Economics Bulletin

Volume 36, Issue 3

A note on the normative content of the Atkinson inequality aversion parameter

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Abstract

The aim of this note is to emphasize that there are clear necessary and sufficient conditions for the Atkinson social welfare functions to satisfy proportional transfer principles. These social welfare functions satisfy the proportional ex post transfer principle if and only if the inequality aversion parameter is no smaller than 1. Moreover, these social welfare functions satisfy the proportional transfer principle if and only if the inequality aversion parameter is greater than 2.

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Citation: Marc Dubois, (2016) "A note on the normative content of the Atkinson inequality aversion parameter", *Economics Bulletin*, Volume 36, Issue 3, pages 1547-1552
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Submitted: April 06, 2016. **Published:** August 11, 2016.

1 Introduction

The Atkinson (1970) inequality indices are defined by a class of social welfare functions (SWFs) exhibiting a downside inequality aversion. Indeed, these SWFs satisfy both the *Pigou-Dalton principle of transfer* and Kolm's (1976) *diminishing transfers principle*. The former requires that a transfer of a positive amount of utility from one individual to a worse-off one reduces inequality, thus increasing the social welfare.[1] The latter principle requires that "one values more such a transfer between persons with given [utility] difference if these [utilities] are lower than if they are higher"[Kolm (1976, p. 87)]. Such SWFs are well-known to contain a measure of the degree of inequality aversion (ε, hereafter). Kolm (1976, pp. 87-88) points out that the Atkinson SWFs satisfy both principles if, and only if, ε is positive.

Roberts (1980) and Blackorby and Donaldson (1982) indicate that the Atkinson SWFs are the only way to make social welfare evaluations whenever the ratio-scale comparability of utility is allowed and inequality aversion is supported. This informational basis asserts that $x\%$ of individual i's utility is more or less valuable than $y\%$ of individual j's utility. Therefore, it is possible to link these SWFs with some principles involving transfers that are *proportional* to the benefits of the agents involved in the transfers.

For that purpose, Fleurbaey and Michel (2001) introduce the *proportional transfer principle* (PP) and the *proportional ex post transfer principle* (PEP). The former requires that the social welfare increases as the result of a transfer that reduces the (pre-transfer) utility of a better off by $x\%$ and increases that of a worse-off by $x\%$. Its main advantage relies on the desirability of such a transfer even if there is some loss in the transferred benefit. The latter principle follows the same idea as the PP, however it entails a lower loss in the transferred benefit because the benefit and the gift are percentages of *post-transfer* utility levels in lieu of pre-transfer utility levels.

We aim to parameterize necessary and sufficient conditions for the Atkinson SWFs to satisfy Fleurbaey and Michel's (2001) principles. Such SWFs satisfy the PEP if, and only if, ε is not lower than 1. Moreover, they satisfy the PP if, and only if, ε is higher than 2. For $\varepsilon > 0$, the Atkinson SWFs satisfy the Pigou-Dalton principle of transfer and the diminishing transfers principle. For $\varepsilon \geqslant 1$, in addition to those principles, the Atkinson SWFs endorse a transfer of a positive amount from an individual to a less well-off one to be desirable even if there is some loss in the transferred benefit. For $\varepsilon > 2$, they endorse a transfer of a positive amount from an individual to a less well-off one to be desirable even if there is a greater loss in the transferred benefit than for $\varepsilon \geqslant 1$.

This note is structured as follows: Section 2 presents the framework and the results; Section 3 provides an example from the income distribution of Montpellier with data of the National Institute of Statistics and Economic Studies of France (INSEE).

[1] In this note, we introduce all principles of transfer in terms of utility in lieu of incomes as usually presented. This allows us to deal with sensitivity to a distribution of utility. Both alternatives yield the same results.

2 Setup and results

Let $\mathbb{R}$ (resp. $\mathbb{R}_{++}$ and $\mathbb{R}_{--}$) denote the set of real numbers (resp. positive real numbers and negative real numbers). The population $N := \{1, \dots, n\}$ is assumed to be the same throughout the note. Consider the Atkinson SWF:

$$\text{(A)} \qquad W(\mathbf{u}) = \begin{cases} \sum_{i=1}^{n} \frac{u_i^{1-\varepsilon}}{1-\varepsilon}, & \text{if } \varepsilon > 0 \text{ and } \varepsilon \neq 1; \\ \sum_{i=1}^{n} \ln(u_i), & \text{if } \varepsilon = 1. \end{cases}$$

