Economics Bulletin

Volume 36, Issue 4

Optimally chosen small portfolios are better than large ones

Francesco Cesarone
Department of Business Studies - University of Roma Tre

Jacopo Moretti
Department of Methods and Models for Economics Territory and Finance - Sapienza University of Rome

Fabio Tardella
Department of Methods and Models for Economics Territory and Finance - Sapienza University of Rome

Abstract

One of the fundamental principles in portfolio selection models is minimization of risk through diversification of the investment. However, this principle does not necessarily translate into a request for investing in all the assets of the investment universe. Indeed, following a line of research started by Evans and Archer almost fifty years ago, we provide here further evidence that small portfolios are sufficient to achieve almost optimal in-sample risk reduction with respect to variance and to some other popular risk measures, and very good out-of-sample performances. While leading to similar results, our approach is significantly different from the classical one pioneered by Evans and Archer. Indeed, we describe models for choosing the portfolio of a prescribed size with the smallest possible risk, as opposed to the random portfolio choice investigated in most of the previous works. We find that the smallest risk portfolios generally require no more than 15 assets. Furthermore, it is almost always possible to find portfolios that are just 1% more risky than the smallest risk portfolios and contain no more than 10 assets. Furthermore, the optimal small portfolios generally show a better performance than the optimal large ones. Our empirical analysis is based on some new and on some publicly available benchmark data sets often used in the literature.

Citation: Francesco Cesarone and Jacopo Moretti and Fabio Tardella, (2016) "Optimally chosen small portfolios are better than large ones", *Economics Bulletin*, Volume 36, Issue 4, pages 1876-1891
Contact: Francesco Cesarone - francesco.cesarone@uniroma3.it, Jacopo Moretti - jacopo.moretti@uniroma1.it, Fabio Tardella - fabio.tardella@uniroma1.it.
Submitted: October 03, 2016. **Published:** October 05, 2016.

1 Introduction

Since the start of Modern Portfolio Theory with the seminal Mean-Variance (MV) model of Markowitz (1952, 1959), the main aim of portfolio selection models was that of reducing the risk of an investment in the stock market through diversification while trying to achieve a satisfactory return. However, Markowitz also realized that, due to high correlations in the stock market, the benefit of diversification would rapidly decline with the size of the portfolio. In his fundamental book (Markowitz, 1959, Pag.102) he observed that: "To understand the general properties of large portfolios we must consider the averaging together of large numbers of highly correlated outcomes. We find that diversification is much less powerful in this case. Only a limited reduction in variability can be achieved by increasing the number of securities in a portfolio."

The first empirical evidence of the sufficiency of small portfolios to achieve almost complete elimination of the diversifiable risk in a market is probably due to a very influential work by Evans and Archer (1968) where, for any given size K from 1 to 40, they randomly picked subsets of K assets from a market of 470 securities and computed some statistics on the standard deviations of the Equally-Weighted portfolios formed with each subset of assets. They found that the *average* standard deviation for each size K was decreasing and rapidly converging to an asymptote and they concluded that no more than around 10 assets were needed to almost completely eliminate the unsystematic variation of a portfolio return.

Thenceforth, several authors contributed to the debate about the right size of a portfolio that almost completely eliminates the diversifiable risk in a market (see, e.g., Newbould and Poon, 1993, and references therein). Furthermore, based on Evans and Archer's and on other similar findings, such *magic* size, or size range, has been recommended in several textbooks on investment management and on corporate finance, as reported by Tang (2004).

There are several reasons for preferring small portfolios to large portfolios. The first and more obvious one concerns the infeasibility of holding large portfolios for small investors. However, even big investors should consider the opportunity cost of holding large portfolios and should identify the threshold where the costs exceed the benefit of risk reduction. Statman (1987) identifies such costs with the transaction costs and, using the cost of holding an index fund that replicates the market as a proxy, finds a threshold around 30-40 assets. Furthermore, there are other sources of cost that depend on the size such as those for monitoring the behavior and fundamentals of all the companies involved in the portfolio. Another important advantage of small portfolios seems to be that of reducing the estimation errors for variances and covariances thus leading to better out of sample performance (see, e.g., Cesarone et al, 2014; DeMiguel et al, 2009a).

In this work we provide further evidence of the benefits of small portfolios both in terms of in-sample risk reduction and in terms of out-of-sample performance. However, our approach is significantly different from the mainstream approach pioneered by Evans and Archer. Indeed, we overcome one of the main weaknesses of their approach which consists in stating results that are valid only on average. In other words, if one picks an arbitrary Equally-Weighted portfolio of a given size in a market, there is no guarantee that its risk will not be much larger than the average risk of all portfolios of the same size in that market.

The conceptually simple solution that we propose here is just to choose the *best* Equally-Weighted portfolio for each given size with respect to variance, and, furthermore, the *optimal* portfolios for each given size with respect to three different and complementary risk measures. In this way for each size we clearly obtain a portfolio which has a risk not greater (and typically quite smaller) than the average risk. The reason why this simple idea was not investigated before is probably due to the computational hardness of the models required to find such best portfolios. Indeed, some of these models have been solved exactly only recently for small to medium size markets (see, e.g., Angelelli et al, 2008; Cesarone et al, 2015, and references therein), and one model is solved here for the first time. Once we have obtained the optimal size of the minimum risk portfolio, we proceed with a sensitivity analysis that allows us to find the smallest size of a portfolio whose risk is not more than 1% larger than that of the minimum risk portfolio, thus finding even smaller portfolios with satisfactory risk level.

Another difference between our approach and the one of Evans and Archer consists in the possibility of using general weights in the selected portfolio instead of equal weights only. For each portfolio size, this clearly allows one to find portfolio with even lower in-sample risk. However, since optimizing weights might also cause the maximization of estimation errors (DeMiguel et al, 2009b; Michaud, 1989), this choice does not necessarily implies better out-of-sample performance. For both weighting schemes and for all risk measures we find results comparable to those of Evans and Archer. More precisely, we identify some ranges of (typically small) sizes where the portfolio risks are minimized and ranges of even smaller sizes where the portfolio risks do not exceed the minimum by more than 1%. The out-of-sample performance of the selected portfolios for each specified size is another important feature of our analysis which is rarely found in previous works on the subject. Also in this case we find that the best performances are generally obtained by portfolios with no more than 15 assets.

As an interesting complement to our findings, we mention that, in a recent and detailed analysis on the empirical behavior of investors and on the performance of their portfolios, Ivković et al (2008) show that portfolios of small investors with low diversification exhibit superior performance with respect to the ones with high diversification.

