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Attractor misspecification and threshold estimation bias

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Abstract

Three regime threshold autoregressive models such as the BAND-TAR and EQ-TAR (Balke & Fomby, 1997) are commonly used when studying arbitrage in the presence of trade frictions because the estimated thresholds represent the size of the impediments to arbitrage. This paper shows that, while commonly overlooked, the attractors in these models play an important role in threshold estimation. In particular, misspecified attractors cause systematic biases in estimated thresholds. This paper proposes a generalized three regime TAR model that nests both the BAND-TAR and the EQ-TAR models and allows the attractor to be freely estimated. Simulations suggest that the generalized model mitigates the biases that arise when the attractor is misspecified.

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1. Introduction

Nonlinear time series models are increasingly being used in applied work. In particular, the threshold autoregressive (TAR) model is one of the more commonly used form of this type of model. For instance, Hansen (2011) has demonstrated the broad influence of the TAR model in economic research. The study of arbitrage in the presence of transportation costs is a specific area that has benefited from the use nonlinear time series analysis. For example, the TAR model has been used to study index arbitrage (see Dwyer, Locke, & Yu, 1996, Martens, Kofman, & Vorst, 1998), foreign exchange arbitrage (see Canjels, Prakash-Canjels, & Taylor, 2004, Chappell, Padmore, Mistry, & Ellis, 1996), international goods market arbitrage (see Obstfeld & Taylor, 1997, Sarno, Taylor, & Chowdhury, 2004), and arbitrage in early financial markets (see Bernholz & Kugler, 2011, Norman & Wills, 2015, Volckart & Wolf, 2006) among others. The three regime TAR model has been useful when studying arbitrage, in part, because the threshold estimates can be interpreted as the value of the price difference at which arbitrage becomes profitable. The distance between thresholds then represents a measure of the impediments to trade.

This paper shows that the attractors in the popularly used Equilibrium-TAR and Band-TAR models proposed by Balke & Fomby (1997) (BF) play an important role in threshold estimation. In particular, it is shown that because the attractors in these two models are fixed, biases in the estimated thresholds can arise if the wrong model is used. This is especially important because economic theory has little to say regarding which of the two models is appropriate in any given situation. This paper proposes using a threshold model which allows the attractor to be part of the estimation process. This generalized model nests both the Equilibrium-TAR and Band-TAR models. Evidence is provided which suggests that allowing the attractor to be estimated mitigates the potential bias in the threshold estimation. A strategy for using these models in applied work is also provided.

2. BAND and EQ TAR models

When examining arbitrage in the presence of trade frictions, TAR models are usually employed in the context of cointegration analysis. The principle of nonlinear cointegration can be demonstrated with a simple bivariate model similar to the one used in van Dijk & Franses (2000),

$$y_t + \beta x_t = z_t, \quad z_t = (\rho_1 + \rho_2 R(z_{t-d}))z_{t-1} + \varepsilon_t \tag{1}$$

$$y_t + \alpha x_t = w_t, \quad w_t = w_{t-1} + \eta_t, \tag{2}$$

for $t = 1, ..., T$ where ε_t and η_t are i.i.d mean zero random variables. Here the cointegrating vector is $(1, \beta)$ and w_t is the common stochastic trend of y_t and x_t. The long run equilibrium is $y_t = -\beta x_t$, and thus the deviation from the equilibrium is z_t. Cointegration between y_t and x_t implies z_t is stationary. When $R(z_{t-d}) = 0$ for all z_{t-d} the model represents conventional linear cointegration. In the linear case y_t and x_t are cointegrated if $|\rho_1| < 1$.

For nonlinear cointegration, the transition function, $R(z_{t-d})$, is a function that is bounded between 0 and 1. The transition variable, z_{t-d}, controls the dynamic behavior of the deviation from equilibrium. When $R(z_{t-d}) = 1$, the rate at which the long run equilibrium is established is based upon the value $\rho_1 + \rho_2$, and when $R(z_{t-d}) = 0$ the rate is based upon the value of just ρ_1.