We interpret $\frac{u_i^{1-\varepsilon}}{1-\varepsilon}$ as the ethical value of individual i's utility u_i ($\ln(u_i)$ being the ethical value if $\varepsilon = 1$). This utility level is an element of the utility distribution $\mathbf{u} := (u_1, \dots, u_n) \in \mathbb{R}^n_{++}$.[2] As in Atkinson (1970), ε is a parameter displaying some degree of inequality aversion. This parameter is also relevant with the Atkinson inequality index:

$$(I_A) \qquad I(\mathbf{u}) = 1 - \frac{W(\mathbf{u})}{\mu(\mathbf{u})},$$

where $\mu(\mathbf{u})$ is the mean utility of the distribution $\mathbf{u}$.

The use of I_A (or directly of a SWF (A)) implies ratio-scale comparability of utility. Thereby, it is possible to introduce transfers of proportions of utility. In this sense, Fleurbaey and Michel (2001) state the proportional *ex post* transfer principle.

Definition 2.1. ***Proportional ex post transfer principle*** **[PEP]**. *For* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_{++}$. *Consider that* $\mathbf{u}$ *is obtained from* $\mathbf{v}$ *by a proportional ex post transfer,* i.e.*: for some* $i, j \in N$ *and* $\delta > 0$, $u_i = v_i + \delta u_i \leqslant u_j = v_j - \delta u_j$ *and for all* $m \in N \setminus \{i, j\}$, $u_m = v_m$. *Then,*

$$W(\mathbf{u}) > W(\mathbf{v}).$$

The PEP is clearly stronger than the Pigou-Dalton principle of transfer. The following proposition examines the constraints that the PEP imposes on the SWFs (A).

Proposition 2.1. *For all* W *defined in (A), the following statements are equivalent:*
(i) W *satisfies the proportional* ex post *transfer principle;*
(ii) $\varepsilon \geqslant 1$.

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[2] Usually the Atkinson SWFs are introduced as follows:

$$W_A(\mathbf{u}) = \begin{cases} \left[\frac{1}{n} \sum_{i=1}^{n} u_i^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}} & \text{if } \varepsilon > 0 \text{ and } \varepsilon \neq 1; \\ \frac{1}{n} \sum_{i=1}^{n} \ln(u_i) \text{ if } \varepsilon = 1. \end{cases} \tag{1}$$

Such SWFs are used for making a ranking of a set of utility distributions. For that purpose, consider $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_{++}$:

$$W_A(\mathbf{u}) > W_A(\mathbf{v}) \iff \ln\left[\frac{1}{n}\sum_{i=1}^{n} u_i^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}} > \ln\left[\frac{1}{n}\sum_{i=1}^{n} v_i^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}} \iff \sum_{i=1}^{n} \frac{u_i^{1-\varepsilon}}{1-\varepsilon} > \sum_{i=1}^{n} \frac{v_i^{1-\varepsilon}}{1-\varepsilon}.$$

That is why, for instance, Roberts (1980) introduces the Atkinson SWF as in (A).

Proof. Fleurbaey and Michel (2001, pp. 7-8) show that a SWF such as $\sum_i^n m(u_i)$ satisfies the PEP if, and only if, there is a concave function $h : \mathbb{R}_{--} \to \mathbb{R}$ such that $m(u) = h(\ln(u))$. Then, $h(\tau) = m(e^\tau)$ with $\tau = \ln(u)$. Let us consider a SWF defined in (A) and $\varepsilon \neq 1$, so:

$$h(\tau) = \frac{(e^\tau)^{1-\varepsilon}}{1-\varepsilon} = \frac{e^{\tau[1-\varepsilon]}}{1-\varepsilon}.$$

The h function is concave if, and only if:

$$\frac{\partial^2\left(\frac{e^{\tau[1-\varepsilon]}}{1-\varepsilon}\right)}{\partial\tau^2} \leqslant 0 \iff \frac{\partial\left(\frac{(1-\varepsilon)e^{\tau[1-\varepsilon]}}{1-\varepsilon}\right)}{\partial\tau} \leqslant 0 \iff (1-\varepsilon)e^{\tau[1-\varepsilon]} \leqslant 0 \iff 1-\varepsilon \leqslant 0 \iff \varepsilon \geqslant 1.$$

Moreover, for all $u_i \in \mathbb{R}_{++}$, the ethical value of u_i is $\ln(u_i)$ whenever $\varepsilon = 1$. This case implies that $h''(\tau) = 0$, so h is concave.

