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Borda elimination rule and monotonicity paradoxes in three-candidate elections

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Abstract

We derive analytical representations for the vulnerability of Borda Elimination Rule (BER) to Monotonicity Paradoxes. These results allow to compare BER vulnerability with that of Plurality Elimination Rule (or Plurality Runoff) and Negative Plurality Elimination Rule (or Coombs Rule), that suffer from the same pathologies. We show that BER performs better than these two rules at avoiding monotonicity failures. The probability model on which our results are based is the Impartial Anonymous Culture condition, often used in this kind of study.

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1. Introduction

Scoring Elimination Rules (SER), that give points to candidates according to their rank in voters' preference orders and eliminate the candidate(s) with the lowest number of points, constitute an important class of voting rules. As we only consider in this note the three-candidate case, we can describe this class of voting systems as follows: in the first round of the choice process, each voter ranks the candidates and the score of each candidate is computed on the basis of a point-system $(1, \lambda, 0)$, with $0 \leq \lambda \leq 1$, that gives 1 point each time a candidate is ranked first in voter's preferences, λ points for a second position and 0 point for a third and last position. The candidate with the lowest score is then eliminated and, in a second round, the two remaining candidates are confronted and the one who obtains the majority of votes wins. Plurality Elimination Rule (PER) is obtained when $\lambda = 0$ and is equivalent to the so-called Alternative Vote or Instant Runoff Voting in the three-candidate case. Taking $\lambda = 1$ gives the Coombs method (or Negative Plurality Elimination Rule - NPER): the candidate who is eliminated in the first round is the one who is ranked last by the largest number of voters. A third well known Scoring Elimination Rule is the Borda Elimination Rule (BER), associated to the case where $\lambda = 1/2$; this rule is of particular interest because it is the only one in the class of Scoring Elimination Rules that always selects the Condorcet winner – i.e. the candidate who beats each other candidate in majority pairwise comparisons – when such a candidate exists (see Smith 1973).

A striking feature of SERs is that getting more votes can cause a candidate to lose an election (the 'More is Less Paradox', MLP) and getting fewer votes can cause a losing candidate to win (the 'Less is More Paradox', LMP). The following example illustrates these two forms of monotonicity failures. Suppose there are 73 voters whose rankings of candidates, *a*, *b* and *c*, are as follows:

Number of voters	**Ranking**
16	$R_1: a \succ b \succ c$
7	$R_2: a \succ c \succ b$
11	$R_3: b \succ a \succ c$
18	$R_4: b \succ c \succ a$
15	$R_5: c \succ a \succ b$
6	$R_6: c \succ b \succ a$

When PER is applied, candidate c is eliminated in the first round (a obtains 23 votes, b 29 votes and c 21 votes) and candidate a beats candidate b in the second round (38 to 35): candidate a is thus the election winner. Suppose now that nine out of the 11 voters whose initial ranking is $b \succ a \succ c$ change their ranking to $a \succ b \succ c$ (thereby increasing a's support). As a result of this change, b (rather than c) is eliminated in the first round and c beats a in the second round (39 to 34), illustrating the More-is-Less-Paradox. Suppose instead that four of the 18 voters whose initial ranking is $b \succ c \succ a$ change their ranking to $c \succ a \succ b$ (thereby decreasing b's support). As a result of this change, a (rather than c) is eliminated in the first round and b beats c in the second round (41 to 32), illustrating the Less-is-More-Paradox. Note that, with the same example, it is easily shown that BER is vulnerable to the two paradoxes, MLP and LMP.

Voting rules that never exhibit this kind of anomaly are said to be *monotonic*. Smith (1973) has shown that the whole class of Scoring Elimination Rules (including BER) is subject to monotonicity failure. It is however often suggested that monotonicity failure, while a mathematical possibility, is highly unlikely to occur. Is it true? The studies conducted by Lepelley *et al*. (1996) for PER and NPER and Miller (2016) for PER suggest a rather negative answer in the three-candidate case. But what about BER? The aim of this note is to provide a response to this question by evaluating the vulnerability of BER to monotonicity failure and by comparing BER performances to those of PER and NPER.

