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## Inseparables: exact potentials and addition

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## Abstract

In both universal classes of exact potential games described in the existing literature, congestion games and games with structured utilities, the players sum up their common local utilities. This paper shows that no other method to aggregate local utilities could guarantee the existence of an exact potential.

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## 1 Introduction

The most natural way to explain the motivation for this paper starts with the following results. Theorems 3.1 and 3.2 of Monderer and Shapley (1996) showed that a finite strategic game admits an exact potential if and only if it is isomorphic to a *congestion game* of Rosenthal (1973); Theorem 3.3 of Voorneveld *et al.* (1999) provides a more intuitive proof of the same fact. Theorem 5 of Kukushkin (2007) showed that a strategic game admits an exact potential if and only if it is isomorphic to a *game with structured utilities*.

In both universal classes of potential games, the players sum up *common local utilities* obtained from the use of certain "facilities"; the two classes differ in exactly what strategic possibilities are open to the players and in some other details. In a sense, those differences are of little import since Kukushkin (forthcoming), generalizing Le Breton and Weber (2011), defined a class of potential games including both as particular cases.

One can, theoretically, imagine similar models where local utilities would be aggregated in any number of different ways. However, Theorem 1 of Kukushkin (2007) showed that addition (possibly combined with monotone transformations of a certain kind) is the only way to ensure the existence of a Nash equilibrium regardless of other characteristics of the "generalized congestion" game, albeit under an *a priori* assumption that only continuous and strictly increasing functions can aggregate local utilities. Theorem 3 of Kukushkin (2007) established the necessity of addition in the same sense for games with structured utilities.

The strict monotonicity assumption was not accidental in either theorem. For instance, the minimum ("weakest link") aggregation also ensures the existence of Nash equilibrium in both cases. Moreover, in games with structured utilities it even ensures the existence of a *strong* Nash equilibrium (Kukushkin, 2017); actually, the minimum aggregation was used in the first models of this kind (Germeier and Vatel', 1974). In "bottleneck congestion games," a strong Nash equilibrium exists if every user negatively affects each facility (Fotakis *et al.*, 2008; Harks *et al.*, 2013; Kukushkin, 2017).

Theorems 2 and 4 of Kukushkin (2007) showed additive aggregation of local utilities to be necessary and sufficient for the ensured existence of an exact potential in a generalized congestion game or, respectively, in a generalized game with structured utilities. However, that *a priori* restriction to continuous and strictly increasing aggregation rules was carried over from Theorems 1 and 3. Thus, the question of whether there may exist other aggregation rules also ensuring the existence of an exact potential remained, strictly speaking, open.

This paper finally closes the gap. The necessity of addition is shown without *any* superfluous assumptions. The following section contains basic definitions. In Section 3, our main result is formulated, and in Section 4 proved.

## 2 Basic definitions

A strategic game $\Gamma$ is defined by a finite set $N$ of players, and, for each $i \in N$, a set $X_i$ of strategies and a real-valued utility function $u_i$ on the set $X_N := \prod_{i \in N} X_i$ of strategy profiles. We denote $\mathcal{N} := 2^N \setminus \{\emptyset\}$ and $X_I := \prod_{i \in I} X_i$ for each $I \in \mathcal{N}$. Given $i, j \in N$, we use notation $X_{-i}$ instead of $X_{N\setminus\{i\}}$ and $X_{-ij}$ instead of $X_{N\setminus\{i,j\}}$.

A function $P : X_N \to \mathbb{R}$ is an *exact potential* of $\Gamma$ (Monderer and Shapley, 1996) if

$$u_i(y_N) - u_i(x_N) = P(y_N) - P(x_N)$$

whenever $i \in N$, $y_N, x_N \in X_N$, and $y_{-i} = x_{-i}$.

The notion of a *game with* (*additive*) *common local utilities* (a *CLU game*) was introduced in Kukushkin (forthcoming). The defining feature of such a game is the following structure of utility functions. There is a set $\mathrm{A}$ of *facilities*; the set of all (nonempty) finite subsets of $\mathrm{A}$ is denoted $\mathcal{B}$. For each $i \in N$, there is a mapping $\mathrm{B}_i \colon X_i \to \mathcal{B}$ describing what facilities player $i$ uses having chosen $x_i$. For every $\alpha \in \mathrm{A}$, we denote $I_\alpha^+ := \{i \in N \mid \exists x_i \in X_i\,[\alpha \in \mathrm{B}_i(x_i)]\}$ (the set of players able to use facility $\alpha$) and $I_\alpha^- := \{i \in N \mid \forall x_i \in X_i\,[\alpha \in \mathrm{B}_i(x_i)]\}$ (the set of players unable *not* to use $\alpha$). Given $x_N \in X_N$, we denote $I(\alpha, x_N) := \{i \in N \mid \alpha \in \mathrm{B}_i(x_i)\}$ (the set of players who actually use $\alpha$ under strategies $x_N$); obviously, $I_\alpha^- \subseteq I(\alpha, x_N) \subseteq I_\alpha^+$.