□

Fleurbaey and Michel (2001, p. 4) introduce another proportional transfer which is part of a stronger principle than the PEP.

Definition 2.2. ***Proportional transfer principle*** **[PP]**. *For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_{++}$. Consider that $\mathbf{u}$ is obtained from $\mathbf{v}$ by a proportional transfer,* i.e.*: for some $i, j \in N$ and $\delta > 0$, $u_i = v_i(1+\delta) \leqslant u_j = v_j(1-\delta)$ and for all $m \in N \setminus \{i,j\}$, $u_m = v_m$. Then,*

$$W(\mathbf{u}) > W(\mathbf{v}).$$

The following proposition examines the constraints that the PP imposes on the SWFs (A).

Proposition 2.2. *For all W defined in (A), the following statements are equivalent:*
(i) W satisfies the proportional transfer principle;
(ii) $\varepsilon > 2$.

Proof. Fleurbaey et Michel (2001, pp. 5-7) show that a SWF such as $\sum_i^n m(u_i)$ satisfies the PP if, and only if, there is a function $h : \mathbb{R}_{--} \to \mathbb{R}$ strictly concave such that $m(u) = h\left(\frac{-1}{u}\right)$. Then, $h(\tau) = m\left(\frac{-1}{\tau}\right)$ with $\tau = \frac{-1}{u_i}$. Let us consider a SWF defined in (A) and $\varepsilon \neq 1$, so:

$$h(\tau) = \frac{\left(-\frac{1}{\tau}\right)^{1-\varepsilon}}{1-\varepsilon} \iff h(\tau) = \frac{(-\tau^{-1})^{1-\varepsilon}}{1-\varepsilon} \iff h(\tau) = \frac{-\tau^{\varepsilon-1}}{1-\varepsilon}.$$

The h function is strictly concave if, and only if:

$$\frac{\partial^2\left(\frac{-\tau^{\varepsilon-1}}{1-\varepsilon}\right)}{\partial\tau^2} < 0 \iff \frac{\partial\left(\frac{-(1-\varepsilon)(-\tau)^{\varepsilon-2}}{1-\varepsilon}\right)}{\partial\tau} < 0 \iff \frac{\partial\tau^{\varepsilon-2}}{\partial\tau} < 0 \iff (\varepsilon-2)\tau^{\varepsilon-3} < 0.$$

Given that $-\frac{1}{u_i} \in \mathbb{R}_{--}$, then $\tau \in \mathbb{R}_{--}$, we obtain:

$$\varepsilon > 2.$$

□

According to Fleurbaey and Michel (2001), the upper bounds on the loss, relative to the donor's gift in the proportional transfer and the proportional *ex post* transfer are respectively:

$$1 - \frac{v_i}{v_j} \text{ and } 1 - \frac{u_i}{u_j}.$$

From Definitions 2.1 and 2.2, u_i and u_j are the donor's and the receiver's post-transfer utilities, respectively. Then:

$$1 - \frac{v_i}{v_j} > 1 - \frac{u_i}{u_j}.$$

In other words, SWFs that satisfy the PP can support a higher loss in the transferred benefit to reduce inequality than SWFs that purely satisfy the PEP.

3 An illustration with INSEE data

To illustrate the results shown in the previous section, we have taken the data collected by INSEE-DGFIP-CNAF-CCMSA[3] covering the income deciles (per consumption unit) for Montpellier in 2012.[4] In this section, income is considered as an indicator of household's utility.[5]

Table 1. Sensitivity of the Atkinson inequality index to ε

ε-value	1.0	1.5	2.0	2.5	3.0	3.5
I_A of Montpellier	0.09	0.14	0.19	0.23	0.27	0.31

Following Atkinson (1970), the inequality indices I_A measure income inequality as a social cost. In Montpellier, for $\varepsilon = 1$, we have $I_A = 0.09$, *i.e.* income inequality is a social cost valued as 9% of the total income. That means that if incomes were equally distributed, the same level of social welfare could be achieved with 91% of the total income. As presented in our results, the upper bound on the loss in the transferred benefit to equalize incomes is increasing over ε. For instance, for $\varepsilon = 2, \varepsilon = 2.5$ and $\varepsilon = 3$, if incomes were equally distributed, the same level of social welfare could be achieved with $81\%, 77\%$ and 73% of the total income, respectively.