2 The portfolio models

In this section we describe the models analyzed and we provide an integer or a mixed-integer linear or quadratic formulation for all models. We first need to introduce some notation. Let $T+1$ be the length of the in-sample period used to estimate the inputs for the models. We use p_{it} to denote the price of the i-th asset at time t, with $t = 0, ..., T$; $r_{it} = \dfrac{p_{it} - p_{i(t-1)}}{p_{i(t-1)}}$ is the i-th asset return at time t, with $t = 1, ..., T$; x is the vector whose components x_i are the fractions of a given capital invested in asset i in the portfolio we are selecting; y is a boolean vector whose components y_i are equal to 1 if asset i is selected, and 0 otherwise. We assume that n assets are available in a market and, adopting linear returns, we have that $R_t(x) = \sum_{i=1}^{n} x_i r_{it}$ is the portfolio return at time t, with $t = 1, ..., T$. The n-dimensional vector μ is used to denote the expected returns of the n risky asset, while Σ denotes their covariance matrix, and u denotes an n-dimensional vector of ones.

2.1 The Equally-Weighted portfolio

The most intuitive way to diversify a portfolio is to equally distribute the capital among all stocks available in the market. In terms of relative weights we have $x_i = 1/n$. This is known as the Equally-Weighted (also called naïve or uniform) portfolio. Clearly the choice of the Equally-Weighted (EW) portfolio does not use any in-sample information nor involve any optimization approach. However, some authors claim that its practical out-of-sample performance is hard to beat on real-world data sets (DeMiguel et al, 2009b). Furthermore, from the theoretical viewpoint, Pflug et al (2012) show that when increasing the amount of portfolio model uncertainty, i.e., the degree of ambiguity on the distribution of the assets returns, the optimal portfolio converges to the EW portfolio. We will thus use this portfolio as a benchmark to compare the performances of the portfolios obtained by the models.

2.2 Fixed-Size Minimum Variance Equally-Weighted portfolios

As already observed, the EW portfolio is the most robust choice when there is a great uncertainty about the distribution of the asset returns. However, the EW portfolio has the drawback of using all available assets, which might be too numerous and not all desirable. A first proposal to overcome this drawback is due to Jacob (1974), who proposes to select a small EW portfolio (with a specified number K of assets) that has minimum variance among all EW portfolios of the same size. The model by Jacob is a nonlinear 0-1 optimization model that has not yet been tested in practice due to its computational complexity. Thanks to the recent advances in solution methods and computing power, we can propose here an empirical study of such Fixed-Size Minimum Variance Equally-Weighted (FSMVEW) model formally described below.

$$\begin{array}{ll} \min & y^T \Sigma y \\ s.t. & u^T y = K \\ & y \in \{0,1\}^n \end{array} \tag{1}$$

This is probably the simplest Fixed-Size portfolio model and has the advantage of not requiring the problematic estimates of the assets expected returns. Furthermore, the effects of the possible estimation errors of the covariance matrix Σ do not result in very large or small weights for some assets, but only influence the choice of the subset of selected assets in the portfolio. From the optimization viewpoint, it falls into the class of pseudoBoolean Quadratic Programming problems which are known to be theoretically hard to solve in the worst case (NP-hard) (Boros and Hammer, 2002). However, due to its special structure, practical problems of this type with several hundreds variables can be actually solved fairly efficiently with available free or commercial codes.

Note that the vector x of weights of the optimal FSMVEW portfolio selected by model (1) is obtained as $x = \frac{1}{K}y$. When $K = n$ the FSMVEW portfolio coincides with the EW portfolio.

2.3 Fixed-Size Minimum Variance portfolios

Another model that does not require the estimates of the assets expected returns is the extreme case of the Markowitz model where we only seek to minimize variance. Within our framework we thus consider the following Fixed-Size Minimum Variance (FSMV) model where only K assets are allowed in the selected portfolio

$$\begin{array}{ll} \min & x^T \Sigma x \\ s.t. & u^T x = 1 \\ & u^T y = K \\ & \ell y \leq x \leq y, \\ & y \in \{0,1\}^n \end{array} \tag{2}$$

The first constraint above is the budget constraint; the second one represents the portfolio fixed-size constraint; u is an n-dimensional vector of ones; y is an n-dimensional vector of binary variables used to select the assets to be included in the portfolio; x is the vector of portfolio weights, and ℓ is a minimum threshold (often called buy-in threshold) for the weights of the selected assets which must be greater than zero (in our experiments we chose $\ell = 0.01$). Without these thresholds, Problem (2) could generate portfolios with less than K assets, which is equivalent to replacing the constraint $u^T y = K$ with $u^T y \leq K$. Note that Problem (2) is a Quadratic Mixed Integer Programming (QMIP) problem that falls again in the class of NP-hard problems. However, also in this case problems with a few hundred variables can be solved fairly efficiently with available free or commercial codes. Furthermore, a recently proposed (Cesarone et al, 2009, 2013) specialized algorithm can solve problems of this type with up to two thousand variables.

2.4 Fixed-Size Minimum CVaR portfolios

The Fixed-Size Minimum CVaR (FSMCVaR) model is a minimum risk model like the previous one, but instead of variance it measures risk with Conditional Value-at-Risk at a specified confidence level ε ($CVaR_\varepsilon$), namely the average of losses in the worst $100\varepsilon\%$ of the cases (Acerbi and Tasche, 2002). In our analysis losses are defined as negative outcomes, and we set ε equal to 0.05. The FSMCVaR model can be written as follows:

$$\begin{array}{ll} \min & CVaR_\varepsilon(x) \\ s.t. & u^T x = 1 \\ & u^T y = K \\ & \ell y \leq x \leq y, \\ & y \in \{0,1\}^n \end{array} \tag{3}$$

where ℓ plays the same role as in (2).

Using a classical approach introduced by Rockafellar and Uryasev (2000) (see also Cesarone et al (2014)), Problem (3) can be reformulated as a Mixed Integer Linear Programming (MILP) problem with $n + T + 1$ continuous variables, n binary variables and $T + n + 3$ constraints. Some recent computational experiences reported in Cesarone et al (2015) on the solution of this model with state-of-the-art commercial solvers show that models with more than a few hundreds variables are hard to solve with general purpose solvers and would probably benefit from more specialized methods.