BF propose the Equilibrium-TAR (EQ-TAR) model:

$$z_t = \rho_1 z_{t-1} + I(|z_{t-d}| \geq \gamma)\rho_2 z_{t-1} + \varepsilon_t. \tag{3}$$

In this case the equilibrium error behaves differently whether z_{t-d} it is within $[-\gamma, \gamma]$ or not. BF focus on the case where $\rho_1 = 1$ and $|\rho_2| < 1$. These values imply that z_t follows a simple random walk when z_{t-d} is within the band and mean reverts to zero when it is outside the band. BF also posit the Band-TAR model:

$$z_t = \begin{cases} \gamma(1-\rho) + \rho z_{t-1} + \varepsilon_t & \text{if } z_{t-d} > \gamma \\ z_{t-1} + \varepsilon_t & \text{if } \gamma \geq z_{t-d} \geq -\gamma \\ -\gamma(1-\rho) + \rho z_{t-1} + \varepsilon_t & \text{if } z_{t-d} < -\gamma. \end{cases} \tag{4}$$

In this model, the time series process follows a unit root when it is smaller in magnitude than the threshold, γ. When the process is larger than γ it reverts towards γ, and when the process is smaller than $-\gamma$ it reverts towards $-\gamma$.

The major difference between these two models is the "attractor" or direction of reversion. While the Band-TAR model shows reversion towards the nearest threshold, the EQ-TAR model reverts to the equilibrium. This is illustrated in Figures 1-2. The Band-TAR autogregressive function is continuous, while the EQ-TAR model exhibits discontinuities at the thresholds. A more generalized model, which nests both the EQ and Band threshold models, can be formulated by adding a parameter which controls the value of the attractor. If κ is the attractor then the TAR model would have the following form,

$$z_t = \begin{cases} \kappa(1-\rho) + \rho z_{t-1} + \varepsilon_t & \text{if } z_{t-d} > \gamma \\ z_{t-1} + \varepsilon_t & \text{if } \gamma \geq z_{t-d} \geq -\gamma \\ -\kappa(1-\rho) + \rho z_{t-1} + \varepsilon_t & \text{if } z_{t-d} < -\gamma. \end{cases} \tag{5}$$

In this paper, the above model will be known as the Gen-TAR model. Note that when $\kappa = \gamma$ then the Gen-TAR model is the same as the Band-TAR and when $\kappa = 0$ it is the EQ-TAR model. While there is a cost of having to estimate one additional parameter, the Gen-TAR model could potentially be used to discriminate between the the EQ and Band TAR models and prevent any complications that could arise from misspecification.[1] This could be done by simple examining the estimated value of κ.

[1]In empirical work, it appears that the Band-TAR model is more commonly used than the EQ-TAR model. Some studies do use both models such as Lo & Zivot (2001), Lo (2008), and Federico (2012).

While there is a clear theoretical interpretation of the threshold in a three regime TAR model, there is no clear economic meaning of the attractor. It is by definition the focus of the direction of the time series when it is outside of the inner band. Economic theory has little if anything to say with respect to what the value of the attractor should be, whether γ, zero, or some value in between. This could be viewed as further justification for the use of the GEN-TAR model where the attractor is freely estimated and not arbitrarily set to a specific value.

3. Monte Carlo Simulations

Monte Carlo simulations were performed to evaluate the performance of the three TAR models in a variety of circumstances. Data was created by setting $y_0 = 0$, iterating the data generating process (DGP) $T + 250$ times, and then discarding the first 250 observations so that the final data set had T observations. The DGP for the EQ-TAR and Band-TAR are given in equations 3 and 4. The error term is an iid standard normal random variable, or $\varepsilon_t \sim N(0, \sigma)$ with $\sigma = 1$. As is noted by van Dijk, Teräsvirta, & Franses (2002), in the case of smooth transition autoregressive models, when γ is divided by σ the estimate of γ become approximately scale free. As a result, only the values of γ will be adjusted between simulations while the distribution of the error term remains unchanged.

