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Optimal consumption and portfolio rules when the asset price is driven by a time-inhomogeneous Markov modulated fractional Brownian motion with multiple Poisson jumps

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Abstract

This paper is aimed at developing a stochastic model to study the behavior of a rational consumer that makes consumption and portfolio decisions when the asset price is driven by a time-inhomogeneous Markov modulated fractional Brownian motion combined with multiple Poisson jumps. We provide closed-form solutions. The addition of a time-inhomogeneous Markov is useful to model structural changes related to the physical trend, the instantaneous volatility and the interest rate, improving the understanding of portfolio dynamic behavior. Multiple jumps can be associated with sudden and unexpected leaps of the price itself, the sector, related markets, and the economic and business atmosphere, which provides a richer environment to the consumer's decision making problem under uncertainty.

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1. Introduction

Since classical Merton's (1969) and (1971) papers, the problem of an infinitely-lived, rational consumer maximizing his/her lifetime discounted utility when the dynamics of the asset returns are shaped with a diffusion process has been widely studied. There is, in the mathematical finance literature, a very long list of extensions in several directions of Merton's seminal proposal. Recently, much of latest research aims at modeling asset prices with Markov modulated process; see, for instance: Bäuerle and Rieder (2004) determining the optimal portfolio allocations when the stock price depends on an external time-homogeneous and finite Markov chain; Sotomayor and Cadenillas (2009) finding explicit solutions for the optimal investment and consumption decisions with a HARA utility function when asset prices are driven by standard Brownian motions combined with a regime switching; and Fei (2013) that provides optimal consumption and portfolio allocation when the inflation rate is driven by a Markov-switching process. Approaches to consumption and portfolio optimal decisions for regime switching models have also been broadly studied; for instance: Stockbridge (2002) providing a mathematical programming formulation of the portfolio optimization problem; Zhang and Yin (2004) offering nearly optimal strategies in a financial market, and Sass and Haussmann (2004) solving numerically the problem of maximizing the investor's expected utility of terminal wealth under a finite time horizon.[1]

In this research, we extend Vallejo-Jiménez *et al*. (2015) and Soriano-Morales *et al*. (2015) in various directions.[2] The main contribution of this paper is to provide analytical solution for the utility maximization problem of a rational consumer-investor when the asset price is driven by a time-inhomogeneous Markov modulated fractional Brownian motion with multiple Poisson jumps. In this sense, jumps are associated with sudden and unexpected leaps of the price itself, the sector, the news, the related market, etc. The Markov chain is related with the different combinations of the physical trend, the instantaneous volatility and the interest rate, all of them taking low, mid, and high levels, which enable us to model structural changes regarding these time and state dependent variables.

It is worth stating a list of what models in the specialized literature are already included in our proposal and also distinguishing what is innovative in it. To do this, we present Table 1. Of course, this table is not intended to be exhaustive at all.

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[1] To the extent of our knowledge, regime switching models were initially proposed by Hamilton (1989) to model stock return time series; however, this approach brings new difficulties due to the additional source of uncertainty affecting the completeness of the market. Moreover, the firsts in dealing with asset prices driven by mixed jump-diffusion processes were: Cox and Ross (1976), Ball and Torous (1985), and Page and Sanders (1986). More recent work on jump-diffusion process can be found in Aït-Sahalia *et al*. (2009) and Lui *et al*. (2005). Fractional Brownian motion is a natural extension of Brownian motion (Mandelbrot, 1968) and its statistical properties are widely used in financial modeling; see, *e.g*., Bender *et al*. (2011) and Hu and Øksendal (2003).

[2] Optimal portfolio selection has also been studied in Biagini and Øksendal (2003), Czichowsky and Schachermayer (2015), Hu and Øksendal. (2003), Hu *et al*. (2003), Jumarie (2005), Karatzas *et al*. (1987), He and Pearson (1991), Karatzas *et al*. (1991), Cvitanić and Karatzas (1996), Cvitanić and Wang (2001), Venegas-Martínez (2001), (2005) and (2009), Venegas-Martínez and González-Aréchiga (2000), and Zariphopoulou (2001), (1999) and (1992).

