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Effects of variations in learning effectiveness on durable goods production

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Abstract

Using a two period monopoly model, effects of changes in learning effectiveness on durable goods output and accumulated production are investigated. As learning increases, a firm may increase output in both periods or increase it in one period and decrease it in the other. However, accumulated production always increases in response to the learning change. Findings can explain why unit costs and accumulated production might be different across firms in an industry, subject to learning effects, if firms are ordered along a learning effectiveness continuum and are spatially separated to create local monopolies.

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1. Introduction

Beginning with Wright's seminal paper (1936) describing learning effects in the aircraft manufacturing industry, research continues to show that unit costs decline with accumulated production in many industries. There is now a substantial literature evidencing these effects. However, past analysis also shows significant variation in unit cost declines as a function of accumulated production across industries and even across firms within the same industry (Thompson 2012). The source of these variations, particularly in the latter case where presumably firms have access to the same technology, are not well understood. Accordingly, this paper explains variations in unit cost declines from the perspective of a profit maximizing monopolist producing and renting durable goods and responding to exogenous changes in learning effectiveness. Model results are consistent with findings from cross-sectional analyses where different plants (firms) can have different unit costs after the same or a similar number of periods of production (Yelle 1979).

Findings differ somewhat from a similar analysis undertaken by Utaka (2001) using a two period model. Utaka shows that with rental of perfectly durable goods, production increases in the second period but decreases in the first period when learning is introduced. He also notes that with production of perishable goods, learning increases output in both periods. Utaka derives his results assuming that production occurs in both periods. By contrast, the present analysis treats goods durability and the presence of learning along a continuum in the same two period framework. Goods can be imperfectly durable and learning effectiveness can increase from a present state where learning already exists. From these two assumptions, we show the possibility of market and non-market entry. With market entry, the monopolist always produces in the first period and serves the market in both periods. The monopolist may or may not produce in the second period, dependent on the degree of durability and learning effectiveness present. These findings extend a previous analysis indicating that a monopolist renting a perfectly durable good with no learning present will only produce in the first period (Bulow 1982).

Further, we show that with two period production, output in both periods can increase in response to increases in learning effectiveness when goods durability is low and learning effectiveness is high. We also show that when this result fails, accumulated production still increases with higher learning effectiveness under reasonable assumptions. The appendix to this paper presents a linear model that validates our findings.

2. Model

2.1 Assumptions

We assume that a profit maximizing monopolist produces and rents a durable good over a two period horizon. The firm makes a one-time durability decision reflected in the portion D of first period production that survives for rental in the second period where $D \in [0,1]$. Thus in our simple two period framework, we define the first and second period number of units rented according to Q_1 and $Q_1D + Q_2$ where Q_1 and Q_2 are first and second period production, respectively. To simplify matters, we treat the selection of D as exogenous to the analysis. In a more complete analysis, the firm would select the value for D that maximizes profits. Accordingly, we have the following first and second period revenue functions $R_1 = R(Q_1)$ and $R_2 = R(Q_1D + Q_2)$. Revenue in each period is increasing and strictly concave with respect to units rented. We also assume that the revenue function is continuous and twice differentiable.

Consequently, we define total discounted revenues as $R_T = R(Q_1) + R(Q_1D + Q_2)J$ where J is a discount factor applied to second period revenues. Then from concavity, we can sign the following second order effects for total revenues: $\frac{\partial^2 R_T}{(\partial Q_1)^2} = R''(Q_1) + R''(Q_1D + Q_2)D^2J < 0$, $\frac{\partial^2 R_T}{(\partial Q_2)^2} = R''(Q_1D + Q_2)J < 0$, $\frac{\partial^2 R_T}{(\partial Q_1)(\partial Q_2)} = \frac{\partial^2 R_T}{(\partial Q_2)(\partial Q_1)} = R''(Q_1D + Q_2)DJ < 0$ and $\frac{\partial^2 R_T}{(\partial Q_1)^2}\frac{\partial^2 R_T}{(\partial Q_2)^2} - \left(\frac{\partial^2 R_T}{(\partial Q_1)(\partial Q_2)}\right)^2 = R''(Q_1)R''(Q_1D + Q_2)J > 0$. Notice that signs for $\frac{\partial^2 R_T}{(\partial Q_1)^2}$ and $\frac{\partial^2 R_T}{(\partial Q_1)^2}\frac{\partial^2 R_T}{(\partial Q_2)^2} - \left(\frac{\partial^2 R_T}{(\partial Q_1)(\partial Q_2)}\right)^2$ establish strict concavity for total revenues with respect to period outputs.

