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A variant of Uzawa's steady-state theorem in a Malthusian model

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Abstract

This paper provides a variant of Uzawa's (1961) steady-state theorem in a Malthusian model. Provided that the Malthusian model possesses a steady-state, technical change must be purely land-augmenting and cannot include labor augmentation.

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1. Introduction

Uzawa's steady-state theorem (1961) says that for a neoclassical growth model to possess a steady-state growth path, technological progress must be Harrod-neutral (purely labor-augmenting), at least in steady state. This result raises the question as to why technological progress cannot be, say, Hicks neutral or Solow neutral. Many authors have explicitly raised this question.[1] However, they have discussed only the requirements imposed on the direction of technical change for the neoclassical growth models, which essentially represents economic growth after the Industrial Revolution. In contrast, the process of economic development during the preindustrial era is captured by the Malthusian model (Malthus, 1798).This raises the question concerning the type of technological progress required to generate a steady-state path in that environment. In particular, does it also imply purely labor-augmenting technical change? This question has not received any attention so far. Answering it is important not just *per*-se, but also for developing a unified growth theory over the entire course of human history (Galor, 2011).

Kremer (1993) constructs and empirically tests a model of long-run world population growth. He combines the idea that a large population is conducive to technological change, as implied by many endogenous growth models, with the Malthusian assumption that a given technology limits population size. Lucas (2002) restated the Malthusian model in a neoclassical framework and proved that even with technological progress and capital accumulation, sustained growth of per-capita income cannot be achieved in that environment. Ashraf and Galor (2011) empirically confirm that technological progress had resulted in larger populations and higher population densities instead of leading to higher per-capita income in the preindustrial era.

While these papers discuss the effects of technological progress in a Malthusian world, they do not question whether a Malthusian model necessitates a restriction on the direction of technical change to generate a steady-state path. Irmen (2004) points out the structural similarities between the Malthusian and the Solow (1956) models, but does not address the aforementioned question either. Different from the existing literature, this paper focuses precisely on that question. We use the same method as Schlicht (2006) to prove that the Malthusian model generates a stationary path, if and only if technical change is purely land-augmenting. In contrast with the neoclassical growth model, a labor-augmenting technical change is inconsistent with a Malthusian steady-state.

The rest of the paper is organized as follows. The second section describes the Malthusian model; the third section states and proves the steady-state theorem in that model; the fourth section discusses the intuition of the steady-state theorem and concludes.

2. The Malthusian Model

Consider an economy with a production function F(•). In particular, this function relates, at any point in time t, the quantity produced, $Y_t > 0$, to labor input, $L_t > 0$, and land input, $T_t > 0$, and is characterized by constant returns to scale (CRS) in these inputs. Due to technological progress, it shifts over time, and we write:

[1] Fellner, 1961; von Weizsäacker, 1962; Kennedy, 1964; Samuelson, 1965; Drandakis and Phelps, 1966; Acemoglu, 2003, 2009; Barro and Sala-i-Martin, 2004; Jones, 2005; Jones and Scrimgeour, 2008; Irmen 2013a, 2013b, 2017; Irmen and Tabakovic, 2015; Grossman, Helpman, Oberfield and Sampson, 2016.

$$\mathrm{Y}_t = \mathrm{F}(\mathrm{T}_t, \mathrm{L}_t, \mathrm{t}) \tag{1}$$

with

$$\mathrm{F}(\lambda\mathrm{T}_t, \lambda\mathrm{L}_t, \mathrm{t}) = \lambda\mathrm{F}(\mathrm{T}_t, \mathrm{L}_t, \mathrm{t}), \quad \text{for all} \quad (\mathrm{T}, \mathrm{L}, \mathrm{t}, \lambda) \in \mathrm{R}_+^4 \tag{2}$$

Land input, *T*, grows exponentially at given rate τ:

$$\mathrm{T}_t = T_0 e^{\tau t} \quad , \tau \geq 0 \tag{3}$$

If τ=0, then there is no change in the quantity of land. However, with $\tau > 0$ the key results of the Malthusian model are still valid.[2]

The labor input, L, changes over time according to the key Malthusian assumption that population growth depends on the level of income per capita. Specifically, the higher per-capita income is, the higher (lower) the birth rate (mortality rate) becomes, implying a higher rate of population growth. Let n_t denote the total population growth rate, and b_t and d_t the birth and mortality rates, respectively. Let per-capita income be $\mathrm{y}_t \equiv Y_t/L_t$. Then the population growth function is defined as

$$\mathrm{n}_t = b(\mathrm{y}_t) - d(\mathrm{y}_t), \quad b'(\mathrm{y}_t) > 0, d'(\mathrm{y}_t) < 0\,. \tag{4}$$

From equation (4), it is obvious that

$$n'(\mathrm{y}_t) = b'(\mathrm{y}_t) - d'(\mathrm{y}_t) > 0. \tag{5}$$

Equation (4) describes the central assumption of the Malthusian model.

