω, $f : X \times \Omega \to \mathbb{R}$ satisfies the single crossing property in (x, ω) if for $x > y$ and $\omega > \omega'$,* [2]

$$\begin{aligned} (i) \quad & f(x, \omega') > f(y, \omega') \Rightarrow f(x, \omega) > f(y, \omega) \\ (ii) \quad & f(x, \omega') \geq f(y, \omega') \Rightarrow f(x, \omega) \geq f(y, \omega). \end{aligned}$$

[2] Given any partially ordered set $(X, \geq)$, and for any two elements $a, b \in X$, $a > b$ means $a \geq b$ and $a \neq b$.

The following comparative statics result is a version of Topkis (1978, Theorem 6.1) and is stated in Milgrom and Shannon (1994, Theorem 5):

Theorem 1. *Let X be a lattice, Ω a partially ordered set, and $f : X \times \Omega \to \mathbb{R}$. If f is supermodular in x and has increasing differences in (x, ω), then $\operatorname{argmax}_{x \in X} f(x, \omega)$ is monotone non-decreasing.*

The following result described in Topkis (1978, Section 7, p. 319), and stated in Milgrom and Shannon (1994, Theorem 6), is a very useful characterization of supermodularity and increasing differences:

Theorem 2. *Let X be an open subset of $\mathbb{R}^n$ and Ω be an open subset of $\mathbb{R}^m$ and let $f : X \times \Omega \to \mathbb{R}$ be twice continuously differentiable. Then, (i) f has increasing differences if and only if*

$$\frac{\partial^2 f}{\partial x_i \partial \omega_j} \geq 0 \text{ for } i = 1, \cdots, n; j = 1, \cdots, m;$$

(ii) f is supermodular in x if and only if

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0 \text{ for } i \neq j.$$

It is clear from definitions 3-6, that any supermodular function is also quasi-supermodular. Moreover, any function which has increasing differences also satisfies the single crossing property (see Milgrom and Shannon (1994)). Remarkably, Theorem 1 was generalized by Milgrom and Shannon (1994, Theorem 4), as follows:

Theorem 3. *Let X be a lattice, Ω a partially ordered set, and $f : X \times \Omega \to \mathbb{R}$. Then*

$$\operatorname*{argmax}_{x \in X} f(x, \omega)$$

is monotone non-decreasing if f is quasi-supermodular in x and satisfies the single crossing property in (x, ω).

The following theorem is due to Milgrom and Shannon (1994, Corollary 2). In the next section we will invoke it to study the structure of the set of equilibrium prices.

Theorem 4. *Let $f : X \times \Omega \to \mathbb{R}$. If S is a sub-lattice of X and f is quasi-supermodular, then $\operatorname{argmax}_{x \in S} f(x, \omega)$ is a sub-lattice of S.*

3 General equilibrium comparative statics

There are N consumers and $L + 1$ consumption goods in the exchange economy, and the price of the goods are denoted by the vector $(x_1, \cdots, x_L, x_{L+1})$. We let prices be strictly positive and normalize the price of one of the goods, which is regarded as numéraire good (for instance, the last good x_{L+1}), to be one. Our results would still hold for any other choice of the numéraire good. With this normalization, the set of prices can be identified with $\mathbb{R}^L_{++}$. Hence, $X = \mathbb{R}^L_{++}$ is a lattice with respect to the canonical component-wise order defined on $\mathbb{R}^L$.

Let $\Omega \subset \mathbb{R}^{(L+1)N}_{++}$ denote the profile of consumer endowments. Ω will be viewed as the set of economies. $\omega \in \Omega$ represents a vector of endowments. In what follows, $\omega^n_l \in \mathbb{R}_{++}$ denotes the endowment of any good l for the generic consumer n.

Given Scarf's counter-examples of unstable economies with at least three commodities (see Scarf (1960)), it is impossible that every ω belonging to $\mathbb{R}^{(L+1)N}_{++}$ gives rise to an excess demand function which is a gradient field satisfying the assumptions below (see Assumption 1). In view of this, we assume that the set of economies is a strict subset of $\mathbb{R}^{(L+1)N}_{++}$. Moreover, given that assumption 1 may be quite stringent, we do not require the set of economies Ω to be open. The aggregate excess demand function of the underlying economy is the map: $Z : X \times \Omega \to \mathbb{R}^L$ with the generic component denoted by Z^i. Competitive equilibrium in this exchange economy is formalized as follows.