2.5 Fixed-Size Minimum Semi-MAD portfolios

The last risk measure that we take into account in our analysis is the downside Mean Semi-Absolute Deviation (Semi-MAD):

$$SMAD(x) = E[\min(0, \sum_{i=1}^{n}(r_{it} - \mu_i)x_i)], \tag{4}$$

This is a concise version of the more famous Mean Absolute Deviation (MAD) risk measure, which is defined as the expected value of the absolute deviation of the portfolio return from its mean (Konno and Yamazaki, 1991). Indeed, Speranza (1993) showed that Semi-MAD leads to a portfolio selection model that is equivalent to the MAD model, but with half the number of constraints. We thus consider the following Fixed-Size Minimum Semi-MAD (FSMSMAD) model

$$\begin{array}{ll} \min & SMAD(x) \\ s.t. & u^T x = 1 \\ & u^T y = K \\ & \ell y \leq x \leq y, \\ & y \in \{0,1\}^n \end{array} \tag{5}$$

where ℓ plays the same role as in (2). Using the linearization approach described in Speranza (1993), we can reformulate this problem as a MILP problem with $n + T$ continuous variables, n binary variables and $n + T + 3$ constraints (see Cesarone et al (2014)). From the computational experiences reported in Cesarone et al (2015) it appears that also this model, although slightly easier than the previous one, cannot easily be solved with general purpose state-of-the-art solvers when more than a few hundreds variables are involved.

3 Empirical behavior of the models

In this section we test the models described above on some publicly available data sets.

The analysis consists of two parts. First, we examine the behavior of the portfolios selected by the models on the in-sample window where we obtain the input parameters of the models. The second part consists in evaluating the out-of-sample performance of the portfolios, which is the aspect that matters most to investors.

	Data set	♯ of assets	time interval	frequency	source
1	FF25	25	07/1963-12/2004	monthly	K. French
2	48Ind	48	07/1963-12/2004	monthly	K. French
3	100Ind	100	01/1969-12/2011	monthly	K. French
4	Stoxx50	32	01/2007-05/2013	weekly	Yahoo Finance
5	FtseMib	34	01/2007-05/2013	weekly	Yahoo Finance
6	Ftse100	63	01/2007-05/2013	weekly	Yahoo Finance

Table 1: List of data sets analyzed.

Since the markets are in continuous evolution, it seems appropriate to rebalance the portfolio from time to time in order to take new information into account. For this purpose, we use a Rolling Time Window procedure (RTW), i.e., we shift the in-sample window (and consequently the out-of-sample window) all over the time length of each data set. More specifically, we consider a time window (in-sample period) of 200 observations for the data sets with weekly frequency, and of 120 observations for the data sets with monthly frequency. The choice of the lengths of the in-sample and of the out-of-sample windows is based on typical settings of portfolio selection problems (see, e.g., Bruni et al, 2012, 2013; Cesarone et al, 2015; DeMiguel et al, 2009a). Then we solve the selection problem for overlapping windows built by moving forward in time with step size 4 (for the weekly data sets) or 1 (for the monthly data sets). The optimal portfolio found w.r.t. an in-sample period is held for the following 4 weeks (out-of-sample period of the weekly data sets) or 1 month (out-of-sample period of the monthly data sets).

The out-of-sample performances of the resulting portfolios are evaluated in different ways by computing some performance measures commonly used in the literature (Rachev et al, 2008). Let $x^* = (x_1^*, \dots, x_n^*)$ denote the allocation of the selected portfolio and $r_t = (r_{1t}, \dots, r_{nt})$ denote the assets returns at time t. Then, in our analysis we consider:

- the **Standard Deviation** of the selected portfolio return;
- the **Sharpe Ratio** as $\frac{E[x^* r_t' - r_f]}{Std[x^* r_t' - r_f]}$, where $r_f = 0$;
- the **Rachev Ratio** as $\frac{CVaR_\alpha[r_f - x^* r_t']}{CVaR_\beta[x^* r_t' - r_f]}$, where $r_f = 0$ and $\alpha = \beta = 0.1$;
- the **Max Drawdown** as $-\min x^* r_t'$ which is the maximum loss achieved by a portfolio during the holding period.

In our analysis we use six data sets, summarized in Table 1. The monthly data sets (FF25, 48Ind, 100Ind) are taken from Ken French's website[1]. The weekly data sets (Stoxx50, FtseMib, Ftse100) are downloaded from http://finance.yahoo.com, and are publicly available at http://host.uniroma3.it/docenti/cesarone/DataSets.htm.

[1] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

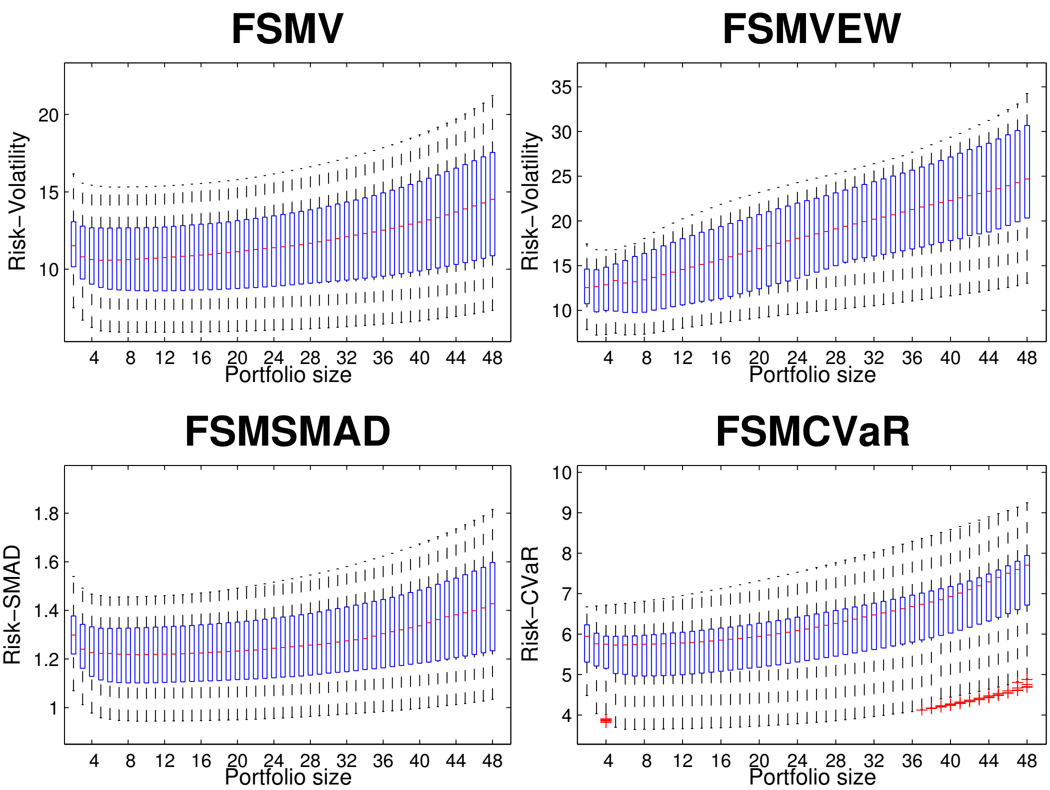


Figure 1: Boxplot of the in-sample risk w.r.t. the portfolio size for 48Ind.