Next, we make the following reasonable assumption: the second order marginal revenue effect from first period output $\left(\frac{\partial^2 R_T}{(\partial Q_1)^2}\right)$ equals or exceeds cross effects in absolute value. That is, we assume $\frac{\partial^2 R_T}{(\partial Q_1)^2} - \frac{\partial^2 R_T}{(\partial Q_1)(\partial Q_2)} = R''(Q_1) + R''(Q_1D + Q_2)DJ(D - 1) \leq 0$. We will use this assumption in establishing the effect on accumulated output from changes in learning effectiveness. Note that if demand is linear, then the decline in marginal revenues is a constant, in which case $R''(Q_1) = R''(Q_1D + Q_2)$ and $\frac{\partial^2 R_T}{(\partial Q_1)^2} - \frac{\partial^2 R_T}{(\partial Q_1)^2}\frac{\partial^2 R_T}{(\partial Q_2)^2} < 0$ follows without further qualification. Additionally, notice that for any demand structure, the term $R''(Q_1D + Q_2)DJ(D - 1)$ disappears for $D = 0,1$. Then $\frac{\partial^2 R_T}{(\partial Q_1)^2} - \frac{\partial^2 R_T}{(\partial Q_1)^2}\frac{\partial^2 R_T}{(\partial Q_2)^2} = R''(Q_1) < 0$ again without qualification.[1]

On the cost side, we consider each period's unit cost as a function of learning from accumulated production through the previous period, the durability factor and a learning effectiveness index denoted as K where $K \in [0,1]$. With $K = 0$, no learning takes place, regardless of the level of accumulated production, but at $K = 1$, learning effectiveness and the corresponding pace of learning is at a theoretical maximum. Without loss of generality for our purposes, we assume production starts in the first period so that any learning affects second

[1] More generally notice that $\lim_{D \to 0,1} R''(Q_1) + R''(Q_1D + Q_2)\,DJ(D - 1) = R''(Q_1) < 0$. As D becomes very large or small, the negative term $R''(Q_1D + Q_2)DJ(D - 1)$ becomes very small in absolute value, while $R''(Q_1)$ approaches a finite, non-zero value. Assume the indicated limit is approached monotonically from both sides. Then if the condition is violated for feasible values of D, there must still exist two values for D, one close enough to zero and the other close enough to one such that $R''(Q_1) + R''(Q_1D^* + Q_2)D^*J(D^* - 1) = 0$. These values, call them $D^* = D_L$ and $D^* = D_H$ for the low value closer to zero and the high value closer to one, respectively, must be such that $R''(Q_1) + R''(Q_1D + Q_2)DJ(D - 1) \leq 0$ for $0 < D \leq D_L$ and $D_H \leq D < 1$. Thus, violation of the condition, if possible, can only be for the subset of D values between D_L and D_H.

period unit costs only. Therefore, we identify first and second period unit costs according to $u_1 = u + g(D)$ and $u_2 = u + g(D) - h(Q_1)K$. The functions $g(D) > 0$ and $-h(Q_1)K < 0$ denote effects on unit costs from positive values for D and Q_1, respectively. Otherwise $g(0) = 0$ and $h(0) = 0$.[2] We assume that durability is a proxy for quality and therefore increasing D increases unit costs at a non-decreasing rate. Thus, we have $g'(D) > 0$ and $g''(D) \geq 0$. Also, consistent with the standard interpretation, we assume that accumulated production is a proxy for learning. More accumulated production creates more knowledge applicable for process improvements that lower unit costs at a non-increasing rate. Thus, we have $-h'(Q_1)K < 0$ and $-h''(Q_1)K \geq 0$ for $K \neq 0$.