Table 1. A summary of models included in our proposal and the proposed extensions

Optimal portfolio when the stock price is driven by:	
Jump-diffusion process	Czichowsky and Schachermayer (2015), Jin and Zhang (2012), Aït-Sahalia *et al*. (2009), Venegas-Martínez (2000) and (2001), Jeanblanc-Picqué and Pontier (1990).
Time-homogeneous Markov chain (regime-switching)	Soriano-Morales *et al*. (2015), Fei (2014), Zhou and Yin (2014), Wu and Li (2011), Elliott *et al*. (2010), Sotomayor and Cadenillas (2009), Çakmak and Özekici (2006), Rieder and Bäuerle (2005), and Bäuerle and Rieder (2004), Sass (2004), Sass and Haussmann (2004), Stockbridge (2002), and Elliot (2002).
Time-inhomogeneous Markov chain (regime-switching)	Vallejo-Jiménez *et al*. (2015), and Rudiger and Backhau (2008).
Fractional Brownian motion modulated by a Time-homogeneous finite Markov chain	Fei and Shu-Juan (2012).
Markov regime switching combined with jump-diffusion processes	Yu (2014), and Elghanjaoui and Karlsen (2012).
Fractional Brownian modulated by a time-inhomogeneous Markov chain combined with multiple jump-diffusion processes	This paper (2017).

Source: Authors' own elaboration.

This research has the following organization: in section 2, we setup the mathematical framework of the proposed model; through section 3, we provide the analytical solution of optimal consumption and asset allocation; in section 4 we revisit some special cases; finally, in section 5, we present the conclusion and acknowledge some limitations.

2. The setting of the model

In considering the problem of determining optimal portfolio and consumption decisions, it is, usually, assumed that the consumer has access to a bond and a risky asset. The randomness in the risky asset returns requires a filtered probability space (or stochastic basis) $(\Omega, \mathscr{F}, \mathfrak{F} = \{\mathscr{F}_t, 0 \le t \le T\}, P)$ where Ω is a sample space, $\mathscr{F}$ is a σ-algebra on Ω, P is a probability measure on $(\Omega, \mathscr{F})$, and $\mathscr{F}_t$ is a filtration containing all available information of the market until time t. The bond price process, b_t, evolves deterministically according to

$$\frac{\mathrm{d}b_t}{b_t} = r_i \,\mathrm{d}t\,. \tag{1}$$

The stock price process, S_t, is driven by the following stochastic differential equation, namely, a time-inhomogeneous Markov modulated fractional Brownian motion with multiple Poisson jumps

$$\frac{\mathrm{d}S_t}{S_t} = \mu_i \,\mathrm{d}t + \sigma_i \,\mathrm{d}B_t^H + \sum_{k=1}^{n} \nu_k \,\mathrm{d}\bar{N}_{t,k}$$

where B_t^H stands for the fractional Brownian motion as a Gaussian zero-mean non-stationary stochastic process indexed by a single scalar parameter $H \in (0,1)$ (Hurst parameter). The usual Brownian motion satisfies $H = 1/2$. It is well known that a fractional market with Hurst parameter $H > 1/2$ allows arbitrage (Bender *et al*. 2011). Hence, this investigation mainly focuses on $H \le 1/2$. The covariance of B_t^H shows that it is non stationary since

$$\mathrm{E}[B_t^H B_s^H] = \frac{\sigma^2}{2}\left(\left|t\right|^{2H} + \left|s\right|^{2H} - \left|t-s\right|^{2H}\right).$$

This, clearly, shows that $\mathrm{Var}[B_t^H] = \sigma^2 \left|t\right|^{2H}$. Here (μ_i, r_i, σ_i) is a continuous time Markov chain changing over time with a finite state space E and a matrix $Q = (q_{ij}(t))_{i,j\in E}$ having time dependent transition probabilities under P with respect to $\mathfrak{F}$. In what follows, it is assumed that $\mu_i,\ r_i,\ \sigma_i : E \to R^+$, and $\mu_i,\ r_i,\ \sigma_i > 0$ for all $i \in E$, allowing regime switching in (μ_i, r_i, σ_i). Consider now a Poisson jump process $\mathrm{d}\bar{N}_{t,k}$ with intensity ϕ_k. That is,

$$P\{\text{one jump during } \mathrm{d}t\} = P\{\mathrm{d}\bar{N}_{t,k} = 1\} = \phi_k \,\mathrm{d}t$$

and

$$P\{\text{more than one jump during } \mathrm{d}t\} = P\{\mathrm{d}\bar{N}_{t.k} > 1\} = o(\mathrm{d}t)\,,$$

so that

$$P\{\text{no jump during } \mathrm{d}t\} = P\{\mathrm{d}\bar{N}_{t,k} = 0\} = 1 - \phi_k \,\mathrm{dt} + o(\mathrm{d}t)\,.$$

where, as usual, $o(\mathrm{d}t)/\mathrm{dt} \to 0$ as $\mathrm{d}t \to 0$. Additionally, it is required that $\mathrm{Cov}(\mathrm{d}\bar{N}_{t,k}, \mathrm{d}B_t^H) = 0$ and $\mathrm{Cov}(\mathrm{d}\bar{N}_{t,k_1}, \mathrm{d}\bar{N}_{t,k_2}) = 0$ for all $k, k_1, k_2 \le n;\ \ k_1 \ne k_2$. It is also, usually, convenient to redefine the process $\mathrm{d}\bar{N}_{t,k}$ in such a way that