Definition 7. *Given an economy $\omega \in \Omega$, $x^* \in X$ is said to be an equilibrium price if $Z(x^*, \omega) = 0$.*

The aggregate excess demand function satisfies the following assumption.

Assumption 1. *For every $\omega \in \Omega$, $Z(\cdot, \omega) = Z_\omega : X \to \mathbb{R}^L$ is continuously differentiable and satisfies the following properties: the Jacobian matrix of Z_ω, $D_x Z(x, \omega)$, is symmetric, negative semidefinite and satisfies the weak gross substitution property, i.e.,*

$$\frac{\partial Z^i_\omega}{\partial x_j} \geq 0 \text{ for } i, j = 1, \cdots, L,\ i \neq j.$$

Remark 2. *Existence of equilibrium is not an issue in this model. We could add a suitable boundary condition to ensure that the set of equilibrium prices is non-empty.*

We apply the following theorem (Apostol (1969, Theorem 10.9)) to show the existence of a potential function for the map $Z_\omega : X \to \mathbb{R}^L$. Note that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a gradient field, then $f = \nabla \varphi$ for some potential function $\varphi : \mathbb{R}^n \to \mathbb{R}$.

Theorem 5. *Let $f = (f_1, \cdots, f_n)$ be a continuously differentiable vector field on an open convex set S in $\mathbb{R}^n$. Then f is a gradient field on S if and only if we have*

$$D_k f_j(x) = D_j f_k(x)$$

for each x in S and all $k, j = 1, 2, \cdots, n$.

Lemma 1. *Let the excess demand function $Z : X \times \Omega \to \mathbb{R}^L$ satisfy assumption 1. Then, there exists a potential function $f : X \times \Omega \to \mathbb{R}$ such that*

$$\nabla_x f(x, \omega) \equiv Z(x, \omega).$$

Moreover, the smooth function $f(\cdot, \omega) = f_\omega : X \to \mathbb{R}$ is concave for each $\omega \in \Omega$.

Proof. The set $X = \mathbb{R}^L_{++}$ is open and convex and $Z_\omega : X \to \mathbb{R}^L$ is continuously differentiable. Given assumption 1, for all $x \in X$

$$\frac{\partial Z^i_\omega}{\partial x_j} = \frac{\partial Z^j_\omega}{\partial x_i} = \text{ for } i, j = 1, \cdots, L,\ i \neq j.$$

Therefore, all of the conditions of Theorem 5 are met. Hence, there exists a function $f : X \times \Omega \to \mathbb{R}$ such that

$$\nabla_x f(x, \omega) \equiv Z(x, \omega). \tag{1}$$

Further, $D_x Z(x, \omega)$ is the Hessian matrix of $f(\cdot, \omega) = f_\omega : X \to \mathbb{R}$. By assumption 1, the Hessian matrix of f_ω is negative semidefinite. Therefore, the function $f_\omega : X \to \mathbb{R}$ is concave for each $\omega \in \Omega$. ∎

The following analysis revolves around the $\text{argmax}_{x \in X} f(x, \omega)$, where f is any potential function for $Z_\omega : X \to \mathbb{R}^L$. Hence, it is important to establish that our approach does not depend on the particular potential function which is selected. To this end, we prove the following claim.

Claim 1. *The set* $\text{argmax}_{x \in X} f(x, \omega)$ *is invariant under the choice of the potential function.*

Proof. Let $g : X \times \Omega \to \mathbb{R}$ be another potential function for Z_ω. Define the function $(g - f) : X \times \Omega \to \mathbb{R}$ and observe that

$$\begin{aligned} \nabla_x \ [g(x, \omega) - f(x, \omega)] &= \nabla_x \ g(x, \omega) - \nabla_x \ f(x, \omega) \\ &\equiv Z(x, \omega) - Z(x, \omega) \equiv 0. \end{aligned}$$

This implies that $g(x, \omega) - f(x, \omega)$ depends only on ω. Therefore we can write

$$g(x, \omega) - f(x, \omega) = \theta(\omega).$$

for some function $\theta : \Omega \to \mathbb{R}$. In other words,

$$g(x, \omega) = f(x, \omega) + \theta(\omega).$$

Therefore, it should be clear that for any $\omega \in \Omega$,

$$\underset{x \in X}{\text{argmax}} \ g(x, \omega) = \underset{x \in X}{\text{argmax}} \ f(x, \omega).$$

∎

We are now ready to state an instrumental lemma.