Hansen (1997) describes the method of estimating TAR models. For a given threshold value, γ, the model can be estimated by OLS. The threshold is chosen among $\gamma \in [\underline{\gamma}, \overline{\gamma}]$ such that a given percentage of the observations are in each of the outer and inner regimes. The threshold associated with the lowest combined sum of squared errors among the three regimes is chosen as the estimated threshold. The values of $\underline{\gamma}$ and $\overline{\gamma}$ are chosen to be the 15th and 85th percentile.

Tables 1 and 2 contain the simulation results for the cases when the DGP is Band-TAR and EQ-TAR respectively. The simulations were run with $T = \{200, 500, 1000\}$, $\rho = \{0.1, 0.5, 0.9\}$, and $\gamma = \{0.5, 1.0, 2.0\}$. In each case, the Band-TAR, EQ-TAR, and Gen-TAR models were estimated with the simulated data. The average values of $\hat{\sigma}$, $\hat{\rho}$, and $\hat{\gamma}$ are reported. The true threshold values were chosen so that they would be well identified with respect to the distribution of the data. In other words, values of γ are set so that they are found within the distribution and not near the edges or outside of the distribution. Figures 3 and 4 show the thresholds plotted against the distribution of the DGP.[2]

When the DGP is Band-TAR and correctly specified, there is a positive bias in the estimated threshold when the value of ρ is large although this bias does appear to diminish as the sample size increases. This positive bias also appears to be inversely related to the size of the threshold. The most likely cause of this bias is the fact that both a larger value of

[2]The distribution of of the DGP was estimated by setting $y_0 = 0$ and then iterating the DGP $100,250$ times. The first 250 observations were then discarded and a distribution was constructed using a normal kernel density based upon the remaining 100,000 observations.

ρ and a smaller value of γ increases the number of observations in the outer regimes. This can be seen in Figure 3. It has been noted by Norman (2008) that an uneven distribution of observations between regimes can cause a small sample bias in threshold estimation.

Under misspecification, when the DGP is the Band-TAR and the EQ-TAR model is estimated, there is also a positive bias in the estimated threshold, but it is much larger and the reduction in the bias is much smaller for large samples. When the Gen-TAR model is estimated the bias is in between that of the Band-TAR model and the EQ-TAR model in almost every case. In other words, the Gen-TAR model performs better that the incorrectly EQ-TAR specified model, but not as well as the correctly specified Band-TAR model. The estimated value of κ is also reported when the Gen-TAR model is estimated. The attractor is best estimated when the sample size is large and ρ is small.

When the DGP is EQ-TAR and there is no misspecification, there is a similar positive bias in estimation for large ρ and small γ. The bias also decreases with increased sample sizes. When the Band-TAR model is estimated with data created from the EQ-TAR model, the estimated threshold is always smaller than when the model is correctly specified. This negative bias is also persistent even when the sample size increases. Interestingly, when the Gen-TAR model is used, the estimated threshold is very close to the thresholds that are estimated from the EQ-Model. This suggests that, in terms of threshold estimation, using the Gen-TAR model and EQ-TAR model will result in about the same performance as when the EQ-TAR model is the DGP. As with the BAND-TAR data, the attractor is best estimated when the sample size is large and ρ is small.

These results suggest that when using these models with actual data, the practitioner who is interested in the value of a threshold might want to estimate all three models. Without knowing the correct model, in general the Band-TAR will tend to give the lowest value of the threshold estimate, the EQ-TAR model tends to give the highest value, and the Gen-TAR model will yield an estimate in between the two. In the absence of economic theory to guide the model choice, estimating all three models will give the researcher a more informed understanding of the range of possible values of a threshold.