3.1 In-sample analysis

For each model described in Section 2 we study the behavior of its optimal value (minimum risk) when varying the number K of assets in the portfolio.

One of our main empirical findings is the scarce effect of diversification in terms of risk reduction when the portfolio size K does not belong to a certain range of values. Indeed, in all analyzed markets, we find that the risk measures, representing the objective functions of the models, achieve minimum values for a range of portfolio sizes corresponding to a significantly limited number of assets w.r.t. the total. Furthermore, these risk measures tend to increase when increasing the portfolio size, thus contrasting the paradigm that the larger the diversification, the lower the risk. In Figs. 1 and 2 we report some empirical evidences of this phenomenon for monthly (48Ind) and for weekly (Stoxx50) data sets. However, this behavior is similar for each data set analyzed. Fig. 1 exhibits the boxplots of the different risk measures w.r.t. all considered in-sample windows by varying the portfolio size K. This means that, e.g., in the case of the 48Ind data set, for a fixed K we have 377 values of risk, one for each in-sample window (i.e., one for each rebalancing of the portfolio). Similarly we obtain Fig. 2, where we examine the Stoxx50 data set. Note however that in the cases of weekly data sets for a fixed K we have 32 in-sample windows (i.e., 32 values of risk). As mentioned above, the boxplot of the in-sample volatility generated by the EW portfolios corresponds to that of the FSMVEW portfolios when $K = n$, and it generally presents the highest median volatility. This feature is common

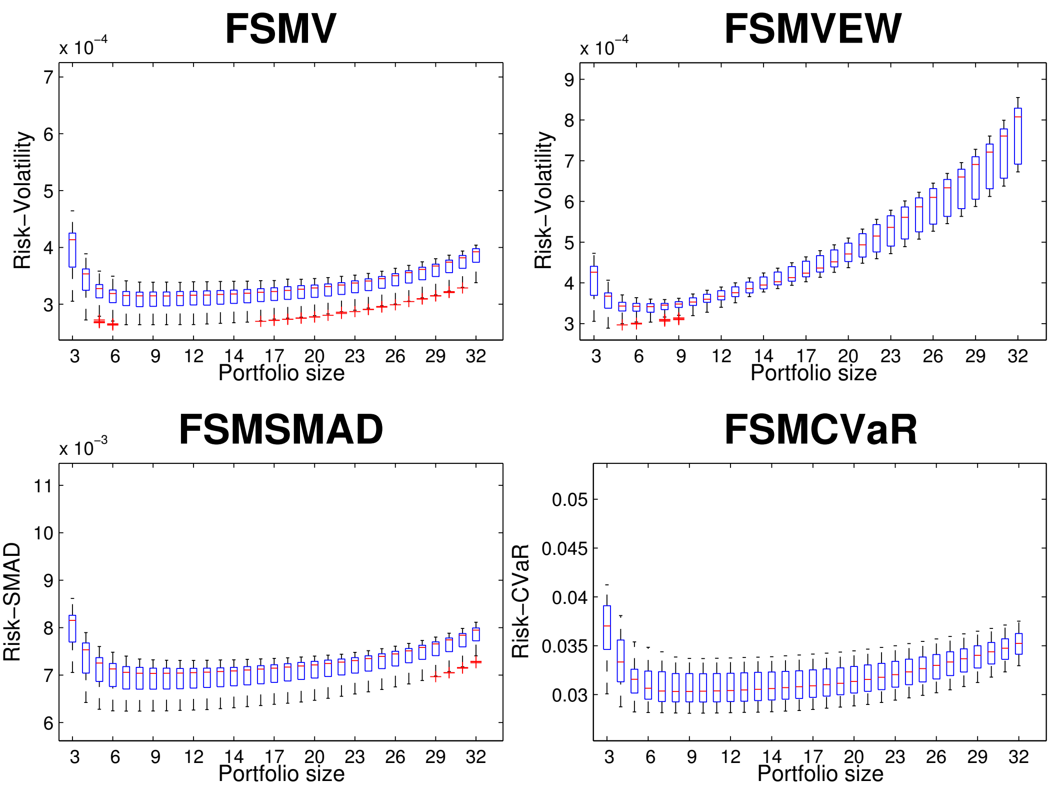


Figure 2: Boxplot of the in-sample risk w.r.t. the portfolio size for Stoxx50.

to all data sets, and it suggests that a greater diversification does not always imply a risk reduction, i.e., increasing the number of assets in the portfolio could worsen its in-sample performance in terms of risk.

The empirical results of the FSMVEW portfolios could be compared to the findings obtained by Evans and Archer, and by further influential experiments in the literature such as the well-known Fama's experiment (Fama, 1976). The author finds that, in a market with 50 stocks, the effect of *naïve diversification* determines a remarkable reduction of the portfolio in-sample volatility, but only when including in the portfolio up to 20 stocks. We refer to *naïve diversification* as an EW strategy with a random selection of K out of n available stocks. Indeed, he observes that adding further stocks in the portfolio does not yield a considerable improvement. More precisely, Fama claims that approximately 95% of the possible reduction deriving from diversification is achieved passing from 1 to 20 assets. However, we point out that our approach is significantly different from that of Fama, as well as from that of Evans and Archer. Indeed, we overcome one of the main weaknesses of their approach which consists in stating results that are valid only on average. In other words, if one picks an arbitrary EW portfolio of a given size in a market, there is no guarantee that its risk will not be much larger than the average risk of all portfolios of the same size in that market. While the results obtained by the FSMVEW portfolios are those corresponding to the best Equally-Weighted portfolios for each given size with respect to volatility. The findings on the FSMVEW model also highlight that when the EW strategy is combined with risk minimization (instead of a randomly selection of K