2. Vulnerability of BER to Monotonicity Paradoxes

We consider elections with a set of n voters and a set of three candidates, $A = \{a, b, c\}$. Each voter's preference is given by one of the six strict rankings R_j $(1 \leq j \leq 6)$ defined in the introductory example. Preferences are supposed to be anonymous; thus we consider *voting situations* represented by six-tuples $x = (n_1, n_2, n_3, n_4, n_5, n_6)$ such that $n_j \geq 0$ $(1 \leq j \leq 6)$ and $\sum_{j=1}^{6} n_j = n$, where n_j is the number of voters with preference ranking R_j. Let $D(n)$ be the set of voting situations with n voters. The *Impartial Anonymous Culture* (IAC) condition, on which our probabilistic results are based, assumes that all the voting situations in $D(n)$ are equally likely to occur. A voting rule is a mapping F from $D(n)$ to A. We are interested here in the class of voting rules introduced in the previous section, i.e. the class of SERs for three-candidate elections. Given $\lambda \in [0, 1]$, we denote by F_λ the SER using the point-system $(1, \lambda, 0)$. For a candidate $w \in A$ and a voting situation $x \in D(n)$, we denote by $S_\lambda(w, x)$ the score obtained by w in the first round, when voters' preferences are described by x and the point-system $(1, \lambda, 0)$ is applied. For example, the score of candidate a for each of the three rules under consideration is given by: $S_0(a, x) = n_1 + n_2$, $S_{0.5}(a, x) = n_1 + n_2 + 0.5(n_3 + n_5)$ and $S_1(a, x) = n_1 + n_2 + n_3 + n_5$. For x and y in $D(n)$ and w in A, we say that y is *an improvement* of the status of w (from x) if w is ranked higher in y by some voters, all else unchanged. Conversely, we say that y is a *deterioration* of the status of w (from x) if w is ranked lower in y by some voters, all else unchanged.

Vulnerability to MLP and LMP can be formulated as follows. A voting system F is vulnerable to (or exhibits) MLP at a voting situation x if there exists an improvement y of the status of $F(x)$ such that $F(y) \neq F(x)$. Similarly, F is vulnerable to LMP at x if there exists a candidate w, $w \neq F(x)$, and a deterioration y of the status of w such that $F(y) = w$. For a monotonicity paradox M (MLP or LMP) and a voting rule F, we define the *vulnerability* of F to M as the probability, $Pr(M, F, n)$, that a situation in $D(n)$ gives rise to M under F. Under the IAC assumption, $Pr(M, F, n)$ is the proportion of voting situations in which F is vulnerable to M: $Pr(M, F, n) = |D(M, F, n)|/|D(n)|$, where $D(M, F, n)$ is the set of all voting situations in $D(n)$ for which F is vulnerable to M. Note that $D(MLP, F, n)$ is the disjoint union of the six subsets $D(MLP, F, n)_{\nearrow(w,w')}$ $(w, w' \in A, w \neq w')$ where $D(MLP, F, n)_{\nearrow(w,w')}$ consists of all voting situations x such that : $F(x) = w$ and there exists an improvement y of the status of w such that $F(y) = w'$. Similarly, $D(LMP, F, n)$ is the disjoint union of the six subsets $D(LMP, F, n)_{\searrow(w,w')}$ $(w, w' \in A, w \neq w')$ where $D(LMP, F, n)_{\searrow(w,w')}$ consists of all voting situations x such that: $F(x) = w$ and there is a deterioration y of the status of w' such that $F(y) = w'$. We also introduce a global measure for the vulnerability of F to monotonicity paradoxes, denoted by $Pr(GMP, F, n)$ and defined as the probability that a voting situation gives rise to MLP or LMP under F. If we denote by $Pr(MLP + LMP, F, n)$ the probability that a voting situation exhibits both MLP and LMP[1], then:

$$Pr(GMP, F, n) = Pr(MLP, F, n) + Pr(LMP, F, n) - Pr(MLP + LMP, F, n) \quad (2.1)$$

It is also easy to see that, by symmetry arguments, we obtain:

[1] Miller (2016) and Felsenthal and Tideman (2014) refer to this kind of voting scenario as "double monotonicity failure".