Lemma 2. *Suppose that Assumption 1 holds. Let $f : X \times \Omega \to \mathbb{R}$ be a potential function for Z_ω. Then, given any $\omega \in \Omega$, $x^* \in X$ is an equilibrium price if and only if*

$$x^* \in \underset{x \in X}{\text{argmax}} \, f(x, \omega).$$

Proof. Fix any $\omega \in \Omega$. Assume first that x^* is a maximizer of $f_\omega : X \to \mathbb{R}$. Then, by the first order necessary condition,

$$\nabla_x f(x^*, \omega) = 0,$$

which implies

$$Z(x^*, \omega) = \nabla_x f(x^*, \omega) = 0.$$

Therefore, x^* is an equilibrium price.

Next, fix any $\omega \in \Omega$ and assume that x^* is an equilibrium price, given ω. Then,

$$Z(x^*, \omega) = \nabla_x f(x^*, \omega) = 0.$$

Observe that by Lemma 1, the potential function f_ω is concave and differentiable. Consider any arbitrary $x \in X$. Then, the concavity of f implies

$$f(x, \omega) - f(x^*, \omega) \leq \nabla_x f(x^*, \omega) \cdot (x - x^*) = 0 \cdot (x - x^*) = 0.$$

Therefore,

$$f(x, \omega) \leq f(x^*, \omega).$$

Because x was arbitrarily chosen, this shows that x^* is a maximizer of the potential function f, given ω.

∎

The equivalence established in Lemma 2 enables us to study comparative statics of equilibrium prices by simply looking at how the solution set of an optimization problem changes with the parameters.

Before we state and prove a general equilibrium comparative statics result, a few more remarks are in order. Given any x and $\hat{x}$ in X, a path from x to $\hat{x}$ is a continuous, piece-wise smooth function $\phi : [0, 1] \to X$ such that $\phi(0) = x$ and $\phi(1) = \hat{x}$. Denoting $\left(\frac{d\phi_1(t)}{dt}, \ldots, \frac{d\phi_L(t)}{dt}\right)$ by $\frac{d\phi(t)}{dt}$, it is possible to define the line integral of Z_ω along any path ϕ as follows:

$$\int_\phi Z_\omega = \int_0^1 Z(\phi(t), \omega) \cdot \frac{d\phi(t)}{dt} dt. \tag{2}$$

Since the excess demand function Z_ω admits a potential function and the set X is an open and connected subset of $\mathbb{R}^L$, Z_ω satisfies the path independence property. That is, for any x and $\hat{x}$ in X, and any two paths ϕ^0 and ϕ^1 from x to $\hat{x}$,

$$\int_{\phi^0} Z_\omega = \int_{\phi^1} Z_\omega$$

(see, e.g., Apostol (1969, Theorem 10.5)). Using (2) and the fundamental theorem of calculus, it is easy to verify that for any x and $\hat{x}$ in X, and for every path $\phi : [0, 1] \to X$ from x to $\hat{x}$, we have that

$$\int_{\phi} Z_{\omega} = \int_0^1 \nabla_x f(\phi(t), \omega) \cdot \frac{d\phi(t)}{dt} dt = \int_0^1 \frac{\partial f(\phi(t), \omega)}{\partial t} dt = f(\hat{x}, \omega) - f(x, \omega). \qquad (3)$$

Incidentally, it's easy to see that $f(\hat{x}, \omega) - f(x, \omega)$ is independent of the choice of the potential function f. We posit that Z^i is non-decreasing in ω_l^n. Essentially, this amounts to assuming that the consumer at hand's demand for non - numéraire goods is normal.