$$\frac{\mathrm{d}S_t}{S_t} = \left(\mu_i + \sum_{k=1}^{n} \phi_k \nu_k\right)\mathrm{d}t + \sigma_i \,\mathrm{d}B_t^H + \sum_{k=1}^{n} \nu_k \,\mathrm{d}N_{t,k} \tag{2}$$

where $\mathrm{d}N_{t,k}$ has the same probability distribution but $\mathrm{E}[\mathrm{d}N_{t,k}] = 0$. From now on, we denote by $\theta_{t,i}$ the proportion of wealth not intended for consumption that is invested in the asset at time t. The process $\theta_{t,i}$ is called a portfolio strategy, and we assume that $\int_0^T \theta_{s,i}^2 \,\mathrm{d}s < \infty$ almost surely. Let us denote by a_t the real wealth process under a self-financing assumption. Thus, real wealth is driven by the following stochastic differential equation

$$\mathrm{d}a_t = a_t(1-\theta_{t,i})\frac{\mathrm{d}b_t}{b_t} + a_t\theta_{t,i}\frac{\mathrm{d}S_t}{S_t} - \mathrm{d}c_t \tag{3}$$

with $a_0 > 0$. Consider a utility function $U:[0,\infty)\to\mathbb{R}$ that satisfies Inada's conditions. The consumer wishes to maximize the total expected discounted utility:

$$\mathrm{E}\left[\int_0^\infty u(c_t)\, e^{-\rho t}\,\mathrm{d}t \mid \mathscr{F}_0\right] \tag{4}$$

where ρ is the subjective discount rate. Under the previous assumptions, equations (1)-(2) and

$$\mathrm{d}c_t = c_t\,\mathrm{d}t \tag{5}$$

are substituted in (3) to obtain consumer's budget constraint in such way that

$$\mathrm{d}a_t = a_t\left(r_i + \theta_{t,i}(\mu_i - r_i + \sum_{k=1}^n \phi_k v_k) - \frac{c_t}{a_t}\right)\mathrm{d}t + a_t\theta_{t,i}\sigma_i\,\mathrm{d}B_t^H + a_t\theta_{t,i}\sum_{k=1}^n v_k\,\mathrm{d}N_{t,k}\,. \tag{6}$$

The market risk premium, adjusted by volatility, is denoted by

$$\lambda_i = \frac{\mu_i - r_i + \sum_{k=1}^n \phi_k v_k}{\sigma_i} \tag{7}$$

Hence, from (4), (6), and (7), we have that the lifetime utility maximization problem is given by

$$\begin{aligned} &\text{Maximize } \mathrm{E}\left[\int_0^\infty u(c_t)e^{-\rho t}\,\mathrm{d}t \mid \mathscr{F}_0\right] \\ &\text{subject to } \mathrm{d}a_t = a_t\left(r_i + \theta_{t,i}\lambda_i\sigma_i - \frac{c_t}{a_t}\right)\mathrm{d}t + a_t\theta_{t,i}\sigma_i\,\mathrm{d}B_t^H + a_t\theta_{t,i}\sum_{k=1}^n v_k\,\mathrm{d}N_{t,k}. \end{aligned} \tag{8}$$

In order to solve problem (8), we define the value function

$$J(a_t,t,i) = \max_{c_s|_{s\in[t,\infty)}} \mathrm{E}\left[\int_t^\infty u(c_s)e^{-\rho s}\,\mathrm{d}s \middle| \mathscr{F}_t\right] \tag{9}$$

Hence,

$$0 = \max_{c_s|_{s\in[t,t+\mathrm{d}t]}} \mathrm{E}\left[u(c_t)e^{-\rho t}\,\mathrm{d}t + o(\mathrm{d}t) + \mathrm{d}J(a_t,t) \middle| \mathscr{F}_t\right] \tag{10}$$

In this case, the stochastic differential satisfies

$$
\begin{aligned}
\mathrm{d}\,J(a_t,t,i) = &\left(\frac{\partial J(a_t,t,i)}{\partial t} + \frac{\partial J(a_t,t,i)}{\partial a_t} a_t \psi_{t,i} + H\frac{\partial^2 J(a_t,t,i)}{\partial a_t^{\,2}} a_t^{\,2}\theta_{t,i}^2\sigma_i^2 t^{2H-1}\right)\mathrm{d}t \\
&+\frac{\partial J(a_t,t,i)}{\partial a_t} a_t\theta_{t,i}\sigma_i \mathrm{d}B_t^H + \left(\sum_{j\in E} q_{ij}(t)\left[J\left(a_t,t,j\right) - J\left(a_t,t,i\right)\right]\right)\mathrm{d}t \\
&+\left(\sum_{k=1}^{n}\left[J\left(a_t(1+\theta_t\nu_k),t,i\right) - J\left(a_t,t,i\right)\right]\phi_k\right)\mathrm{d}t
\end{aligned}
\tag{11}
$$