4. Conclusion

This paper has demonstrated that the attractors in Equilibrium-TAR and Band-TAR models affect threshold estimation. It is shown that because the attractors in these two models are not estimated, biases in the thresholds estimates are observed when the incorrect model is used. This paper proposes using a generalized threshold model which allows the attractor to be part of the estimation process. Evidence was given that suggests that allowing the attractor to be estimated reduces the bias in the threshold estimation that stems from model misspecification. It is suggested that those interested in estimating thresholds should estimate all three models to better understand what the true value of the threshold is.

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5. Tables

Table 1: Band-TAR DGP Monte Carlo Simulations

		T=200			T=500			T=1000		
	$\rho =$	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9
					$\gamma = 0.5$					
Band	$\hat{\sigma}$	0.99	0.98	0.98	0.99	0.99	0.99	1.00	1.00	1.00
	$\hat{\rho}$	0.07	0.45	0.80	0.09	0.48	0.87	0.10	0.49	0.88
	$\hat{\gamma}$	0.51	0.56	1.36	0.50	0.52	0.97	0.50	0.50	0.79
EQ	$\hat{\sigma}$	0.98	0.98	0.98	1.00	0.99	0.99	1.00	1.00	1.00
	$\hat{\rho}$	0.34	0.60	0.89	0.36	0.62	0.90	0.36	0.62	0.91
	$\hat{\gamma}$	0.83	0.97	2.04	0.81	0.89	1.73	0.79	0.85	1.51
Gen	$\hat{\sigma}$	0.97	0.97	0.97	0.99	0.99	0.99	0.99	0.99	0.99
	$\hat{\rho}$	0.19	0.59	0.90	0.14	0.54	0.92	0.13	0.53	0.92
	$\hat{\gamma}$	0.66	0.88	1.98	0.58	0.74	1.72	0.54	0.66	1.50
	$\hat{\kappa}$	0.26	-0.22	1.15	0.40	0.19	-4.33	0.45	0.37	2.78
					$\gamma = 1$					
Band	$\hat{\sigma}$	0.98	0.98	0.98	0.99	0.99	0.99	1.00	1.00	1.00
	$\hat{\rho}$	0.07	0.44	0.80	0.09	0.48	0.86	0.09	0.49	0.88
	$\hat{\gamma}$	0.99	1.02	1.76	1.00	1.00	1.35	1.00	1.00	1.17
EQ	$\hat{\sigma}$	0.98	0.98	0.98	1.00	0.99	0.99	1.01	1.00	1.00
	$\hat{\rho}$	0.48	0.67	0.90	0.50	0.69	0.91	0.51	0.70	0.92
	$\hat{\gamma}$	1.38	1.49	2.44	1.39	1.46	2.20	1.39	1.45	2.03
Gen	$\hat{\sigma}$	0.97	0.97	0.97	0.99	0.99	0.99	0.99	0.99	0.99
	$\hat{\rho}$	0.20	0.57	0.89	0.15	0.54	0.92	0.14	0.53	0.92
	$\hat{\gamma}$	1.01	1.17	2.26	0.98	1.07	2.06	0.98	1.01	1.86
	$\hat{\kappa}$	-2.10	0.57	15.00	0.86	0.59	0.45	0.93	0.84	-0.43
					$\gamma = 2$					
Band	$\hat{\sigma}$	0.98	0.98	0.98	0.99	0.99	0.99	1.00	1.00	1.00
	$\hat{\rho}$	0.05	0.43	0.79	0.08	0.47	0.86	0.09	0.49	0.88
	$\hat{\gamma}$	1.97	1.97	2.58	1.99	1.98	2.24	2.00	1.99	2.09
EQ	$\hat{\sigma}$	0.99	0.98	0.98	1.00	0.99	0.99	1.01	1.00	0.99
	$\hat{\rho}$	0.66	0.77	0.91	0.67	0.78	0.93	0.67	0.79	0.93
	$\hat{\gamma}$	2.28	2.40	3.25	2.36	2.48	3.14	2.39	2.52	3.06
Gen	$\hat{\sigma}$	0.97	0.97	0.97	0.99	0.99	0.99	0.99	0.99	0.99
	$\hat{\rho}$	0.17	0.54	0.86	0.17	0.55	0.91	0.15	0.54	0.91
	$\hat{\gamma}$	1.75	1.82	2.79	1.89	1.88	2.65	1.95	1.92	2.50
	$\hat{\kappa}$	2.01	2.26	2.05	1.81	1.41	-2.53	1.89	1.76	5.67