$$Pr(MLP,F,n) = \frac{6|D(MLP,F,n)_{\nearrow(a,c)}|}{|D(n)|} \quad (2.2) \quad \text{and} \quad Pr(LMP,F,n) = \frac{6|D(LMP,F,n)_{\searrow(a,b)}|}{|D(n)|} \quad (2.3)$$

Finally, we can write $Pr(MLP + LMP, F, n)$ in the same way:

$$Pr(MLP + LMP, F, n) = \frac{6|D(MLP,F,n)_{\nearrow(a,c)} \cap D(LMP,F,n)_{\searrow(a,b)}|}{|D(n)|} \quad (2.4)$$

Lepelley *et al.* (1996) provided analytical expressions for $Pr(MLP,F,n)$ and $Pr(LMP,F,n)$ for $F = F_0$ (PER) and $F = F_1$ (NPER). The aim of our study is to complement their results by extending these representations to the case $F = F_{0.5}$ (BER) and by computing the global vulnerability to monotonicity paradoxes for each of the three classical SER's. The first step in such calculations is to characterize the situations belonging respectively to $D(MLP,F,n)_{\nearrow(a,c)}$ and $D(LMP,F,n)_{\searrow(a,b)}$ for each F under consideration. The characterization of these sets for F_0 and F_1 is given in Lepelley *et al.* (1996). The following proposition (the proof of which is given in appendix) provides characterizations of all voting situations belonging to $D(MLP,F_{0.5},n)$ and to $D(LMP,F_{0.5},n)$. As in Lepelley *et al.* (1996), to simplify calculations, we ignore the problem of tied elections: we assume that one and only one alternative is eliminated in the first stage as well as in the second (this assumption alters the results only for small values of n).

Proposition 1. A voting situation $x = (n_1, n_2, n_3, n_4, n_5, n_6)$ belongs to $D(MLP,F_{0.5},n)_{\nearrow(a,c)}$ (resp. to $D(LMP,F_{0.5},n)_{\searrow(a,b)}$) if and only if it satisfies system $(S1)$ (resp. $(S2)$):

$$\begin{cases} -2n_1 - n_2 - n_3 + n_4 + n_5 + 2n_6 < 0 \\ -n_1 + n_2 - 2n_3 - n_4 + 2n_5 + n_6 < 0 \\ -n_1 - n_2 + n_3 + n_4 - n_5 + n_6 < 0 \\ n_1 - n_2 + n_3 + n_4 - 2n_5 - 2n_6 < 0 \\ n_1 + n_2 + n_3 - n_4 - n_5 - n_6 < 0 \end{cases} \quad (S1) \qquad \begin{cases} -2n_1 - n_2 - n_3 + n_4 + n_5 + 2n_6 < 0 \\ -n_1 - n_2 + n_3 + n_4 - n_5 + n_6 < 0 \\ n_1 + n_2 - n_4 - n_6 < 0 \end{cases} \quad (S2)$$

The second step of calculation is now to count the exact number of integer solutions for each of the two systems given by the previous proposition. Note that all (in)equalities in these systems are linear and have integer coefficients on the variables n_j and on the parameter n. We know from Lepelley *et al.* (2008) and Wilson and Pritchard (2007) that there is a well-established mathematical theory and efficient algorithms to calculate the number of integer solutions of such systems. Indeed, by Ehrhart's theorem (Ehrhart 1977), this number is a quasi-polynomial in n, i.e. a polynomial expression $f(n)$ of the form $f(n) = \sum_{k=0}^{d} c_k(n) n^k$, where d is the degree of $f(n)$ and where the coefficients $c_k(n)$ are rational periodic numbers in n. A rational periodic number of period q on the integer variable n is a function $u: \mathbb{Z} \to \mathbb{Q}$ such that $u(n) = u(n')$ whenever $n \equiv n' \pmod q$. Each coefficient $c_k(n)$ can have its own period, but we can always write $f(n)$ in a form where the coefficients have a common period called the period of the quasi-polynomial $f(n)$ and defined as the least common multiple of the periods of all coefficients. To calculate the quasi-polynomials associated with the systems of Proposition 1, we use the program proposed by Verdoolaege *et al.* (2005) based on Barvinok's algorithm (Barvinok 1994).