What is new in our analysis is the introduction of the index K to reflect the quality of learning realized for any level of accumulated production. The value for the index can be interpreted as the extent of new knowledge created by labor and management based on a given level of production experience (accumulated production). In the context of our model this means that as K increases, knowledge gained from a given Q_1 increases and more knowledge means a further lowering of second period unit costs. Thus, we have $\frac{\partial u_2}{K} = -h(Q_1) < 0$.

With the specified unit costs, we get the following period and total cost functions: $C_1 = [u + g(D)]Q_1$, $C_2 = [u + g(D) - h(Q_1)K]Q_2$ and $C_T = [u + g(D)]Q_1 + [u + g(D) - h(Q_1)K]Q_2J$. As with revenues, we assume the cost functions are continuous and twice differentiable. We then get the following second order effects on total costs: $\frac{\partial^2 C_T}{(\partial Q_1)^2} = -h'' Q_2 KJ \geq 0$, $\frac{\partial^2 C_T}{(\partial Q_2)^2} = 0$, $\frac{\partial^2 C_T}{(\partial Q_1)(\partial Q_2)} = \frac{\partial^2 C_T}{(\partial Q_2)(\partial Q_1)} - h'KJ < 0$, and $\frac{\partial^2 C_T}{(\partial Q_1)^2}\frac{\partial^2 C_T}{(\partial Q_2)^2} - \left(\frac{\partial^2 C_T}{(\partial Q_1)(\partial Q_2)}\right)^2 = (h'KJ)^2 > 0$. The first and last conditions establish convexity for total costs with respect to period outputs.

2.2. Profits Maximization and Output Effects from Changes in Learning Effectiveness

We use the revenue and cost functions to define the monopolist's total profits as the following sum of first and discounted second period profits:

$$\pi^T(Q_1, Q_2, D, K, J) = \pi^1(Q_1, D) + \pi^2(Q_1, Q_2, D, K)J$$

where:

$$\pi^1(Q_1, D) = R(Q_1) - [u + g(D)]Q_1$$

$$\pi^2(Q_1, Q_2, D, K) = R(Q_1 D + Q_2) - [u + g(D) - h(Q_1)K]Q_2.$$

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[2] First period unit costs in full form are $u_1 = u + g(D) - h(0)K$ which reduces to the indicated expression. Since accumulated production is zero in the first period, there is no separate learning effect on unit costs for that period.

Using π^T we can succinctly define the firm's profit maximizing objective according to:

$$\underset{Q1,Q2}{\text{MAX}}\, \pi^T(Q_1, Q_2, D, K, J)$$

subject to $Q_1 \geq 0$ and $Q_2 \geq 0$.

Then the following Kuhn-Tucker first order conditions define the optimal outputs:

$$\pi^T{}_{Q1} \leq 0, Q_1 \geq 0, \pi^T{}_{Q1}Q_1 = 0 \text{ and } \pi^T{}_{Q2} \leq 0, Q_2 \geq 0, \pi^T{}_{Q2}Q_2 = 0,$$

where:

$$\pi^T{}_{Q1}(Q_1, Q_2, D, K, J) = R'(Q_1) - u - g(D) + [R'(Q_1D + Q_2)D + h'(Q_1)Q_2K]J$$

$$\pi^T{}_{Q2}(Q_1, Q_2, D, K, J) = [R'(Q_1D + Q_2) - (u + g(D) - h(Q_1)K)]J.$$

Notice that strict concavity for total revenues and convexity for total costs means that total profits are strictly concave with respect to outputs. This coupled with the output restrictions means that the Kuhn-Tucker conditions are also sufficient for profit maximization. Inspection of these conditions leads immediately to the following proposition.

Proposition 1: There are three feasible market outcomes for a profit maximizing monopolist intent on producing and renting a durable good that is subject to learning economies of scale. The monopolist can either: a) produce and enter the market in the first period and produce or not produce in the second, or b) not produce in either period. It is never profit maximizing for the monopolist to produce in the second period alone. Thus, the monopolist's market entry and rental in both periods is wholly dependent on profitable first period production.