Table 2: EQ-TAR DGP Monte Carlo Simulations

		T=200			T=500			T=1000		
	$\rho =$	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9
					γ =0.5					
Band	$\hat{\sigma}$	1.00	0.99	0.99	1.01	1.00	0.99	1.01	1.00	1.00
	$\hat{\rho}$	-0.08	0.38	0.81	-0.06	0.40	0.86	-0.06	0.41	0.88
	$\hat{\gamma}$	0.25	0.29	1.04	0.23	0.26	0.68	0.22	0.24	0.54
EQ	$\hat{\sigma}$	0.98	0.98	0.98	0.99	0.99	0.99	1.00	0.99	1.00
	$\hat{\rho}$	0.09	0.48	0.87	0.10	0.49	0.89	0.10	0.50	0.89
	$\hat{\gamma}$	0.49	0.57	1.66	0.49	0.51	1.30	0.50	0.50	1.04
Gen	$\hat{\sigma}$	0.97	0.97	0.97	0.99	0.99	0.99	0.99	0.99	1.00
	$\hat{\rho}$	0.16	0.57	0.91	0.12	0.53	0.93	0.11	0.52	0.92
	$\hat{\gamma}$	0.50	0.58	1.66	0.49	0.52	1.37	0.49	0.50	1.10
	$\hat{\kappa}$	-0.18	-0.66	6.39	-0.06	-0.20	-2.11	-0.03	-0.09	-1.74
					γ =1					
Band	$\hat{\sigma}$	1.02	1.00	0.99	1.04	1.01	1.00	1.04	1.01	1.00
	$\hat{\rho}$	-0.33	0.28	0.81	-0.32	0.30	0.86	-0.32	0.30	0.87
	$\hat{\gamma}$	0.61	0.57	1.06	0.61	0.57	0.73	0.61	0.58	0.61
EQ	$\hat{\sigma}$	0.98	0.98	0.98	0.99	0.99	0.99	1.00	1.00	0.99
	$\hat{\rho}$	0.09	0.49	0.87	0.10	0.49	0.89	0.10	0.50	0.89
	$\hat{\gamma}$	0.99	0.98	1.71	1.00	0.99	1.41	1.00	1.00	1.21
Gen	$\hat{\sigma}$	0.97	0.97	0.97	0.99	0.99	0.99	0.99	0.99	0.99
	$\hat{\rho}$	0.12	0.54	0.91	0.11	0.51	0.92	0.11	0.51	0.92
	$\hat{\gamma}$	0.96	0.92	1.71	0.99	0.96	1.45	1.00	0.99	1.23
	$\hat{\kappa}$	-0.28	-0.63	211.29	-0.08	-0.25	255.97	-0.04	-0.11	-1.45
					γ =2					
Band	$\hat{\sigma}$	1.06	1.01	0.99	1.07	1.02	0.99	1.08	1.03	1.00
	$\hat{\rho}$	-0.95	-0.07	0.79	-0.93	-0.05	0.84	-0.91	-0.04	0.85
	$\hat{\gamma}$	1.54	1.49	1.45	1.54	1.51	1.23	1.54	1.51	1.17
EQ	$\hat{\sigma}$	1.01	0.98	0.98	1.01	0.99	0.99	1.01	1.00	1.00
	$\hat{\rho}$	0.14	0.49	0.87	0.13	0.50	0.89	0.12	0.50	0.90
	$\hat{\gamma}$	1.92	1.95	2.17	1.95	1.99	2.05	1.97	2.00	1.99
Gen	$\hat{\sigma}$	1.00	0.97	0.97	1.01	0.99	0.99	1.01	1.00	0.99
	$\hat{\rho}$	-0.17	0.47	0.89	-0.09	0.50	0.91	-0.05	0.51	0.91
	$\hat{\gamma}$	1.85	1.83	1.99	1.94	1.97	1.89	1.96	1.99	1.87
	$\hat{\kappa}$	0.19	0.48	6.07	0.29	-0.90	3.53	0.30	-0.32	-1.27