Proposition 2 (**BER**) For $n \equiv 1$ [12] (i.e. n = 13, 25, 37…), we have:

$$Pr(MLP, F_{0.5}, n) = \frac{(n-1)(53n^4 + 188n^3 - 1482n^2 + 9388n - 139475)}{1728(n+1)(n+2)(n+3)(n+4)(n+5)},$$

$$Pr(LMP, F_{0.5}, n) = \frac{(n-1)(n-7)(3n^2 - 6n - 109)}{144(n+1)(n+2)(n+3)(n+4)},$$

$$Pr(MLP + LMP, F_{0.5}, n) = \frac{(n-1)(n-13)(13n^3 - 163n^2 - 1801n + 8575)}{1728(n+1)(n+2)(n+3)(n+4)(n+5)},$$

$$Pr(GMP, F_{0.5}, n) = \frac{(n-1)(19n^3 - n^2 - 1051n + 889)}{432(n+1)(n+2)(n+3)(n+4)}.$$

The proof of this result is immediate. Using Barvinok's algorithm, we calculate quasi polynomials describing the numbers $|D(MLP, F_{0.5}, n)_{\nearrow(a,c)}|$ and $|D(LMP, F_{0.5}, n)_{\searrow(a,b)}|$ as functions of n. The number $|D(n)|$ is known and given by $|D(n)| = \binom{n+5}{n}$ for $n \geq 1$; it then suffices to apply formulas (2.2) and (2.3) to obtain the analytical expressions for $Pr(MLP, F_{0.5}, n)$ and $Pr(LMP, F_{0.5}, n)$. To calculate $Pr(GMP, F_{0.5}, n)$, we first calculated $Pr(MLP + LMP, F_{0.5}, n)$ and then we applied formula (2.1). The calculation of $Pr(MLP + LMP, F_{0.5}, n)$ is done in three steps: (i) characterization of all voting situations belonging to $D(MLP, F_{0.5}, n)_{\nearrow(a,b)} \cap D(LMP, F_{0.5}, n)_{\searrow(a,c)}$ that are simply all voting situations that jointly satisfy the two systems of Proposition 1, (ii) use of Barvinok's algorithm to obtain the quasi-polynomial giving the expression of $|D(MLP, F_{0.5}, n)_{\nearrow(a,c)} \cap D(LMP, F_{0.5}, n)_{\searrow(a,b)}|$ and (iii) application of formula (2.4). Note that the obtained quasi-polynomials are of degree 5 and period 12. For simplicity, we have only exhibited here the expression of these quasi-polynomials for integers n that are congruent to 1 modulo 12. However, complete formulas for the probabilities calculated in this proposition for any congruence modulo 12 are available and can be provided on request from the authors.

3. Comparison with other Scoring Elimination Rules

We begin by complementing the results obtained by Lepelley *et al.* (1996) for PER and NPER (recall that the formulas they give in their study only deal with the vulnerability to MLP and LMP and ignore MLP+LMP and GMP). The following propositions are easily deduced from the characterization results proposed by these authors for F_0 and for F_1.

Proposition 3 (PER) For $n \equiv 1\ [12]$ (i.e. n = 13, 25, 37…), we have:

$$Pr(MLP + LMP, F_0, n) = \frac{(n-1)(n+11)(n-13)(17n^2 + 56n - 25)}{2304(n+1)(n+2)(n+3)(n+4)(n+5)},$$

$$Pr(GMP, F_0, n) = \frac{(n-1)(397n^4 + 1292n^3 - 35298n^2 - 142228n - 142115)}{6912(n+1)(n+2)(n+3)(n+4)(n+5)}.$$

Proposition 4 (NPER) For $n \equiv 1\ [12]$ (i.e. n = 13, 25, 37…), we have:

$$Pr(MLP + LMP, F_1, n) = \frac{5(n-1)(n-13)(2n^3 - 15n^2 - 228n - 623)}{2592(n+1)(n+2)(n+3)(n+4)(n+5)},$$

$$Pr(GMP, F_1, n) = \frac{(n-1)(302n^4 + 2017n^3 + 2217n^2 - 3053n - 17035)}{2592(n+1)(n+2)(n+3)(n+4)(n+5)}.$$

The following Tables display some values of $Pr(M, F_\lambda, n)$ for $M \in \{MLP, LMP, MLP + LMP, GMP\}$, $\lambda \in \{0, \frac{1}{2}, 1\}$ and n = 13 (Table 1), n = 109 (Table 2), $n = \infty$ (Table 3).