Proposition 1. *Suppose that Assumption 1 holds. Assume, further, that Z^i is non–decreasing in ω_l^n, for every $i = 1, \ldots, L$. Then, the set of (normalized) equilibrium prices is monotone non–decreasing in ω_l^n.*

Proof. By weak gross substitution, $\frac{\partial Z_{\omega}^i}{\partial x_j} \geq 0$ for $i, j = 1, \ldots, L, i \neq j$. By definition of potential function f, this implies that $\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$, for $i \neq j$. Thus, by part (ii) of Theorem 2, $f_{\omega} : X \to \mathbb{R}$ is supermodular. Now, pick any x and $\hat{x}$ in X such that $\hat{x} \geq x$. By (2) and (3), and by path independence, we have that

$$f(\hat{x}, \omega) - f(x, \omega) = \int_0^1 Z(\phi(t), \omega) \cdot \frac{d\phi(t)}{dt} dt, \qquad (4)$$

where the above equality holds, in particular, for the straight path ϕ, from x to $\hat{x}$, given by $\phi(t) = t\hat{x} + (1-t)x$. But for such a path, clearly $\frac{d\phi(t)}{dt} = \hat{x} - x \geq 0$.

Therefore, because by assumption Z^i is non-decreasing in ω_l^n, (4) readily implies that $f(\hat{x}, \omega) - f(x, \omega)$ is monotone non-decreasing in ω_l^n. Thus, by Definition 3, f has increasing differences in (x, ω_l^n). Hence, all of the sufficient conditions of Theorem 1 are met. Therefore, $\text{argmax}_{x \in X} f(x, \omega)$ is monotone non-decreasing in ω_l^n and, by virtue of Lemma 2, so is the set of equilibrium prices. ∎

Here is another interesting implication of Lemma 2 which pertains to the structure of the set of equilibrium prices.

Corollary 1. *Suppose that Assumption 1 holds. Then, for every $\omega \in \Omega$ the set of (normalized) equilibrium prices is a convex lattice.*

Proof. Observe that by Lemma 2, given any $\omega \in \Omega$, the set of equilibrium prices coincides with the set $\text{argmax}_{x \in X} f(x, \omega)$. Hence, it will suffice to prove that the latter is a convex lattice. The proof consists of three steps.

Step 1 $f_{\omega} : X \to \mathbb{R}$ is concave (see Lemma 1), and X is convex. Therefore, the set $\text{argmax}_{x \in X} f(x, \omega)$ is convex.

Step 2 As for the lattice structure of the set $\text{argmax}_{x \in X} f(x, \omega)$, we know that $f_{\omega} : X \to \mathbb{R}$ is supermodular in x (see proof of Proposition 1), and thus it is also quasi-supermodular.

Step 3 Theorem 4 yields the desired result. ∎

Remark 3. *Since we stick to the canonical component-wise order in $\mathbb{R}^L$, as a corollary of Proposition 1, one has that if equilibrium is unique, as ω_l^n goes up then every relative price either stays the same or increases (see Remark 1). Hence, we obtain a very easy interpretation of the comparative statics changes.*

4 Concluding remarks

In the present paper the following questions have been left unanswered for future research. Firstly, what are the economies that generate an aggregate excess demand function satisfying Assumption 1? So, what type of economies do our results (Proposition 1 and Corollary 1) apply to? To address these questions, one should spell out from the outset hypotheses on preferences and endowments that give rise to an excess demand function which is a gradient vector field. In the special case of two-consumer, two-commodity economies, it should not be difficult to construct an economy whose excess demand for, say, the first good admits a potential function on an arbitrarily large compact subset of the non-negative real numbers. This is an easy consequence of Dierker (1974, Theorem 6.1).

Also, is it possible to find a map (that is not necessarily a potential function) whose set of critical points coincides with the set of equilibrium prices? The interest in this question lies in the fact that the existence of such a map would enable the analyst to drop the assumption that the Jacobian of Z is symmetric.

Finally, it would be worth improving upon Proposition 1 to extend our main theorem to a broader class of economies. A tentative road map on how to achieve this goal is as follows: one could formalize conditions on Z, milder than weak gross substitution and the assumption that Z^i is non–decreasing in ω_l^n, that result in the potential function f being quasi-supermodular and satisfying the single crossing property. Then, using Theorem 3 above it should be possible to prove the same result as Proposition 1.

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