6. Figures

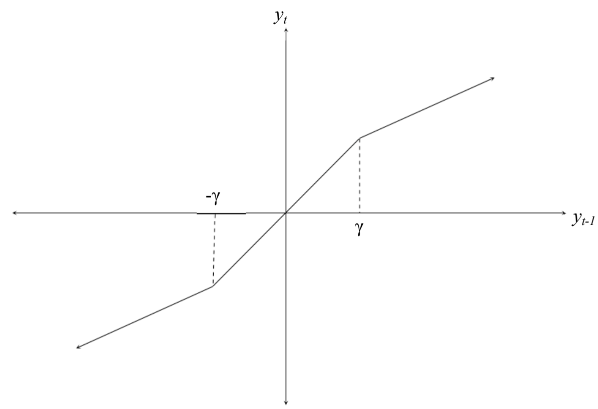

Figure 1: Band-TAR Autogregressive Function

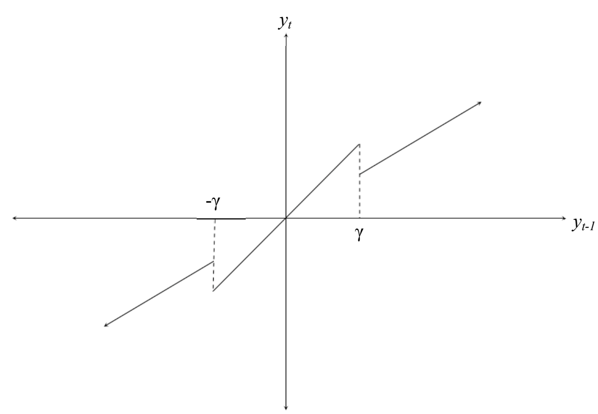

Figure 2: EQ-TAR Autogregressive Function