Table 1: Vulnerability for PER (F_0), NPER (F_1) and BER ($F_{0.5}$), $n = 13$

	MLP	LMP	MLP+LMP	GMP
F_0	0	5/476 = 1.05%	0	5/476 =1.05%
$F_{0.5}$	4/357 = 1.12%	1/357 = 0.28%	0	5/ 357= 1.40%
F_1	8/357 = 2.24%	9/238 = 3.78%	0	43/714=6.02%

Table 2: Vulnerability for PER (F_0), NPER (F_1) and BER ($F_{0.5}$), $n = 109$

	MLP	LMP	MLP+LMP	GMP
F_0	475593/12233606 = 3.89%	63981/3495316 = 1.83%	78021/12233606 = .64%	16143/317756 =5.08%
$F_{0.5}$	411/15029 = 2.73%	5559/321937 = 1.73%	30648/6116803 = .50%	12750/312937 = 3.96%
F_1	15789/321937 = 4.90%	3555/58534 = 6.07%	16572/6116803 = .27%	187119/1747658=10.71%

Table 3: Vulnerability for PER (F_0), NPER (F_1) and BER ($F_{0.5}$), $n \to \infty$

	MLP	LMP	MLP+LMP	GMP
F_0	13/288 = 4.51%	17/864 = 1.97%	17/2304 = .74%	397/6912 =5.74%
$F_{0.5}$	53/1728 = 3.07%	3/144 = 2.08%	13/1728 = .75%	19/432 = 4.40%
F_1	1/18 = 5.56%	7/108 = 6.48%	5/1296 = .39%	151/1296=11.65%

The computed values show that, for the three rules under consideration, the vulnerability to monotonicity paradoxes increases with the number of voters and, with the exception of double monotonicity paradox, this vulnerability reaches values that cannot be considered as negligible. NPER - or Coombs rule (F_1) clearly exhibits the poorest performance for almost each type of monotonicity failure and each value of n, with a GMP probability close to 12% when n tends to infinity. However, a noticeable exception is observed for double monotonicity paradox for which the vulnerability of NPER is lower than the vulnerability of both PER and BER. Finally, it turns out that BER and PER perform similarly for LMP and MLP+LMP but BER dominates PER for MLP and GMP.

4. Concluding remark

In a companion paper, we have extended the current study to consider the whole class of SER $(1, \lambda, 0)$ with $0 < \lambda < 1$.The results we have obtained indicate that, when the number of voters tends to infinity: i) the vulnerability to monotonicity failures is minimized for a value of λ slightly lower than 1/2, and ii) the BER vulnerability is very close to the optimal value.

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Appendix

Proof of Proposition 1.