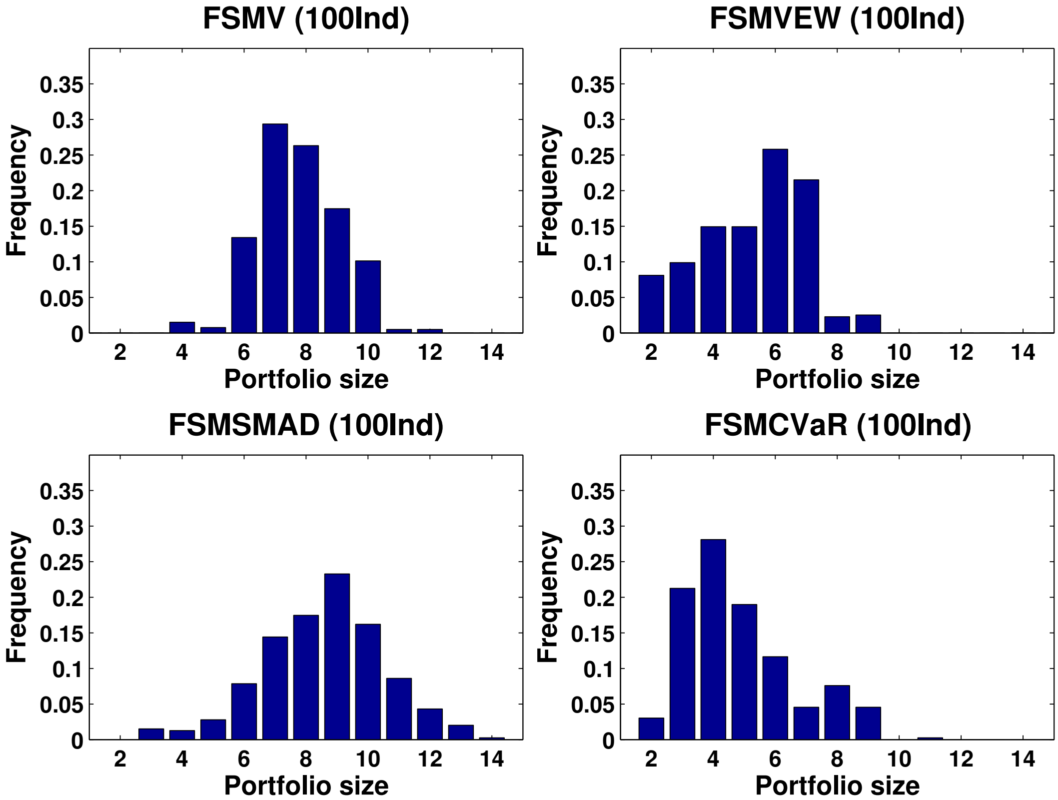


Figure 3: Distribution of the portfolio size corresponding to the global minimum risk.

out of n available stocks) the selected small portfolios show an improvement both in terms of volatility and of robustness of its values obtained on all in-sample windows.

In addition, once we have obtained the optimal size of the minimum risk portfolio for each in-sample window, we examine the range spanned by these optimal sizes. In Fig. 3 we show for the 100Ind data set the distribution of the optimal portfolio sizes (i.e., corresponding to the global minimum risk) for all models analyzed w.r.t. all in-sample windows. We can see that the global minimum risk portfolio never exceeds 15 stocks for the 100Ind data set. However, this behavior is almost the same in all the other considered data sets, with the only exception of Ftse100, where the optimal portfolio size is seldom around 20 stocks. Furthermore, given the global minimum risk on an in-sample window, we detect the smallest size of a portfolio whose risk is not more than 1% larger than that of the minimum risk portfolio, thus finding even smaller portfolios with satisfactory risk level. Then, we repeat this procedure for each in-sample window and for each portfolio model. In Fig. 4 we report the distribution of these *101% min-risk* optimal portfolio sizes for each model analyzed w.r.t. all in-sample windows. More precisely, for each in-sample window we consider all the cardinalities for which the corresponding portfolio has a risk at most 1% greater than that of the minimum risk portfolio. As highlighted from the four sub-figures (one for each portfolio), the *101% min-risk* portfolios generally show a significant risk reduction with 10 stocks for 100Ind data set. Furthermore, in most of the cases we can achieve it with just 6 stocks. However, the *101% min-risk* portfolio for the

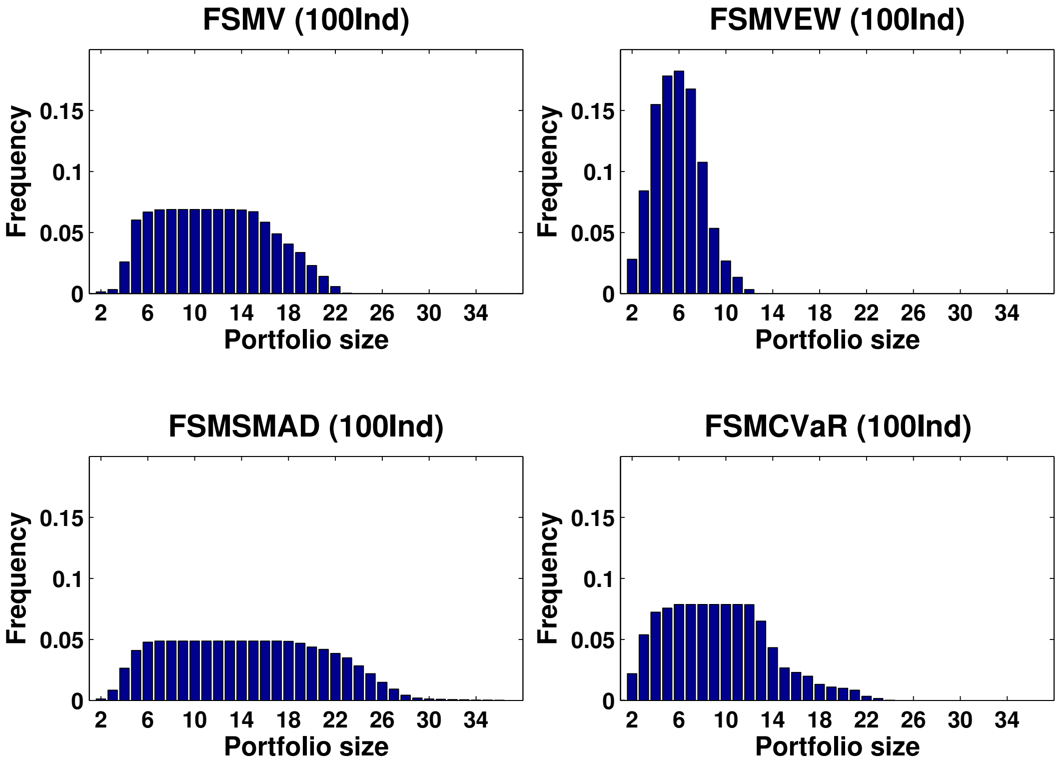


Figure 4: Distribution of the *101% min-risk* portfolio.