Table 2. Social welfare variations as the result of proportional transfers of 4%

ε-value	1.0	1.5	2.0	2.5	3.0	3.5
Montpellier (PP4)	**−0.020**%	+0.230%	+0.432%	+0.574%	+0.652%	+0.672%
Montpellier (PEP4)	+0.020%	+0.268%	+0.469%	+0.609%	+0.685%	+0.703%

[3]National Institute of Statistics and Economic Studies (INSEE), Public Finances Directorate General (DGFIP), National Family Allowances Office (CNAF), Agricultural Social Insurance Mutual Benefit Fund (CCMSA).

[4]Located on the Mediterranean coast, Montpellier is among the fastest growing cities in France.

[5]It may be worth noting that the following results are purely illustrative; the income deciles are used to make a simplified distribution in which the poorest 10% and the richest 10% of the population are discarded. An income distribution is composed by 8 income levels that are the midpoints of the extreme incomes of 8 groups formed by the deciles. For instance, the group with lowest income in Montpellier embodies 10% of the population with annual income of at least $5,698$ Euros and less than $10,068$ Euros in 2012. The simplification we make assumes that all households in this group earned $7,883$ euros in 2012 and so on for the other groups.

In Table 2, "Montpellier (PP4)"indicates an income distribution obtained from the original distribution of Montpellier by a *proportional* transfer of 4% in which donors are the households in the ninth decile and recipients are those in the second one. For instance, for $\varepsilon = 3$, the social welfare in Montpellier increases by 0.652% as the result of a transfer that reduces incomes in the ninth decile by 4% and increases incomes in the second one by 4%. Moreover, "Montpellier (PEP4)"indicates an income distribution obtained from the original distribution of Montpellier by a proportional *ex post* transfer of 4% involving the very same households as in "Montpellier (PP4)". For example, for $\varepsilon = 1$, the social welfare in Montpellier increases by 0.02% as the result of a transfer that reduces post-transfer incomes in the ninth decile by 4% and increases post-transfer incomes in the second one by 4%.

For $\varepsilon \geqslant 1$, all SWFs (A) satisfy the PEP and, by definition, they yield Montpellier (PEP4) as a better distribution than that of Montpellier (original). However, not all SWFs (A) satisfy the PP, and some of them do not yield Montpellier (PP4) as a better distribution than the original one. Indeed, for $\varepsilon = 1$, the social welfare *decreases* by 0.02% as the result of a transfer that reduces incomes in the ninth decile by 4% and increases incomes in the second one by 4% (see the bold value in Table 2).

Table 3. Losses and upper bounds on the losses (relative to the donor gift)

	Montpellier (PP4)	Montpellier (PEP4)
Loss in the transferred benefit	66.9%	64.1%
Upper bound on the loss determined by the PEP	**64.1**%	64.1%
Upper bound determined by the PP	66.9%	66.9%

Following the PEP, the loss in the transferred benefit must not exceed 64.1% of the donor's gift to consider "Montpellier (PP4)"as a better distribution than that of Montpellier (original) (see the bold value in Table 3). In fact, the proportional transfer of 4% involves a loss of 66.9% of the donor's gift.[6]

For $\varepsilon > 2$, all SWFs (A) satisfy the PP (and so the PEP); they yield Montpellier (PEP4) and Montpellier (PP4) as better distributions than that of Montpellier (original).

4 Conclusion

We have provided parametric results derived from Fleurbaey and Michel (2001). The Atkinson SWFs with a parameter $\varepsilon \geqslant 1$ satisfy the proportional *ex post* transfer principle and

[6] To reach the upper bound on the loss in the transferred benefit, we have made:

$$1 - \frac{12282.4}{34218.2} = 0.641,$$

where the numerator is the households' income in the second decile after the transfer and the denominator is the households' income in the ninth decile after the transfer. To reach the loss of the proportional transfer of 4%, we have made:

$$\frac{35644 - 34218.2 + 11810 - 12282.4}{35644 - 34218.2} = 0.669,$$

where 11,810 is the households' income (in Euros) in the second decile, and 35,644 is the households' income in the ninth decile.

such SWFs with $\varepsilon > 2$ satisfy the proportional transfer principle. Studying the example of Montpellier suggests that some Atkinson SWFs with $\varepsilon \geqslant 1$ yield inequality-reducing reforms as social welfare decreases if such reforms entail higher losses in the transferred benefit than the upper bound determined by the proportional *ex post* transfer principle.

By analyzing the reforms implemented by a given social planner, it is possible to determine the upper bound on the loss in the transferred benefit he defends. This feature is helpful to estimate his degree of inequality aversion in terms of ε-value.

References

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