For simplicity, we denote by F the SER $F_{0.5}$ and by $S(w,x)$ the score $S_{0.5}(w,x)$. For a voting situation x in $D(n)$ and candidates w, w' in A, we denote by $P_x(w,w')$ the number of voters who prefer w to w' (for example, $P_x(a,b) = n_1 + n_2 + n_5$ and $P_x(b,a) = n_3 + n_4 + n_6$).
1) Let $x = (n_1, n_2, n_3, n_4, n_5, n_6)$ be a voting situation. Suppose that x belongs to $D(MLP, F, n)_{\nearrow(a,c)}$. By definition, $F(x) = a$ and there exists an improvement y of the status of a such that $F(y) = c$. Candidate a is present in the second round both at x and y since $F(x) = a$ and y is an improvement of the status of a from x. Candidate c is in the second round at y since $F(y) = c$ and candidate b is in the second round at x (if c is present in the second round at x, he will win against a since he wins after improvement of the status of a). Thus, in one hand we have $S(a,x) > S(c,x)$ (1), $S(b,x) > S(c,x)$ (2) and $P_x(a,b) > P_x(b,a)$ (3). In the other hand, $S(a,y) > S(b,y)$ (4), $S(c,y) > S(b,y)$ (5) and $P_y(c,a) > P_y(a,c)$ (6).
Clearly, the improvement of the status of a from x to y is only intended to decrease the score of b at the first round in such a way that c is now qualified for the second round at y and wins against a. With BER, turning $R_4 = bca$ into $R_1 = abc$ or $R_6 = cba$ into $R_2 = acb$ will decrease both the score of b and the score of c by the same amount; and turning $R_4 = bca$ into $R_3 = bac$ or $R_5 = cab$ into $R_2 = acb$ will only decrease the score of c. These operations have no contribution to the election of c at y. Hence, with BER, the only changes (in favor of a) that allow to move from (1)-(3) to (4)-(6) consist in moving from $R_3 = bac$ to $R_1 = abc$ and from $R_6 = cba$ to $R_5 = cab$. Note that when it is possible to move from (1)-(3) to (4)-(6) by changing the preferences of some voters of type R_3 or R_6, it is obviously possible to move from (1)-(3) to (4)-(6) by changing the preferences of all these voters. We can therefore take $y = (n_1 + n_3, n_2, 0, n_4, n_5 + n_6, 0)$. Then, writing conditions (1), (2), (3), (5) and (6) yields (in order) the five inequalities of system $(S1)$. This shows that the conditions described by system $(S1)$ are necessary.
To see that these conditions are also sufficient, let $x = (n_1, n_2, n_3, n_4, n_5, n_6)$ be a voting situation satisfying system $(S1)$. Let $y = (n_1 + n_3, n_2, 0, n_4, n_5 + n_6, 0)$. It is obvious that y is an improvement of the status of a from x and that the five inequalities of $(S1)$ describe (in order) the five following conditions: $S(a,x) > S(c,x)$ (1), $S(b,x) > S(c,x)$ (2), $P_x(a,b) > P_x(b,a)$ (3), $S(c,y) > S(b,y)$ (5) and $P_y(c,a) > P_y(a,c)$ (6). Now, note that $S(a,x) > S(c,x)$ implies $S(a,y) > S(c,y)$, and since $S(c,y) > S(b,y)$, we must have $S(a,y) > S(b,y)$ (4). So we have $F(x) = a$ (by (1)-(3)) and $F(y) = c$ (by (4)-(6)). Therefore $x \in D(MLP, F, n)_{\nearrow(a,c)}$.
2) Let $x = (n_1, n_2, n_3, n_4, n_5, n_6)$ be a voting situation. Suppose that x belongs to $D(LMP, F, n)_{\searrow(a,b)}$. By definition, $F(x) = a$ and there exists a deterioration y of the status of

b such that $F(y) = b$. Candidate a is present in the second round at x, since $F(x) = a$. Candidate b is present in the second round at y (since $F(y) = b$) and in the second round at x (the score of b increases from y to x). Candidate c is present in the second round at y (otherwise c is eliminated at the first round at y; and a will defeat b at the second round since a is already winning against b at x before the deterioration of the status of b). Thus, in the one hand we have $S(a,x) > S(c,x)$ (1), $S(b,x) > S(c,x)$ (2) and $P_x(a,b) > P_x(b,a)$ (3). In the other hand, $S(b,y) > S(a,y)$ (7), $S(c,y) > S(a,y)$ (8) and $P_y(b,c) > P_y(c,b)$ (9). The deterioration of the status of b from x to y is only intended to increase the score of c at the first round in such a way that c is now qualified for the second round at y and loses against b. With BER, turning $R_3 = bac$ into $R_2 = acb$ or $R_4 = bca$ into $R_5 = cab$ will increase both the score of c and the score of a by the same amount; and turning $R_3 = bac$ into $R_1 = abc$ or $R_6 = cba$ into $R_5 = cab$ will only increase the score of a. These operations have no contribution to the election of b at y. Hence, with BER, the only changes (to the detriment of b) that allow to move from (1)-(3) to (7)-(9) consist in moving from $R_1 = abc$ to $R_2 = acb$ and from $R_4 = bca$ to $R_6 = cba$. Each of these changes removes 0.5 point to $S(b,x)$ and adds it to $S(c,x)$. Similarly, each of these changes removes one point to $P_x(b,c)$ and adds it to $P_x(c,b)$. To move from (1)-(3) to (7)-(9), the score of c must increase to exceed that of a, while maintaining the score of b higher than the score of a and without changing the majority decision between b and c. For this to be possible, inequalities (1)-(3) and the additional following inequalities must be satisfied: $S(b,x) > S(a,x)$ (10), $S(b,x) - S(a,x) > S(a,x) - S(c,x)$ (11), $P_x(b,c) - 2[S(a,x) - S(c,x)] > P_x(c,b) + 2[S(a,x) - S(c,x)]$ (12). Writing conditions (1), (3) and (11) gives us (in order) the three inequalities of system $(S2)$. This shows that the conditions described by system $(S2)$ are necessary.