Proof: We proceed in three parts.

(1) First, assume that first period production alone is profitable by $\pi^T{}_{Q1}(0,0,D,K,J) = R'(0) + R'(0)DJ - u - g(D) > 0$. Then by concavity of the profit function $\pi^T{}_{Q1}(Q_1, 0, D, K, J) = R'(Q_1) + R'(Q_1D)DJ - u - g(D) = 0$ is feasible and $Q_1 > 0$ solving $\pi^T{}_{Q1} = 0$ is profit maximizing, given $Q_2 = 0$.

(2) Then it follows that at $Q_2 = 0$, we have $\pi^T{}_{Q2}(Q_1, 0, D, K, J) = (R'(Q_1D) - u - g(D) + h(Q_1)K)J$, where Q_1 solves $\pi^T{}_{Q1}(Q_1, 0, D, K, J) = 0$. Substituting for $u + g(D)$ from $\pi^T{}_{Q1} = 0$ gives $\pi^T{}_{Q2} = (R'(Q_1D)(1 - DJ) - R'(Q_1) + h(Q_1)K)J$ which can be positive, zero or negative. If positive, then the Kuhn Tucker condition $\pi^T{}_{Q2} \leq 0$ at $Q_2 = 0$ is violated and production in both periods is optimal. Otherwise, production in the first period only remains optimal.

(3) Finally, consider $\pi^T{}_{Q1}(0,0,D,K,J) = R'(0)+R'(0)DJ - u - g(D) \leq 0$. In this case, $Q_1 = 0$ is optimal, given $Q_2 = 0$. But then we have $\pi^T{}_{Q2}(0,0,D,K,J) = [R'(0) - u - g(D)]J \leq 0$ which is optimal as well and the market remains unserved in both periods. Thus, market entry occurs only if $\pi^T{}_{Q1} = R'(0)+R'(0)DJ - u - g(D) > 0$. This completes the proof.

Given market entry, the intuition behind the decision of whether or not to produce in the second period is straightforward. With durability, first period output substitutes for second period output. That is carry-over of rental goods from the first period, lowers marginal revenues from second period production which lowers the incentive to produce in this period.[3] However, as D decreases, marginal revenue increases and as K increases marginal cost decreases from more learning. Eventually the difference between marginal revenues and marginal costs turns positive and second period production becomes profitable. Thus in general, high (low) values for D and K support first period production only and low (high) values for D and K support production in both periods.[4]

We can now examine the effects of changes in learning effectiveness on outputs when production occurs in both periods. Consider the following proposition.

Proposition 2: Assume first and second period production. Then if ${\pi^T}_{Q1Q2} = {\pi^T}_{Q2Q1} \geq 0$, *small increases(decreases) in K yield increases (decreases) in first and second period outputs. However, small increases (decreases) in K always yield increases(decreases) in accumulated output.*

Proof: Using the first order conditions:

$${\pi^T}_{Q1} = R'(Q_1) - u - g(D) + [R'(Q_1D + Q_2)D + h'(Q_1)Q_2K]J = 0$$

$${\pi^T}_{Q2} = [R'(Q_1D + Q_2) - (u + g(D) - h(Q_1)K)]J = 0$$

we establish the following second order conditions:

$${\pi^T}_{Q1Q1} = R''(Q_1) + [R''(Q_1D + Q_2)D^2 + h''(Q_1)Q_2K]J$$

$${\pi^T}_{Q2Q2} = R''(Q_1D + Q_2)J$$

$${\pi^T}_{Q1Q2} = {\pi^T}_{Q2Q1} = [R''(Q_1D + Q_2)D + h'(Q_1)K]J.$$

Then differentiating the two first order conditions $\pi^T_{Q1}(Q_1, Q_2, D, K, J) = \pi^T_{Q2}(Q_1, Q_2, D, K, J) = 0$ with respect to K yields:

$${\pi^T}_{Q1Q1}\frac{dQ_1}{dK} + {\pi^T}_{Q1Q2}\frac{dQ_2}{dK} = -{\pi^T}_{Q1K}$$

$${\pi^T}_{Q2Q1}\frac{dQ_1}{dK} + {\pi^T}_{Q2Q2}\frac{dQ_2}{dK} = -{\pi^T}_{Q2K}$$

and solving for marginal effects gives:

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[3] With perfect durability and no learning ($D = 1, K = 0$), the substitution is complete and second period production is completely displaced. To see this, assume first period production so that ${\pi^T}_{Q2} = (R'(Q_1D)(1 - DJ) - R'(Q_1) + h(Q_1)K)J$ at $Q_2 = 0$ applies to evaluate for second period production, as indicated above. Then substituting for D and K yields $-R'(Q_1)J^2 < 0$ and $Q_2 = 0$ is profit maximizing.

[4] In the appendix, we use a linear model to show two period and one period production solutions based on a range of values for D and K. Results show reductions in second period production as D increases for any given K value. Eventually, the disincentive to produce is strong enough to yield $Q_2 = 0$.

$$\frac{dQ_1}{dK} = \frac{-\pi^T{}_{Q1K}\pi^T{}_{Q2Q2} + \pi^T{}_{Q2K}\pi^T{}_{Q1Q2}}{D}$$

$$\frac{dQ_2}{dK} = \frac{-\pi^T{}_{Q2K}\pi^T{}_{Q1Q1} + \pi^T{}_{Q1K}\pi^T{}_{Q2Q1}}{D}$$

where:

$D = \pi^T{}_{Q1Q1}\pi^T{}_{Q2Q2} - \pi^T{}_{Q1Q2}\pi^T{}_{Q2Q1}.$

Clearly, $\pi^T{}_{Q1Q1} < 0$ and $\pi^T{}_{Q2Q2} < 0$, and $D > 0$ from concavity of the profit function. Also, note: a) $\pi^T{}_{Q1K} = h'(Q_1)Q_2J > 0$ and $\pi^T{}_{Q2K} = h(Q_1)J > 0$, but b) $\pi^T{}_{Q1Q2} = \pi^T{}_{Q2Q1}$ can be positive, negative or zero. Therefore, if $\pi^T{}_{Q1Q2} = \pi^T{}_{Q2Q1} \geq 0$, then $\frac{dQ_1}{dK} > 0$ and $\frac{dQ_2}{dK} > 0$ follow.

With respect to effects of K on second period accumulated output, add the two marginal effects to get:

$$\frac{dQ_1}{dK} + \frac{dQ_2}{dK} = \frac{-\pi^T{}_{Q1K}(\pi^T{}_{Q2Q2} - \pi^T{}_{Q2Q1}) - \pi^T{}_{Q2K}(\pi^T{}_{Q1Q1} - \pi^T{}_{Q1Q2})}{D}$$

Notice that $\pi^T{}_{Q2Q2} - \pi^T{}_{Q2Q1} = R''(Q_1D + Q_2)J(1 - D) - h'(Q_1)KJ < 0$ and $\pi^T{}_{Q1Q1} - \pi^T{}_{Q1Q2} = R''(Q_1) + R''(Q_1D + Q_2)DJ(D - 1) + (h''(Q_1)Q_2K - h'(Q_1)K)J$. By previous assumption, $R''(Q_1) + R''(Q_1D + Q_2)DJ(D - 1) \leq 0$ so $\pi^T{}_{Q1Q1} - \pi^T{}_{Q1Q2} < 0$ too. Therefore $\frac{dQ_1}{dK} + \frac{dQ_2}{dK} > 0$ follows. This completes the proof.

We can make several observations regarding these results. With respect to the individual marginal effects, clearly particular values for K and D affect outcomes through the indicated cross effect on profits. Note that for non-durable goods ($D = 0$), $\pi^T{}_{Q1Q2} = \pi^T{}_{Q2Q1}$ is non-negative and therefore $\frac{dQ_1}{dK} > 0$ and $\frac{dQ_2}{dK} > 0$ always. With all other combinations of K and D yielding positive production in both periods, the individual marginal effects cannot be signed in a general model such as this one. However, we confirm in the linear model in the appendix that high values for learning effectiveness and low values for durability support positive cross effects and therefore corresponding increases in both outputs with increases in K.