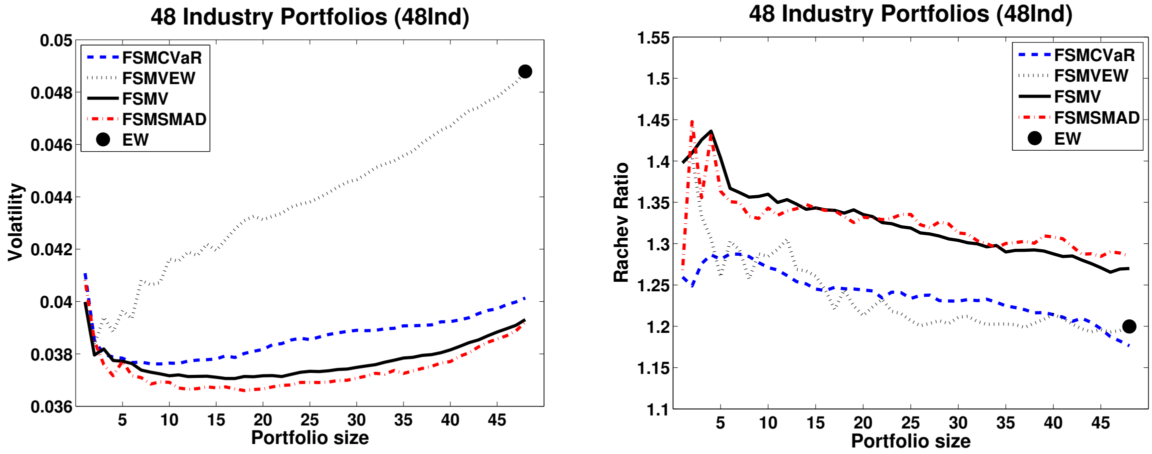
other data sets never exceeds a size of 15, and generally are needed at most 10 stocks.

The most compelling result emerging from the in-sample analysis is the existence of a portfolio size range (whose location could depend on the number of assets for each market) where one can generally find the lowest values of risk for all models considered. Indeed, we find that the smallest risk portfolios generally require no more than 15 assets. Furthermore, it is almost always possible to find portfolios that are just 1% more risky than the smallest risk portfolios and contain no more than 10 assets.

3.2 Out-of-sample analysis

The second part of our analysis concerns the out-of-sample behavior of the portfolios. Our main goal is to confirm the finding, emerged from the in-sample analysis, that we can improve performances without investing in a large number of stocks.

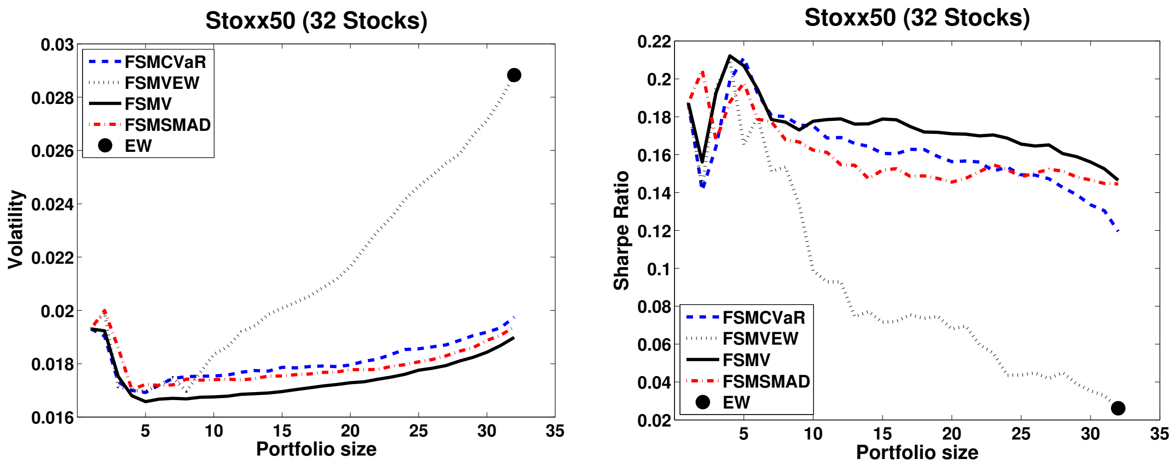
Again, we consider the EW portfolio as a benchmark and, instead of focusing only on volatility reduction, we also compute the performance indices described in Section 3. We start by verifying the behavior of the out-of-sample standard deviation. More precisely, we check whether this performance measure reaches an optimal value, or at least a good value, for small-size portfolios. We can see in Figs. 5a and 6a that, both for monthly (48Ind) and for weekly (Stoxx50) data sets, the standard deviations of the portfolios returns reach their minima for small sizes. For larger sizes, the portfolios volatility tends to increase with different growth rates. These increases, except for the FSMVEW, are due to the buy-



(a) Volatility.

(b) Rachev ratio.

Figure 5: Analysis of the out-of-sample portfolio returns for 48Ind.



(a) Volatility.

(b) Sharpe ratio.

Figure 6: Analysis of the out-of-sample portfolio returns for Stoxx50.

in threshold constraints. Without these constraints we should expect nearly flat curves. However, the buy-in threshold constraints are necessary to eliminate unrealistically small trades that can otherwise be included in an optimal portfolio. In Figs. 5b and 6b we show the values of two other performance measures, namely the Rachev and Sharpe ratios for the same data sets. As for the standard deviation, each model generally tends to provide the best values of the latter performance measures for small sizes. Furthermore, these values almost always decay when the portfolio size approaches n. This behavior provides a further support to the idea of improving the performances of a portfolio by limiting the number of its stocks.