where

$$
\mathrm{d}a_t = a_t\psi_{t,i}\,\mathrm{d}t + a_t\theta_{t,i}\sigma_i\,\mathrm{d}B_t^H + a_t\theta_{t,i}\sum_{k=1}^{n}\nu_k\,\mathrm{d}N_{t,k} \qquad \text{and} \qquad \psi_{t,i} = r_i + \theta_{t,i}\lambda_i\sigma_i - \frac{c_t}{a_t} \tag{12}
$$

By substituting (11) and (12) in (10), and simplifying, it is obtained

$$
\begin{aligned}
0 = \max_{c_s|_{s\in[t,\,t+\mathrm{d}t]}} \mathrm{E}\Bigg[\quad & u(c_t)e^{-\rho t}\,\mathrm{d}t + \left(\frac{\partial J(a_t,t,i)}{\partial t} + \frac{\partial J(a_t,t,i)}{\partial a_t} a_t\psi_{t,i} + H\frac{\partial^2 J(a_t,t,i)}{\partial a_t^{\,2}} a_t^{\,2}\theta_{t,i}^2\sigma_i^2 t^{2H-1}\right)\mathrm{d}t \\
&+\left(\sum_{j\in E} q_{ij}(t)\left[J\left(a_t,t,j\right) - J\left(a_t,t,i\right)\right]\right)\mathrm{d}t + \left(\sum_{k=1}^{n}\left[J\left(a_t(1+\theta_t\nu_k),t,\mathrm{i}\right) - J\left(a_t,t,i\right)\right]\phi_k\right)\mathrm{d}t \quad \Bigg]
\end{aligned}
\tag{13}
$$

If c_t and $\theta_{t,i}$ are both optimal, then

$$
\begin{aligned}
0 = &u(c_t)e^{-\rho t} + \frac{\partial J(a_t,t,i)}{\partial t} + \frac{\partial J(a_t,t,i)}{\partial a_t} a_t\left(r_i + \theta_{t,i}\lambda_i\sigma_i - \frac{c_t}{a_t}\right) + H\frac{\partial^2 J(a_t,t,i)}{\partial a_t^{\,2}} a_t^{\,2}\theta_{t,i}^2\sigma_i^2 t^{2H-1} \\
&+\sum_{j\in E} q_{ij}(t)\left[J\left(a_t,t,j\right) - J\left(a_t,t,i\right)\right] + \sum_{k=1}^{n}\left(\left[J\left(a_t(1+\theta_t\nu_k),t,i\right) - J\left(a_t,t,i\right)\right]\phi_k\right)
\end{aligned}
\tag{14}
$$

The proposed candidate for solving the above equation is

$$
S\left(a_t,t,i\right) = \beta_0 e^{-\rho t} + \beta_1 u(a_t)\,e^{-\rho t} + g(t,i)\,e^{-\rho t} \tag{15}
$$

By substituting (15) in (14), and simplifying, it is obtained that

$$
\begin{aligned}
0 = &u(c_t) - \rho\left(\beta_0 + \beta_1 u(a_t) + g(t,i)\right) + \frac{\partial g(t,i)}{\partial t} \\
&+\beta_1 u'(a_t)a_t\left(r_i + \theta_t\lambda_i\sigma_i - \frac{c_t}{a_t}\right) + H\beta_1 u''(a_t)a_t^{\,2}\theta_{t,i}^2\sigma_i^2 t^{2H-1} \\
&+\sum_{j\in E} q_{ij}(t)\left[g\left(t,j\right) - g\left(t,i\right)\right] + \beta_1\sum_{k=1}^{n}\left(\left[u\left(a_t(1+\theta_t\nu_k)\right) - u\left(a_t\right)\right]\phi_k\right)
\end{aligned}
\tag{16}
$$

After taking partial derivatives of (16) with respect to c_t and $\theta_{t,i}$, it follows that

$$
u'(c_t) = \beta_1 u'(a_t) \tag{17.a}
$$

$$
0 = u'(a_t)\lambda_{t,i}\sigma + 2H\; u''(a_t)a_t\theta_{t,i}\sigma_i^2 t^{2H-1} + \sum_{k=1}^{n} u'(a_t(1+\theta_t\nu_k))\nu_k\phi_k \tag{17.b}
$$