Last, it is important to note that $Proposition$ 2 provides only a sufficient condition for positive marginal effects from K on both outputs. It is still possible for $\pi^T{}_{Q1Q2} = \pi^T{}_{Q2Q1} < 0$ to produce $\frac{dQ_1}{dK} > 0$ and $\frac{dQ_2}{dK} > 0$ because of the presence of the positive terms $-\pi^T{}_{Q1K}\pi^T{}_{Q2Q2}$ and $-\pi^T{}_{Q2K}\pi^T{}_{Q1Q1}$ in the numerators of $\frac{d_{Q1}}{dK}$ and $\frac{d_{Q2}}{dK}$, respectively. We demonstrate this possibility in the appendix.

3. Conclusion

We have shown in the context of a two period model that increases in learning effectiveness for durable goods produce gains in accumulated production under reasonable assumptions when output occurs in both periods. If the cross effect on marginal profit is positive, then output increases in both periods. Otherwise, output decreases in one period will cause more than offsetting increases in the remaining period. Model results are consistent with observed data indicating that individual firms, even with the same industry, can have very different accumulated outputs and unit costs after a similar number of periods of production. Firms learning at different rates will have dissimilar unit costs for the same accumulated output. Those that learn at faster rates will have incentives to expand accumulated production to lower unit costs and increase profits even further than those learning at slower rates.

A more complete analysis would expand on the two period framework and determine effects on outputs over an extended period of time. Also, it might be useful to include product durability as a control variable and examine effects on output and durability from changes in learning effectiveness. It is expected that these results would also be consistent with the data and explain findings using a more realistic framework.

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APPENDIX

In this appendix, we present a linear model to arrive at more detailed results. We also validate previous conclusions through a numerical example. For brevity in the description, we categorize a two period production solution as an "interior" solution and a first period production only solution as a "corner" solution.

Assume the following inverse demand and unit cost functions for the two periods: $P_1 = a - bQ_1$, $P_2 = a - b(Q_1D + Q_2)$, $u_1 = u + fD$, and $u_2 = u - eQ_1K + fD$. Then we can write total profits as $\pi^T = \pi^1 + \pi^2 J$ where $\pi^1 = aQ_1 - b(Q_1)^2 - (u + fD)Q_1$ and $\pi^2 J = [a(Q_1D + Q_2) - b(Q_1D + Q_2)^2 - (u - eQ_1K + fD)Q_2]J$.

The first order profit maximizing conditions are then:

$$\pi^T_{Q1} = a - 2bQ_1 - (u + fD) + [a - 2b(Q_1D + Q_2)]DJ + eKJQ_2 = 0$$

and

$$\pi^T_{Q2} = [a - 2b(Q_1D + Q_2) - (u - eQ_1K + fD)]J \leq 0, Q_2 \geq 0, \pi^T_{Q2}Q_2 = 0.$$

For convenience, we can rearrange to show:

$$Q_1 = \frac{a(1 + DJ) - (u + fD)}{2b(1 + D^2J)} + \frac{(eK - 2bD)JQ_2}{2b(1 + D^2J)}$$

and

$$Q_2 \geq \frac{a - (u + fD)}{2b} + \frac{(eK - 2bD)Q_1}{2b}, Q_2 \geq 0, \left[Q_2 - \frac{a - (u + fD)}{2b} - \frac{(eK - 2bD)Q_1}{2b}\right]Q_2 = 0.$$

Second order effects are $\pi^T_{Q1Q1} = -2b(1 + D^2J)$, $\pi^T_{Q2Q2} = -2bJ$, and $\pi^T_{Q1Q2} = \pi^T_{Q2Q1} = (ek - 2bD)J$. From concavity, we require $\pi^T_{Q1Q1} < 0$ and $\pi^T_{Q1Q1}\pi^T_{Q2Q2} - \pi^T_{Q1Q2}\pi^T_{Q2Q1} > 0$ as the second order profit maximizing conditions or by substitution $-2b(1 + D^2J) < 0$ and $(2b)^2(1 + D^2J)J - (ek - 2bD)^2J^2 > 0$. Dividing by $(2b)^2(1 + D^2J)J$, the last can be expressed more usefully as $1 - \frac{\left[\frac{(ek-2bD)}{2b}\right]^2 J}{1+D^2J} > 0$. Last, we assume $a(1 + DJ) - (u + fD) > 0$ to ensure first period profits and therefore the existence of a solution other than zero values for both outputs.