3. The Steady-State Theorem in the Malthusian Model

Definition 1: A steady state in the Malthusian model is a path along which the quantities $\{\mathrm{Y_t}, \mathrm{T_t}, \mathrm{L_t}\}$ grow at constant exponential rates (possibly zero) for all $\mathrm{t} \geq 0$. That is, Y_t, T_t and L_t are all nonnegative and grow at constant rates, $\mathbf{g} \geq \mathbf{0},\ \boldsymbol{\tau} \geq \mathbf{0}$ and $\mathbf{n} \geq \mathbf{0}$, respectively.[3]

Theorem: If the system (1)-(4) possesses a steady state solution, then the production function must take the form:

$$\mathrm{F}(\mathrm{T}_t, \mathrm{L}_t, t) = G(e^{(g-\tau)t} T_t, L_t) \tag{6}$$

In words, the theorem states that a steady state path in the Malthusian model requires that technological progress is purely land-augmenting, with a rate of progress of $g - \tau$. In the steady state, the rate of output growth, g, is equal to the growth rate of labor, n. As a result, per capita income must be constant. The latter result, obtained despite the fact that output growth is positive, may be interpreted as the Malthusian trap in a world with technical change.

Proof: Assume that the steady-state growth path is attained. By assumption we have

$$\mathrm{Y}_t = \mathrm{Y}_0 e^{gt}, \tag{7}$$

and

$$\mathrm{L}_t = \mathrm{L}_0 e^{nt}. \tag{8}$$

From equation (7) and (8) above, we can obtain

$$\mathrm{y}_t = \mathrm{y}_0 e^{(g-n)t} \tag{9}$$

where the initial per-capita income, $\mathrm{y}_0 \equiv \mathrm{Y}_0/\mathrm{L}_0 > 0$ is given.

Taking the time derivative of equation (4) yields

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[2] The amount of land is usually fixed in a Malthusian model. However, in the preindustrial era, whether for individual countries or humanity as a whole, farming land was increasing. The purpose of the assumption that land can grow is to show that fixing the amount of land is not a necessary conditions for the Malthusian trap. And it is not indispensable for the crucial result of this paper.

[3] Suggestions of anonymous referees have helped us improve the exposition of this section.

$$0 = \mathrm{b}'(y_t)\dot{y}_t - d'(y_t)\dot{y}_t \tag{10}$$

Taking also the time derivative of equation (9), we have:

$$\dot{y}_t = \mathrm{y}_0 e^{(g-n)t}(g-n) \tag{11}$$

Substituting equation (11) into (10), we obtain:

$$0 = [\mathrm{b}'(y_t) - d'(y_t)]\mathrm{y}_0 e^{(g-n)t}(g-n) \tag{12}$$

By equation (5), we know that $\mathrm{b}'(y_t) - d'(y_t) > 0$. Therefore, we must have:

$$g - n = 0. \tag{13}$$

Define

$$\mathrm{G(T, L)} := \mathrm{F(T, L, 0)}. \tag{14}$$

As $\mathrm{Y_0} = \mathrm{G(T_0, L_0)}$, and $\mathrm{Y}_t = \mathrm{Y}_0 e^{gt}$, we can rewrite Y_t as

$$\mathrm{Y}_t = \mathrm{G(T_0, L_0)}e^{gt}. \tag{15}$$

Replacing $\mathrm{L_0}$ with $\mathrm{L}_t e^{-nt}$, and $\mathrm{T_0}$ with $T_t e^{-\tau t}$, we further get:

$$\mathrm{Y}_t = \mathrm{G}(T_t e^{-\tau t}, \mathrm{L}_t e^{-nt})e^{gt}. \tag{16}$$

As G(•) is homogeneous of degree 1, from equation (16) we obtain:

$$\mathrm{Y}_t = \mathrm{G}\left(T_t e^{(g-\tau)t}, \mathrm{L}_t e^{(g-n)t}\right). \tag{17}$$

Since $g = n$, the theorem is proved.■

4. Conclusion

This paper provides a counterpart steady-state theorem in a Malthusian model which is analogous to Uzawa's theorem in a neoclassical environment. In particular, for a Malthusian model to possess a steady-state path, technical change must be purely land-augmenting, and the case of labor augmentation is excluded.