Solving for $\theta_{t,i}$ in (17.b), it follows

$$\theta_{t,i}=\frac{\left(\dfrac{\lambda_i}{2H\,t^{2H-1}\sigma_i}\right)}{\left(-\dfrac{u''(a_t)a_t}{u'(a_t)}\right)}+\frac{\left(\dfrac{\sum_{k=1}^{n}u'(a_t(1+\theta_t\nu_k))\nu_k\phi_k}{2H\,t^{2H-1}u'(a_t)\sigma_i^2}\right)}{\left(-\dfrac{u''(a_t)a_t}{u'(a_t)}\right)} \tag{18}$$

where λ_i is now defined as the risk premium in the state i, thus λ_i/σ_i should be renamed as the market risk premium adjusted by variance, and $-u''(a_t)a_t/u'(a_t)$ stands for the relative degree of risk aversion; this being the elasticity of the marginal utility of wealth. Observe that (18) differs from standard results, for example, from the classic mean-variance approach, because the optimal proportion, $\theta_{t,i}$, changes with t since the variance is now modified by the factor t^{2H-1}. The dependence of $\theta_{t,i}$ on the state i is due to the regime-switching.

3. Analytic solution for logarithmic utility

Considering logarithmic utility, $u(c)=\ln(c)$, in (17.a) and (17.b), it follows that a constant proportion $1/\beta_1$ of wealth is always consumed, *i.e.*,

$$c_t=a_t/\beta_1 \tag{19}$$

and

$$\theta_{t,i}=\frac{\lambda_i}{2H\,t^{2H-1}\sigma_i}+\sum_{k=1}^{n}\frac{\nu_k\phi_k}{2H\,t^{2H-1}\sigma_i^2(1+\theta_{t,i}\nu_k)} \tag{20}$$

In order to obtain a closed-form solution of $\theta_{t,i}$ in the above equation, the size of all jumps will be fixed, equal to $\nu_0>0$, and intensities will be modified to compensate the jump change for each k. That is, if the original size is less than ν_0, then the intensity will increase and *vice versa*. To do this redefine ϕ_k^* for each k as $\phi_k^*=\nu_k\phi_k/\nu_0$, then

$$\theta_{t,i}=\frac{\dfrac{\lambda_i}{2H\,t^{2H-1}}-\dfrac{\sigma_i}{\nu_0}\pm\sqrt{\left(\dfrac{\lambda_i}{2H\,t^{2H-1}}+\dfrac{\sigma_i}{\nu_0}\right)^2+\dfrac{2\sum_{k=1}^{n}\phi_k^*}{H\,t^{2H-1}}}}{2\sigma_i}. \tag{21}$$

If $\phi_k^*=0$, then $\theta_{t,i}=\lambda_i/\sigma_i$. On the other hand, by substituting (19) in (16), it is obtained that

$$\begin{aligned}0=&-\rho\beta_0-\ln(\beta_1)+(1-\rho\beta_1)\ln(a_t)+\frac{\partial g(t,i)}{\partial t}-\rho g(t,i)\\&+\beta_1 r_i+\beta_1\left(\theta_{t,i}\lambda_i\sigma_i-H\sigma_i^2\theta_{t,i}^2t^{2H-1}\right)-1+\sum_{j\in E}q_{ij}(t)\left[g(t,j)-g(t,i)\right]\\&+\beta_1\sum_{k=1}^{n}\left(\phi_k\ln(1+\theta_{t,i}\nu_k)\right)\end{aligned} \tag{22}$$

This equation must hold for all a_t, then

$$\beta_1=1/\rho \tag{23}$$

Hence, the optimal consumption rule satisfies

$$c_t = \rho a_t \tag{24}$$

By substituting (23) in (22), it is obtained that

$$\begin{aligned}0 = -\rho\beta_0 + \ln(\rho) - 1 + \frac{r_i}{\rho} + \frac{\left(\theta_{t,i}\lambda_i\sigma_i - H\sigma_i^2\theta_{t,i}^2 t^{2H-1}\right)}{\rho} + \sum_{j\in E} q_{ij}(t)\left[g(t,j) - g(t,i)\right] \\ + \frac{1}{\rho}\sum_{k=1}^{n}\left(\phi_k \ln(1+\theta_{t,i}\nu_k)\right) + \frac{\partial g(t,i)}{\partial t} - \rho g(t,i).\end{aligned} \tag{25}$$

Now, observe that there are terms in (25) that do not depend on the state, then the equation can be split in two parts both equal to zero:

Part 1

$$0 = -\rho\beta_0 + \ln(\rho) - 1 \tag{26}$$

Part 2

$$\begin{aligned}0 = \frac{r_i}{\rho} + \frac{\left(\theta_{t,i}\lambda_i\sigma_i - H\sigma_i^2\theta_{t,i}^2 t^{2H-1}\right)}{\rho} + \sum_{j\in E} q_{ij}(t)\left[g(t,j) - g(t,i)\right] \\ + \frac{1}{\rho}\sum_{k=1}^{n}\left(\phi_k \ln(1+\theta_{t,i}\nu_k)\right) + \frac{\partial g(t,i)}{\partial t} - \rho g(t,i).\end{aligned} \tag{27}$$

Solving (26) for β_0, it follows that

$$\beta_0 = \frac{\ln(\rho) - 1}{\rho} \tag{28}$$

In order to solve (27), let

$$h_i = \frac{r_i + \theta_{t,i}\lambda_i\sigma_i + \sum_{k=1}^{n}\left(\phi_k \ln(1+\theta_{t,i}\nu_k)\right)}{\rho} \qquad \text{and} \qquad k_i(t) = -\frac{H\sigma_i^2\theta_{t,i}^2 t^{2H-1}}{\rho}.$$

Hence,

$$0 = \frac{\partial g(t,i)}{\partial t} - \rho g(t,i) + h_i + k_i(t) + \sum_{j\in E} q_{ij}(t)\left[g(t,j) - g(t,i)\right].$$

Therefore,

$$g(t,i) = \int_t^\infty h_i e^{-\rho(s-t)}\mathrm{d}s + \sum_{j\in E}\int_t^\infty q_{ij}(s)\left[g(s,j) - g(s,i) + k_i(s)\right]e^{-\rho(s-t)}\mathrm{d}s \tag{29}$$

where $q_{ij}(t)$ and $g(t,i)$ are integrable for every interval in $[0,\infty)$, and $k_i(t)$ is integrable for $t\in[0,\infty)$. An alternative form for writing $g(t,i)$ is given by

$$g(t,i) = h_i e^{\rho t}\int_t^\infty e^{-\rho s}\mathrm{d}s + e^{\rho t}\sum_{j\in E}\int_t^\infty q_{ij}(s)\left[g(s,j) - g(s,i) + k_i(s)\right]e^{-\rho s}\mathrm{d}s. \tag{30}$$

The partial derivative of (30) with respect to *t* leads to

$$\begin{aligned}\frac{\partial g(t,i)}{\partial t} = \rho\left(\int_t^\infty \rho h_i e^{-\rho(s-t)}\mathrm{d}s + \sum_{j\in E}\int_t^\infty \rho q_{ij}(s)\left[g(s,j) - g(s,i) + k_i(s)\right]e^{-\rho(s-t)}d\ s\right) \\ -\left(h_i + \sum_{j\in E} q_{ij}(t)\left[g(t,j) - g(t,i) + k_i(t)\right]\right)\end{aligned} \tag{31}$$

Substituting and rearranging $g(t,i)$ in (31), leads to

$$\frac{\partial g(t,i)}{\partial t}-\rho g(t,i)+h_i+k_i(t)+\sum_{j\in E}q_{ij}(t)\left[g\left(t,j\right)-g\left(t,i\right)\right]=0\cdot$$

Hence, function in (29) fulfills necessary conditions. And with this, we have provided closed-form solutions for the allocation problem of an infinitely-lived rational consumer-investor, equipped with logarithmic utility, and assuming that the asset price is guided by a time-inhomogeneous Markov modulated fractional Brownian motion with multiple Poisson jumps.

4. Revisiting some special cases

The particular case for the optimal proportion θ_t when the stock price is driven by a mixed jump-diffusion process is obtained from equation (21) as

$$\theta_t=\frac{\lambda-\sigma'+\sqrt{(\lambda+\sigma')^2+4\phi}}{2\sigma}$$

where $\sigma'=\sigma/\nu$. If $\phi=0$, then $\theta_t=\lambda/\sigma$. Compare this result with that from Téllez-León *et al*. (2011) and Venegas-Martínez and Rodríguez-Nava (2010).[3]

Next, we characterize optimal decisions when the asset price is driven by a time-inhomogeneous Markov modulated Brownian motion without Poisson jumps. Notice that necessary conditions for a maximum lead to

$$c_t=\beta_1 a_t \qquad \text{and} \qquad \theta_t=\frac{\lambda_i}{\sigma_i}. \tag{32}$$

We observe that θ_t does not depend on t, it only depends on the state i, thus it is convenient to change notation to $\theta_i=\lambda_i/\sigma_i$. Hence, from (32), we get

$$\begin{aligned}0&=r\beta_1-\ln(\beta_1)-1-\rho\,\beta_0+(1-\rho\beta_1)\ln(a_t)\\&\quad+\frac{\partial g(t,i)}{\partial t}-\rho\,\mathrm{g}(t,i)+\frac{1}{2}\beta_1\lambda_i^2+\sum_{j\in E}q_{ij}(t)\left[g\left(t,j\right)-g\left(t,i\right)\right]\end{aligned}\tag{33}$$