We can proceed to obtain solution values for the outputs as follows. First to simplify notation let $A = \frac{a-(u+fD)}{2b}$, $B = \frac{eK}{2b} - D$, $C = \frac{a(1+DJ)-(u+fD)}{2b(1+D^2J)}$ and $E = \frac{J}{(1+D^2J)}$. Then the first order conditions can be expressed more succinctly as $Q_1 = C + BEQ_2$ and $Q_2 \geq A + BQ_1$, $Q_2 \geq 0$, $(Q_2 - A - BQ_1)Q_2 = 0$. Next, we substitute for Q_1in $Q_2 \geq A + BQ_1$ to get $Q_2 \geq A + B(C + BEQ_2)$ or $Q_2 \geq \frac{A+BC}{1-B^2E}$. Thus the solution for Q_2 can expressed as:

$$Q_2 \geq \frac{A + BC}{1 - B^2E}, Q_2 \geq 0, \left[Q_2 - \frac{A + BC}{1 - B^2E}\right]Q_2 = 0.$$

Notice that the term $1 - B^2E$ is simply $1 - \frac{\left[\frac{(ek-2bD)}{2b}\right]^2 J}{1+D^2J}$ which must be positive by the second order requirement. Also, it is clear that $A, C,$ and E are always positive but B can be either positive or negative. Thus an interior solution must be indicated by $A + BC > 0$ and a corner solution by $A + BC \leq 0$.

With respect to first period output, there are also two possible solutions. From $Q_1 = C + EQ_2$, we get the corner solution $Q_1 = C$. Otherwise, we substitute $\frac{A+BC}{1-B^2E}$ for Q_2 to get $Q_1 = \frac{C+BEA}{1-B^2E}$ for an interior solution. Inspection of terms shows that $C + BEA > A + BC$ when $B < 0$. Thus the optimal Q_1 is always positive and greater than the optimal Q_2 when $B < 0$.

Several points can be made regarding solution characteristics. First note that $\pi^T_{Q1Q2} = \pi^T_{Q2Q1} = (ek - 2bD)J = 2bBJ$. Thus B takes on the same sign as $\pi^T_{Q1Q2} = \pi^T_{Q2Q1}$ and therefore $\pi^T_{Q1Q2} = \pi^T_{Q2Q1} \geq 0$ always yields an interior solution. Also B is the only term containing K and its value varies directly with this variable. Thus low (high) enough values for K yield corner (interior) solutions as described earlier. We also note that the B value is limited by the second order condition $1 - B^2E > 0$. Since $0 < E \leq 1, -1 < B < 1$ meets this condition regardless of the E value.

Finally, we can show that linearity yields $\frac{dQ_2}{dK} > 0$ regardless of the sign for $\pi^T_{Q1Q2} = \pi^T{}_{Q2Q1}$. First as noted before, if $\pi^T_{Q1Q2} = \pi^T{}_{Q2Q1} < 0$, then $\pi^T{}_{Q1Q1} < \pi^T{}_{Q2Q1}$ and $Q_1 > Q_2$. Therefore $\pi^T{}_{Q2K} \geq \pi^T{}_{Q1K}$ or $h(Q_1) \geq h'(Q_1)Q_2$ is sufficient for $\frac{dQ_2}{dK} = (-\pi^T{}_{Q2K}\pi^T{}_{Q1Q1} + \pi^T{}_{Q1K}\pi^T{}_{Q2Q1})/D > 0$ when $\pi^T_{Q1Q2} = \pi^T{}_{Q2Q1} < 0$. But concavity of $h(Q_1)$ requires $h(Q_1) \geq h'(Q_1)Q_1$ so that indeed $h(Q_1) \geq h'(Q_1)Q_2$ by $Q_1 > Q_2$. Therefore $\frac{dQ_2}{dK} > 0$ when $\pi^T_{Q1Q2} = \pi^T{}_{Q2Q1} < 0$. We already know $\frac{dQ_2}{dK} > 0$ if $\pi^T_{Q1Q2} = \pi^T{}_{Q2Q1} \geq 0$, therefore $\frac{dQ_2}{dK} > 0$ always.