In addition to the graphical evidence, where only the most representative results are shown, we also performed an extensive comparative analysis on all data sets considered. Since describing the results for all data sets and for all portfolio sizes is impractical, we

$\boldsymbol{K}$**=5**	**FF25**	**48Ind**	**100Ind**	**Ftse100**	**FtseMib**	**Stoxx50**
FSMCVaR	0.0448	0.0378	*0.0556*	**0.0134**	0.0185	0.0169
FSMVEW	0.0431	0.0397	**0.0424**	0.0139	0.0181	0.0170
FSMV	0.0431	**0.0377**	0.0443	0.0141	**0.0169**	**0.0166**
FSMSMAD	**0.0430**	**0.0377**	0.0446	0.0136	0.0177	0.0172
EW	*0.0509*	*0.0488*	0.0512	*0.0210*	*0.0296*	*0.0288*
$\boldsymbol{K}$**=10**	**FF25**	**48Ind**	**100Ind**	**Ftse100**	**FtseMib**	**Stoxx50**
FSMCVaR	0.0447	0.0376	*0.0536*	0.0133	0.0189	*0.0175*
FSMVEW	0.0441	0.0416	**0.0428**	**0.0125**	0.0203	0.0184
FSMV	**0.0432**	0.0372	0.0438	0.0126	0.0170	**0.0168**
FSMSMAD	**0.0432**	**0.0369**	0.0440	0.0128	**0.0167**	0.0174
EW	*0.0509*	*0.0488*	0.0512	*0.0210*	*0.0296*	0.0288
$\boldsymbol{K}$**=15**	**FF25**	**48Ind**	**100Ind**	**Ftse100**	**FtseMib**	**Stoxx50**
FSMCVaR	0.0448	0.0378	*0.0537*	0.0134	0.0188	*0.0179*
FSMVEW	0.0458	0.0419	**0.0435**	0.0128	0.0218	0.0201
FSMV	**0.0433**	0.0371	0.0439	0.0127	0.0173	**0.0170**
FSMSMAD	0.0434	**0.0367**	0.0436	**0.0126**	**0.0171**	0.0175
EW	*0.0509*	*0.0488*	0.0512	*0.0210*	*0.0296*	0.0288

Table 2: Standard Deviation of the out-of-sample returns.

report here the out-of-sample analysis for only three fixed sizes: $K = 5, 10, 15$. This choice is based on the observation that $K = 5, 10, 15$ generally belong to the optimal ranges in which the various models achieve the in-sample lowest risk for each data set. In Table 2 we provide the standard deviation of the out-of-sample returns for $K = 5, 10, 15$ for each model and data set analyzed. It is remarkable that the EW portfolio has almost always the worst performance, with the single exception of the 100Ind market, where the FSMCVaR portfolios generate the highest standard deviation. In Table 3 we report the Sharpe ratio of the out-of-sample returns for the same portfolio sizes of the previous table and for each model and data set analyzed. Note that when the portfolio excess return is negative some gain-to-risk ratios have no meaning, thus we report "-". Again we observe that the EW portfolio yields the worst performances compared with those of the other models, with the exception of the FSMCVaR portfolios for the 100Ind and FF25 markets.

Similar considerations can be made about the Rachev Ratio of the out-of-sample returns shown in Table 4. Indeed, again the EW portfolio tends to be the worst choice, with the only exception of the 100Ind data set. We also observe that for $K = 10$ the FSMV model seems to be preferable since it provides the best results for 4 data sets out of 6, while for $K = 15$ it presents the best performances for 5 data sets out of 6.

The last performance measure considered in our analysis is the Max drawdown, which is the worst out-of-sample loss achieved by a portfolio, as described in Section 3. Table 5 shows that, again, the EW portfolio always has the worst performance for the prescribed sizes $K = 5, 10, 15$, with the exception of the 100Ind market, where the FSMCVaR portfolios provide the worst loss. On the other hand, although there is not a clear superiority of a single model, we observe that the FSMV portfolios present the best values for 3 data sets out of 6 for $K = 5$, and for 4 data sets out of 6 for $K = 10$ and for $K = 15$.

K=5	FF25	48Ind	100Ind	Ftse100	FtseMib	Stoxx50
FSMCVaR	*0.244*	0.279	*0.133*	**0.334**	0.124	**0.211**
FSMVEW	0.267	0.285	**0.249**	0.247	0.119	0.165
FSMV	0.268	**0.293**	**0.249**	0.257	**0.126**	0.207
FSMSMAD	**0.276**	0.293	0.247	0.244	0.086	0.198
EW	0.264	*0.242*	0.215	*0.173*	–	*0.026*
K=10	**FF25**	**48Ind**	**100Ind**	**Ftse100**	**FtseMib**	**Stoxx50**
FSMCVaR	*0.248*	0.278	*0.142*	**0.298**	0.120	0.175
FSMVEW	**0.290**	0.268	**0.256**	0.284	0.091	0.099
FSMV	0.27	0.289	0.242	0.276	0.112	**0.178**
FSMSMAD	0.276	**0.294**	0.253	0.266	**0.122**	0.162
EW	0.264	*0.242*	0.215	*0.173*	–	*0.026*
K=15	**FF25**	**48Ind**	**100Ind**	**Ftse100**	**FtseMib**	**Stoxx50**
FSMCVaR	*0.250*	0.274	*0.143*	**0.281**	**0.124**	0.161
FSMVEW	**0.296**	0.260	0.249	0.278	0.042	0.072
FSMV	0.272	0.289	0.242	0.269	0.098	**0.179**
FSMSMAD	0.280	**0.293**	**0.257**	0.260	0.109	0.152
EW	0.264	*0.242*	0.215	*0.173*	–	*0.026*

Table 3: Sharpe Ratio of the out-of-sample returns.

K=5	FF25	48Ind	100Ind	Ftse100	FtseMib	Stoxx50
FSMCVaR	1.216	1.281	*0.877*	**1.618**	1.162	1.167
FSMVEW	1.201	1.260	**1.049**	1.614	1.048	1.101
FSMV	1.245	**1.403**	1.043	1.514	**1.164**	1.205
FSMSMAD	**1.25**	1.363	1.032	1.382	1.07	**1.218**
EW	*1.150*	*1.200*	0.992	*1.007*	*0.861*	*0.910*
K=10	**FF25**	**48Ind**	**100Ind**	**Ftse100**	**FtseMib**	**Stoxx50**
FSMCVaR	1.215	1.271	*0.892*	1.459	1.145	1.104
FSMVEW	1.199	1.284	**1.047**	1.477	0.963	1.007
FSMV	**1.244**	**1.360**	1.027	**1.514**	**1.147**	1.150
FSMSMAD	1.234	1.343	1.033	1.333	1.147	**1.185**
EW	*1.150*	*1.200*	0.992	*1.007*	*0.861*	*0.910*
K=15	**FF25**	**48Ind**	**100Ind**	**Ftse100**	**FtseMib**	**Stoxx50**
FSMCVaR	1.217	1.245	*0.896*	1.397	1.123	1.064
FSMVEW	1.195	1.261	**1.052**	1.347	0.861	*0.907*
FSMV	**1.244**	**1.343**	1.024	**1.426**	**1.135**	**1.135**
FSMSMAD	1.234	1.343	1.047	1.320	1.103	1.132
EW	*1.150*	*1.200*	0.992	*1.007*	*0.861*	0.910

Table 4: Rachev Ratio of the out-of-sample returns.