Equation (33) holds for any value of a_t, then $1-\rho\beta_1=0$ or $\beta_1=\rho^{-1}$, thus, $c_t=\rho a_t$. Moreover,

$$\begin{aligned}0&=\frac{r}{\rho}+\ln(\rho)-1-\rho\,\beta_0+\frac{\partial g(t,i)}{\partial t}-\rho g(t,i)\\&\quad+\frac{1}{2\rho}\lambda_i^2+\sum_{j\in E}q_{ij}(t)\left[g\left(t,j\right)-g\left(t,i\right)\right].\end{aligned}\tag{34}$$

Now, it is clear that there is a part of the equation that does not depend on the state i, then the equation can be split in two parts which are equal to zero. That is,

Part 1: $$0=\frac{r}{\rho}+\ln(\rho)-1-\rho\,\beta_0 \tag{35}$$

[3] There is an extensive literature on the modeling of jumps in the underlying asset in pricing contingent claims; see, for example: Cox y Ross (1976), Ball y Torous (1985), Page y Sanders (1986), Cao (2001) and Chandrasekhar Reddy Gukhal (2004).

Part 2: $$0=\frac{\partial g(t,i)}{\partial t}-\rho g(t,i)+\frac{1}{2\rho}\lambda_i^2+\sum_{j\in E}q_{ij}(t)\left[g(t,j)-g(t,i)\right] \tag{36}$$

After solving (35) for β_0, we get

$$\beta_0=\rho^{-1}\left(\frac{r}{\rho}+\ln(\rho)-1\right) \tag{37}$$

In order to solve (36), we propose as a candidate of solution

$$\begin{aligned} g(t,i)&=\int_t^\infty \frac{1}{2\rho}\lambda_i^2 e^{-\rho(s-t)}\mathrm{d}s+\sum_{j\in E}\int_t^\infty q_{ij}(s)\left[g(s,j)-g(s,i)\right]e^{-\rho(s-t)}\mathrm{d}s\\ &=\frac{1}{2\rho^2}\lambda_i^2 e^{\rho t}\int_t^\infty \rho e^{-\rho s}\mathrm{d}s+e^{\rho t}\sum_{j\in E}\int_t^\infty q_{ij}(s)\left[g(s,j)-g(s,i)\right]e^{-\rho s}\mathrm{d}s \end{aligned} \tag{38}$$

The partial derivative of (38) with respect to t leads to

$$\begin{aligned} \frac{\partial g(t,i)}{\partial t}&=\rho\left(\int_t^\infty \frac{1}{2\rho}\lambda_i^2 e^{-\rho(s-t)}\mathrm{d}s+\sum_{j\in E}\int_t^\infty q_{ij}(s)\left[g(s,j)-g(s,i)\right]e^{-\rho(s-t)}\mathrm{d}s\right)\\ &-\left(\frac{1}{2\rho}\lambda_i^2+\sum_{j\in E}q_{ij}(t)\left[g(t,j)-g(t,i)\right]\right). \end{aligned} \tag{39}$$

By substituting $g(t,i)$ in the above expression, we have

$$\frac{\partial g(t,i)}{\partial t}-\rho g(t,i)+\frac{1}{2\rho}\lambda_i^2+\sum_{j\in E}q_{ij}(t)\left[g(t,j)-g(t,i)\right]=0\cdot \tag{40}$$

Hence, the proposed function fulfills the conditions to solve analytically the stated decision making problem.

We now study a specific case of a time-dependent Markov chain with transition probabilities that are stabilized as $t\to\infty$. In particular, consider a two-state set E with transition probabilities defined by

$$q_{11}(t)=1-\mathrm{e}^{-\xi_1 t},\quad q_{12}(t)=\mathrm{e}^{-\xi_1 t},\quad q_{21}(t)=\mathrm{e}^{-\xi_2 t}\ \text{ and }\ q_{22}(t)=1-\mathrm{e}^{-\xi_2 t} \tag{41}$$

with $\xi_i>0,\ i=1,2.$ Notice that the transition probabilities are stabilized at rate ξ_i as time grows, specifically $\lim_{t\to\infty}q_{11}(t)=\lim_{t\to\infty}q_{22}(t)=1$ and $\lim_{t\to\infty}q_{12}(t)=\lim_{t\to\infty}q_{21}(t)=0.$ In this case, the proposed function $g(t,i)$ is given by

$$g(t,i)=\int_t^\infty \frac{1}{2\rho}\lambda_i^2 e^{-\rho(s-t)}\mathrm{d}s+\int_t^\infty e^{-\xi_i s}\left[g(s,j)-g(s,i)\right]e^{-\rho(s-t)}\mathrm{d}s \tag{42}$$