A sensitivity analysis was conducted to validate the conclusions presented above using the described linear structure. TABLES 1 and 2 demonstrate both internal and corner solutions according to a range of learning and durability factors under non-discounted and discounted scenarios. We use following parameter values to develop results: $a = 10, b = .004, u = 8, f = .004$ and $e = .006$. Although other values are certainly possible, the selected values show plainly how outputs vary according to described patterns.

Two Period Outputs by Learning Effectiveness
and Durability Factors Without Discounting

J = 1	D = 0			D = .5			D = 1		
K	Q1	Q2	Q1+Q2	Q1	Q2	Q1+Q2	Q1	Q2	Q1+Q2
0	250	250	500	700	0	700	750	0	750
0.2	294	294	588	698	5	704	750	0	750
0.4	357	357	714	682	113	795	750	0	750
0.6	455	455	909	691	215	906	750	0	750
0.8	625	625	1,250	726	322	1,048	750	0	750
1	1,000	1,000	2,000	789	447	1,236	742	64	806

TABLE 1

Two Period Outputs by Learning Effectiveness
and Durability Factors with Discounting

J = .8	D = 0			D = .5			D = 1		
K	Q1	Q2	Q1+Q2	Q1	Q2	Q1+Q2	Q1	Q2	Q1+Q2
0	250	250	500	625	0	625	694	0	694
0.2	285	293	578	617	34	651	694	0	694
0.4	334	350	684	608	128	736	694	0	694
0.6	406	433	838	617	219	836	694	0	694
0.8	520	562	1,081	646	314	960	694	0	694
1	727	795	1,523	695	424	1,119	685	78	764

TABLE 2

First notice for the undiscounted case shown in TABLE 1, if goods are non-durable ($D = 0$), B is non-negative for all K values and therefore both outputs increase throughout as K increases. Outputs are the same in this case because the terminal marginal cost (for period two) applies to both first order conditions.[5] With imperfect durability ($D = .5$), note that first period output first decreases and then increases with higher K values as B turns from negative to positive. The value for B is zero at $K = .667$ but first period output actually starts increasing at

[5] With $D = 0$ and $J = 1$, first order conditions for internal solutions are: $\pi^T{}_{Q1} = R'(Q_1) - (u - h'(Q_1)Q_2K) = 0$ and $\pi^T{}_{Q2} = R'(Q_2) - (u - h(Q_1)K) = 0$ where $u - h(Q_1)K$ is the terminal (period 2) marginal cost. With linear unit costs, the concavity requirement evaluates as an equality, so we have $h(Q_1) = h'(Q_1)Q_1$ and $Q_1 = Q_2$ satisfies both first order conditions.

a lower K where B is negative. However, second period and total outputs increase throughout, as described before. With perfect durability $(D = 1)$, as expected we have mostly corner solutions.

The discounted case shown in TABLE 2 follows the same general pattern but the distribution of output for each K and D combination changes. Compared to TABLE 1, first period production decreases in all cases but second period output moves in both directions. The former holds because decreases in J cause second period marginal profits to have less current (present) value at the pre-existing first period output. Therefore, first period outputs must decrease from comparable values at $J = 1$ to increase total marginal profits back to zero via the concavity requirement. The one exception is with $K = 0, D = 0$ where first period output is unaffected because it has no effect on second period profits. In the case of internal solutions, the movement in second period output is explained by the cross profit effect π^T_{Q2Q1}. If positive, then outputs move in the same direction and if negative movements are in opposing directions. Thus when $B < 0$ $(B > 0)$, second period output increases (decreases).