The intuition behind this theorem is quite similar to that of Uzawa's theorem which was pointed out by Jones and Scrimgeour (2008). In the neoclassical setting, capital accumulates and "automatically" inherits the trend in output while labor does not. Therefore, as the Uzawa's theorem shows, in a neoclassical growth model along a stationary path technical change must be purely labor-augmenting. In the Malthusian setting, it is labor that accumulates and inherits the trend in output, while land does not. To further elaborate, consider equation (1). Using the CRS property, it can be written as $1 = \mathrm{F(T_t/Y_t, L_t/Y_t, t)}$. Because labor accumulates and inherits the trend in output, $\mathrm{L_t/Y_t}$ is constant in steady state. However, land does not inherit the trend in output, so that $\mathrm{T_t/Y_t}$ falls in steady state. To compensate for that, technical change must exactly offset the decline in $\mathrm{T_t/Y_t}$. That is, technical change must be purely land-augmenting with a progress rate $\mathrm{g} - \tau$. In both cases, technical change cannot augment the factor which accumulates and inherits the trend of output.[4]

Uzawa's steady-state theorem and its Malthusian variant apply only in post- and pre-Industrial Revolution eras, respectively. Armed with these two theorems, the future research agenda is to develop a unified steady-state theorem to illuminate what type of technical change is required to obtain a steady-state path under different circumstances and how such a path evolves.

[4] We thank anonymous referees for suggesting the model's intuitions.

References:

Acemoglu, Daron (2003) “Labor- and Capital-Augmenting Technical Change” *Journal of European Economic Association,* Vol.1 (1), pp. 1-37.

Acemoglu, Daron (2009) *Introduction to Modern Economic Growth*, Princeton University Press.

Ashraf, Quamrul, and Oded Galor (2011) “Dynamics and Stagnation in the Malthusian Epoch” *The American Economic Review*, Vol. 101(5), pp. 2003-2041.

Barro, Robert and Xavier Sala-i-Martin (2004) *Economic Growth*, MIT Press, Cambridge, MA.

Drandakis, E. M., and Edmund S. Phelps (1966) “A Model of Induced Invention, Growth, and Distribution” *Economic Journal*, Vol. 76 (304), pp. 823-840.

Fellner, William (1961) “Two Propositions in the Theory of Induced Innovations” *Economic Journal*, Vol. 71(282), pp. 305-308.

Galor,Oded (2011) *Unified Growth Theory*, Princeton University Press, Princeton and Oxford.

Grossman, Gene M., Elhanan Helpman, Ezra Oberfield, and Thomas Sampson (2016) “Balanced Growth Despite Uzawa” *NBER Working Paper* No. 21861

Irmen, Andreas (2004) “Malthus and Solow − a Note on Closed−Form Solutions” *Economics Bulletin*, Vol.10, No. 6, pp. 1−6.

Irmen, Andreas (2013a) “Adjustment Cost in a Variant of Uzawa’sSteady-State Growth Theorem” *Economics Bulletin*, Vol 33 (4), pp.2860-2873.

Irmen, Andreas (2013b) “A Generalized Steady-State Growth Theorem” CESifo Working Paper No. 4477, CESifo Group Munich.

Irmen, Andreas (2017) “Capital- and Labor-Saving Technical Change in an Aging Economy” *International Economic Review,* forthcoming.

Irmen, A., and A. Tabakovic (2015) “Endogenous Capital- and Labor-Augmenting Technical Change in the Neoclassical Growth Model” CESifoWorking Paper No. 5643, CESifo Group Munich.

Jones, Charles I. (2005) “The Shape of Production Functions and the Direction of Technical Change” *Quarterly Journal of Economics* 2: 517–549.

Jones, Charles I., and Dean Scrimgeour (2008) “A New Proof of Uzawa’s Steady-State Growth Theorem” *Review of Economics and Statistics*, Vol. 90(1), pp. 180-182.

Kennedy, Charles M. (1964) “Induced Bias in Innovation and the Theory of Distribution” *Economic Journal*, Vol. 74 (295), pp. 541-547.

Kremer, Michael (1993) “Population Growth and Technological Change: One Million b.c. to 1990” *Quarterly Journal of Economics* 108: 681–716.

Lucas, Robert E. (2002) “The Industrial Revolution: Past and Future” In *Lectures in Economic Growth,* Cambridge, Mass.: Harvard University Press.

Malthus, Thomas R. (1798) *An Essay on the Principle of Population,* London: W. Pickering.

Samuelson, Paul A. (1965) “A Theory of Induced Innovation along Kennedy-Weizsäcker Lines” *Review of Economics and Statistics*, Vol. 47(4), pp. 343-356.

Schlicht, Ekkehart (2006) “A Variant of Uzawa’s Theorem” *Economics Bulletin*, Vol. 5 (6), pp. 1-5.

Solow, Robert M. (1956) “A Contribution to the Theory of Economic Growth” *Quarterly*

Journal of Economics, Vol. 70, February, pp. 65-94.

Uzawa, Hirofumi (1961) "Neutral Inventions and the Stability of Growth Equilibrium" *Review of Economic Studies*, Vol. 28, February, pp. 117-124.

von Weizsäcker,, C. C. (1962) "A New Technical Progress Function" *German Economic Review (2010)*, 11, 248–265.