The partial derivative of (42) with respect to *t* leads to

$$\begin{aligned} \frac{\partial g(t,i)}{\partial t}&=\rho\left(\int_t^\infty \frac{1}{2\rho}\lambda_i^2 e^{-\rho(s-t)}\mathrm{d}s+\int_t^\infty e^{-\xi_i s}\left[g(s,j)-g(s,i)\right]e^{-\rho(s-t)}\mathrm{d}s\right)\\ &\qquad-\left(\frac{1}{2\rho}\lambda_i^2+e^{-\xi_i t}\left[g(t,j)-g(t,i)\right]\right) \end{aligned} \tag{43}$$

By substituting $g(t,i)$ in (42), the above equation leads to

$$\frac{\partial g(t,i)}{\partial t}-\rho g(t,i)+\frac{1}{2\rho}\lambda_i^2+e^{-\xi_i t}\left[g(t,j)-g(t,i)\right]=0 \tag{44}$$

Therefore, the proposed candidate fulfills all the required conditions to solve the analytically the stated utility maximization problem.

Finally, we examine a time-dependent Markov chain with transition probabilities that do not have defined periods. To do that, we consider the logistic mapping

$$x_{n+1} = 4x_n(1-x_n), \tag{45}$$

which has a closed-form solution

$$x_n = \sin^2(2^{n-1}\cos^{-1}(1-2x_0)). \tag{46}$$

The above equation is a mapping taking values in $[0,1]$, which is useful for providing no periods. In particular, consider a two-state set E with transition probabilities defined by: $q_{11}(t) = 1-\sin^2(2^t\xi_1)$, $q_{12}(t) = \sin^2(2^t\xi_1)$, $q_{21}(t) = \sin^2(2^t\xi_2)$, and $q_{22}(t) = = 1-\sin^2(2^t\xi_2)$ with speed parameters $\xi_i \in (0,1), i = 1,2$. In this case, the proposed function $g(t,i)$ is given by

$$g(t,i) = \frac{1}{2\rho^2}\lambda_i^2 e^{\rho t}\int_t^\infty \rho e^{-\rho s}\mathrm{d}s + e^{\rho t}\int_t^\infty \sin^2(2^s\xi_i)\left[g(s,j)-g(s,i)\right]e^{-\rho s}\mathrm{d}s. \tag{47}$$

The partial derivative of (47) with respect to *t* leads to

$$\begin{aligned}\frac{\partial g(t,i)}{\partial t} &= \rho\left(\int_t^\infty \frac{1}{2\rho}\lambda_i^2 e^{-\rho(s-t)}ds + \int_t^\infty \sin^2(2^s\xi_i)\left[g(s,j)-g(s,i)\right]e^{-\rho(s-t)}ds\right) \\ &\quad -\left(\frac{1}{2\rho}\lambda_i^2 + \sin^2(2^s\xi_i)\left[g(t,j)-g(t,i)\right]\right).\end{aligned} \tag{48}$$

After substituting $g(t,i)$ in (48), and rearranging terms, we obtain

$$\frac{\partial g(t,i)}{\partial t} - \rho g(t,i) + \frac{1}{2\rho}\lambda_i^2 + \sin^2(2^s\xi_i)\left[g(t,j)-g(t,i)\right] = 0. \tag{49}$$

Hence, the proposed function $g(t,i)$ accomplishes all required conditions. Compare the above result with those from Vallejo-Jiménez *et al*. (2015). All the analyzed special cases highlight the benefits of the additional proposed structure, which substantially improves the understanding portfolio behavior.

5. Conclusions

The addition of both a time-inhomogeneous Markov chain and multiple Poisson jumps generalize previous results regarding optimal consumption and portfolio rules under uncertain environments. Furthermore, all desirable's statistical properties of the fractional Brownian motion widely used in financial modeling are now included in our proposal. Finally, in the developed model multiple jumps can be associated with sudden and unexpected leaps of the price itself, the sector, the news, the related market, etc. Needles to say, all of this provides a much richer and realistic environment to the consumer's decision making problem in risky environments.

We have also provided a summary of all the models included in our proposal and the extensions developed in this research. Several special cases were revisited and discussed with respect to the benefits of the additional structure, which improves the understanding of portfolio dynamics behavior.

A limitation of our proposal is that in order to obtain closed-form solutions it was assumed that the sizes of all jumps are fixed; however, all the parameter intensities are modified to compensate the jump size change. More work in this route will be done in the future. Of course, the developed model also enables us to calibrate, in future research, structural changes related to the physical trend, the instantaneous volatility and the interest rate.

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