K=5	FF25	48Ind	100Ind	Ftse100	FtseMib	Stoxx50
FSMCVaR	0.209	0.150	*0.578*	0.027	0.042	0.031
FSMVEW	0.214	0.175	**0.201**	**0.029**	0.045	0.036
FSMV	**0.198**	0.123	0.206	0.030	**0.037**	**0.031**
FSMSMAD	0.201	**0.121**	0.206	0.039	0.041	0.032
EW	*0.261*	*0.259*	0.262	*0.052*	*0.064*	*0.061*
K=10	**FF25**	**48Ind**	**100Ind**	**Ftse100**	**FtseMib**	**Stoxx50**
FSMCVaR	0.210	0.145	*0.501*	0.033	0.042	0.034
FSMVEW	0.219	0.201	**0.201**	0.029	0.041	0.041
FSMV	**0.198**	0.126	0.209	**0.023**	**0.039**	**0.031**
FSMSMAD	0.203	**0.125**	0.212	0.031	0.039	0.034
EW	*0.261*	*0.259*	0.262	*0.052*	*0.064*	*0.061*
K=15	**FF25**	**48Ind**	**100Ind**	**Ftse100**	**FtseMib**	**Stoxx50**
FSMCVaR	0.212	0.141	*0.493*	0.029	0.040	0.036
FSMVEW	0.235	0.206	0.213	**0.027**	0.050	0.045
FSMV	**0.199**	0.129	**0.210**	0.027	**0.039**	**0.031**
FSMSMAD	0.203	**0.127**	**0.210**	0.027	0.040	0.035
EW	*0.261*	*0.259*	0.262	*0.052*	*0.064*	*0.061*

Table 5: Max drawdown of the out-of-sample returns.

4 Conclusions

The concept of diversification is not well-defined and the measures of diversification are continuously evolving (see, e.g., Fragkiskos, 2013; Meucci, 2009, and references therein). However, the qualitative idea of diversification is to not overly concentrate the investments in very few stocks. Indeed, the role of diversification is to reduce risk by diversifying it as much as possible.

In this work we investigated the possible benefits and disadvantages of enlarging the portfolio size in several portfolio selection models with respect to various measures of performance. Similar to various previous findings, but with a substantially different approach, our empirical results show that in most cases limiting the size of the selected portfolio improves both the in-sample and the out-of-sample performance. We might call this a "*small portfolio effect*". These results are somewhat in line with the tendency described by DeMiguel et al (2009a), where an improved out-of-sample performance is often observed for the 1-norm-constrained minimum-variance portfolios. The analogy is based on the observation that the 1-norm is often regarded as an approximation of the 0-norm, i.e., the size of the portfolio.

Further studies are underway to investigate the validity of this *small portfolio effect* with respect to other risk and performance measures and in larger markets.

References

Acerbi C, Tasche D (2002) On the coherence of expected shortfall. J Bank Financ 26:1487–1503

Angelelli E, Mansini R, Speranza M (2008) A comparison of MAD and CVaR models with real features. J Bank Financ 32:1188–1197

Boros E, Hammer PL (2002) Pseudo-boolean optimization. Discrete Appl Math 123:155–225

Bruni R, Cesarone F, Scozzari A, Tardella F (2012) A new stochastic dominance approach to enhanced index tracking problems. Econ Bull 32:3460–3470

Bruni R, Cesarone F, Scozzari A, Tardella F (2013) No arbitrage and a linear portfolio selection model. Econ Bull 33:1247–1258

Cesarone F, Scozzari A, Tardella F (2009) Efficient algorithms for mean-variance portfolio optimization with hard real-world constraints. G Ist Ital Attuari 72:37–56

Cesarone F, Scozzari A, Tardella F (2013) A new method for mean-variance portfolio optimization with cardinality constraints. Ann Oper Res 205:213–234

Cesarone F, Moretti J, Tardella F (2014) Does Greater Diversification Really Improve Performance in Portfolio Selection? Available at SSRN: http://ssrncom/abstract=2473630

Cesarone F, Scozzari A, Tardella F (2015) Linear vs. quadratic portfolio selection models with hard real-world constraints. Comput Manage Sci 12:345–370

DeMiguel V, Garlappi L, Nogales F, Uppal R (2009a) A generalized approach to portfolio optimization: Improving performance by constraining portfolio norms. Manage Sci 55:798–812

DeMiguel V, Garlappi L, Uppal R (2009b) Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy? Rev Financ Stud 22:1915–1953

Evans J, Archer S (1968) Diversification and the Reduction of Dispersion: An Empirical Analysis. J Financ 23:761–767

Fama E (1976) Foundations of finance: Portfolio decisions and securities prices. Basic Books New York

Fragkiskos A (2013) What is Portfolio Diversification? Available at SSRN: http://ssrncom/abstract=2331475

Ivković Z, Sialm C, Weisbenner S (2008) Portfolio concentration and the performance of individual investors. J Financ Quant Anal 43:613–655

Jacob N (1974) A limited-diversification portfolio selection model for the small investor. J Financ 29:847–856

Konno H, Yamazaki H (1991) Mean-absolute deviation portfolio optimization model and its application to Tokyo stock exchange. Manage Sci 37:519–531

Markowitz H (1952) Portfolio selection. J Financ 7:77–91

Markowitz H (1959) Portfolio selection: Efficient diversification of investments. Cowles Foundation for Research in Economics at Yale University, Monograph 16, John Wiley & Sons Inc., New York

Meucci A (2009) Managing diversification. Risk 22:74–79

Michaud R (1989) The Markowitz Optimization Enigma: Is 'Optimized' Optimal? Financ Anal J 45:31–42

Newbould G, Poon P (1993) The minimum number of stocks needed for diversification. Finan Practice Educ 3:85–87

Pflug G, Pichler A, Wozabal D (2012) The 1/N investment strategy is optimal under high model ambiguity. J Bank Financ 36:410–417

Rachev S, Stoyanov S, Fabozzi F (2008) Advanced stochastic models, risk assessment, and portfolio optimization: The ideal risk, uncertainty, and performance measures, vol 149. Wiley

Rockafellar R, Uryasev S (2000) Optimization of Conditional Value-at-Risk. J Risk 2:21–42

Speranza M (1993) Linear programming models for portfolio optimization. Finance 14:107–123

Statman M (1987) How many stocks make a diversified portfolio? J Financ Quant Anal 22:353–363

Tang G (2004) How efficient is naive portfolio diversification? an educational note. Omega 32:155–160