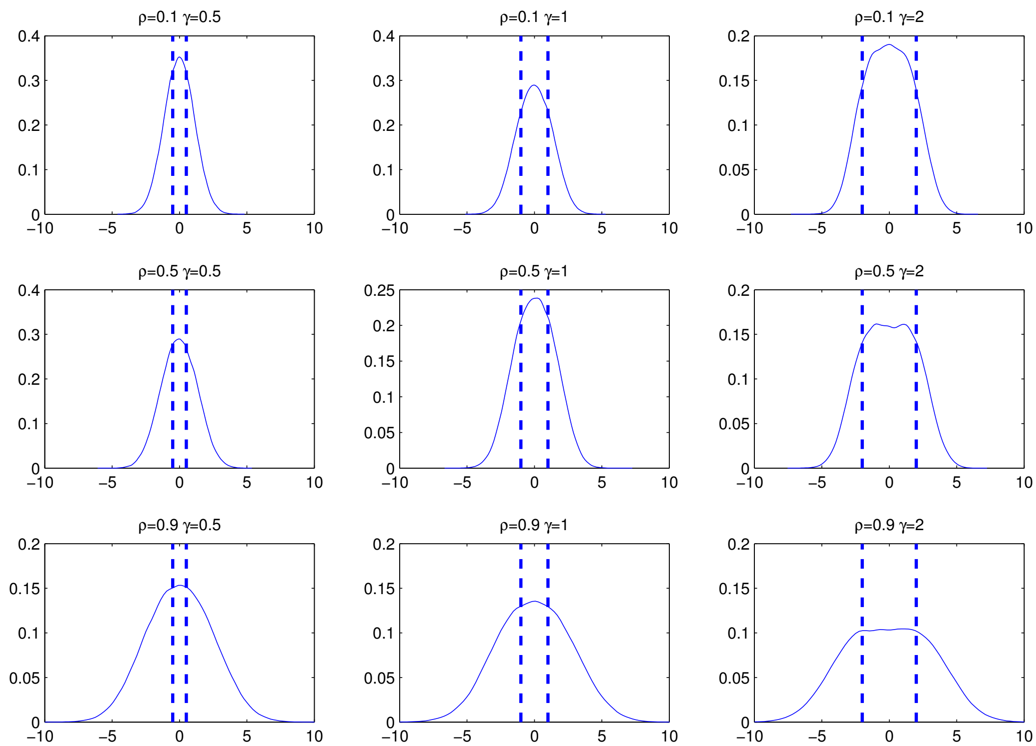

Figure 3: Band-TAR Distributions

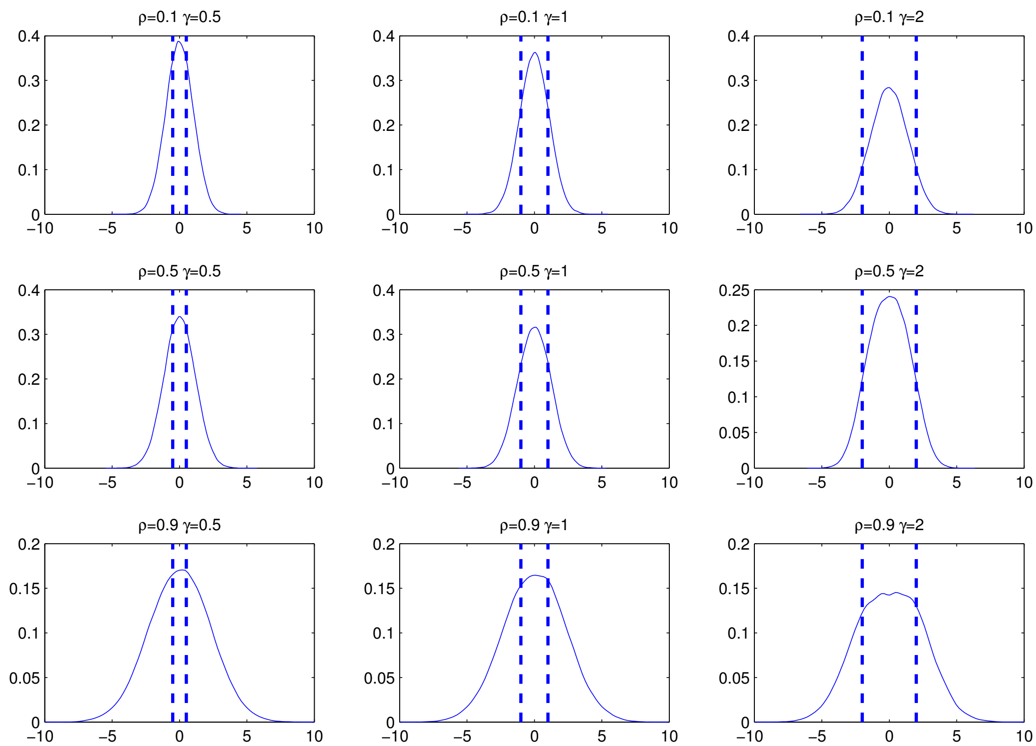

Figure 4: EQ-TAR Distributions