Conversely, let $x = (n_1, n_2, n_3, n_4, n_5, n_6)$ be a voting situation satisfying system $(S2)$. It is easy to see that the inequalities of $(S2)$ describe (in order) the three following conditions: $S(a,x) > S(c,x)$ (1), $P_x(a,b) > P_x(b,a)$ (3) and $S(b,x) - S(a,x) > S(a,x) - S(c,x)$ (11). Notice that, Since $S(b,x) - S(a,x) - [S(a,x) - S(c,x)] = -1.5(n_1 + n_2 - n_4 - n_6)$ and $n_1 + n_2 - n_4 - n_6$ is an integer, then (11) implies $S(b,x) - S(a,x) - [S(a,x) - S(c,x)] > 1$ (13).

Let $y = (n_1 - \alpha, n_2 + \alpha, n_3, n_4 - \beta, n_5, n_6 + \beta)$ whith $0 \le \alpha \le n_1$, $0 \le \beta \le n_4$ and $\alpha + \beta = 2[S(a,x) - S(c,x)] + 1$. This is possible because $n_1 + n_4 > 2[S(a,x) - S(c,x)]$. Indeed, let $\delta = n_1 + n_4 - 2[S(a,x) - S(c,x)]$. It is easy to verify that $\delta = 2[S(b,x) - S(a,x)] - 2[S(a,x) - S(c,x)] + P_x(a,b) - P_x(b,a) + n_4 + n_6$. So from (3) and (11), $\delta > 0$.

Now, note that (1) and (11) imply $S(b,x) > S(a,x)$ (10); and (1) and (10) imply $S(b,x) > S(c,x)$ (2). By a simple calculation of the scores of a and c, we also have $S(a,y) = S(a,x)$ and $S(c,y) = S(c,x) + 0.5(\alpha + \beta) > S(c,x) + [S(a,x) - S(c,x)] = S(a,x)$. Thus, we have $S(c,y) > S(a,y)$ (7). We have also $S(b,y) - S(a,y) = S(b,x) - 2[S(a,x) - S(c,x)] - 1 - S(a,x) = S(b,x) - S(a,x) - [S(a,x) - S(c,x)] - 1$. By (13) this expression is strictly positive. Thus, we have $S(b,y) > S(a,y)$ (2). Finally, we have $P_y(c,b) - P_y(b,c) = P_x(c,b) + 2[S(a,x) - S(c,x)] + 1 - (P_x(b,c) - 2[S(a,x) - S(c,x)] - 1)$; and it can be checked that $P_x(c,b) + 2[S(a,x) - S(c,x)] + 1 - (P_x(b,c) - 2[S(a,x) - S(c,x)] - 1) = P_x(b,a) - P_x(a,b) + \frac{8}{3}([S(a,x) - S(c,x)] - [S(b,x) - S(a,x)]) + 2$ (by simply expanding both the left hand side and the right hand side of this equality). Thus (3) and (13) imply that $P_y(c,b) - P_y(b,c) < 0 - \frac{8}{3} + 2 < 0$. Hence, $P_y(b,c) > P_y(c,b)$ (9).

To conclude, inequalities (1)-(3) show that $F(x) = a$, inequalities (7)-(9) show that $F(y) = b$ and it is obvious that y is a deterioration of the status of b (from x). Hence, $x \in D(LMP, F, n)_{\searrow(a,b)}$.