For each $i \in I_\alpha^+$, we denote $X_i^\alpha := \{x_i \in X_i \mid \alpha \in \mathrm{B}_i(x_i)\}$. Then we define $\mathcal{I}_\alpha := \{I \in \mathcal{N} \mid I_\alpha^- \subseteq I \subseteq I_\alpha^+\}$ and $\Xi_\alpha := \{\langle I, x_I\rangle \mid I \in \mathcal{I}_\alpha \ \&\ x_I \in X_I^\alpha\}$. The *local utility function* at $\alpha \in \mathrm{A}$ is $\varphi_\alpha \colon \Xi_\alpha \to \mathbb{R}$. The *total utility function* of each player $i$ in a CLU game is

$$u_i(x_N) := \sum_{\alpha \in \mathrm{B}_i(x_i)} \varphi_\alpha(I(\alpha, x_N), x_{I(\alpha,x_N)}). \tag{1}$$

A facility $\alpha \in \mathrm{A}$ is *trim* if $\varphi_\alpha$ only depends on the number of users unless the facility is used by *all* potential users. More technically, $\alpha \in \mathrm{A}$ is trim if there is a real-valued function $\psi_\alpha(m)$ defined for integer $m$ between $\min_{I \in \mathcal{I}_\alpha} \#I = \max\{1, \#I_\alpha^-\}$ and $\#I_\alpha^+ - 1$ such that

$$\varphi_\alpha(I, x_I) = \psi_\alpha(\#I)$$

whenever $I \in \mathcal{I}_\alpha$, $I \neq I_\alpha^+$, and $x_I \in X_I^\alpha$. A CLU game is *trim* if every facility is trim.

By Theorem 1 of Kukushkin (forthcoming), every trim CLU game admits an exact potential.

Both congestion games (Rosenthal, 1973) and games with structured utilities (Kukushkin, 2007) are trim CLU games. In the former case, $X_i \subseteq \mathcal{B}$ for each $i \in N$, each $\mathrm{B}_i$ is an identity mapping, and hence the second argument of $\varphi_\alpha$ can be dropped; besides, $\varphi_\alpha$ only depends on $\#I$. In the latter case, for each $i \in N$, the set $\mathrm{B}_i(x_i)$ is the same for all $x_i \in X_i$; hence $I(\alpha, x_N)$ does not depend on the second argument and hence the first argument of $\varphi_\alpha$ can be dropped.

The definition of a *generalized game with common local utilities* (a *GCLU game*) differs from that of a CLU game in just one point: each player's total utility is an arbitrary aggregate of local utilities over $\mathrm{B}_i(x_i)$. In other words, for every $i \in N$ and $x_i \in X_i$, there is an *aggregation rule* $U_i^{x_i} \colon \mathbb{R}^{\mathrm{B}_i(x_i)} \to \mathbb{R}$. The *total utility function* of each player $i$ in a GCLU game is

$$u_i(x_N) := U_i^{x_i}\big(\big\langle \varphi_\alpha(I(\alpha, x_N), x_{I(\alpha,x_N)})\big\rangle_{\alpha \in \mathrm{B}_i(x_i)}\big).$$

## 3 Main Theorem

The same symbol "$\sum$" can be employed in (1) regardless of the set $\mathrm{B}_i(x_i)$ because addition is commutative and associative. If we want to contemplate the replacement of addition with another function, or other functions, possibly devoid of those properties, we have to fix some technical details.

An *abstract aggregation rule* over a set $R \subseteq \mathbb{R}$ is a mapping from a (finite) Cartesian power of $R$ to $\mathbb{R}$, $U\colon R^{\Sigma(U)} \to \mathbb{R}$, where $\Sigma(U)$ is a finite set (of the names for the arguments). To use an abstract aggregation rule $U$ as $U_i^{x_i}$ (given a GCLU game, $i \in N$, and $x_i \in X_i$), we need two conditions to hold: First, $\Sigma(U)$ and $\mathrm{B}_i(x_i)$ must have the same cardinality. Second, all values of $\varphi_\alpha(I(\alpha, x_N), x_{I(\alpha,x_N)})$ for all $\alpha \in \mathrm{B}_i(x_i)$ must belong to $R$. If both conditions hold, a bijection $\mu\colon \Sigma(U) \to \mathrm{B}_i(x_i)$ has to be chosen, which would indicate, for each position among the arguments of $U$, which value of $\varphi_\alpha(I(\alpha, x_N), x_{I(\alpha,x_N)})$ should take this position; a complete description of relevant formalism can be found in Kukushkin (2017, Section 6).

Given a set of abstract aggregation rules $\mathfrak{U}$ and a GCLU game $\Gamma$, we say that a player $i$ *uses aggregation rules from* $\mathfrak{U}$ in $\Gamma$ if, for every $x_i \in X_i$, there are $U \in \mathfrak{U}$ and a bijection $\mu_i^{x_i}\colon \Sigma(U) \to \mathrm{B}_i(x_i)$ such that $U_i^{x_i}$ is obtained from $U$ by the substitution described by $\mu_i^{x_i}$.

**Theorem 1.** *Let $N$ be a finite set with $\#N \geq 2$. Let $R$ be a subset of $\mathbb{R}$. Let $\mathfrak{U}_i$, for each $i \in N$, be a nonempty set of abstract aggregation rules over $R$. Let $\mathfrak{U}$ denote $\bigcup_{i\in N} \mathfrak{U}_i$. Then the following conditions are equivalent.*

**1.1.** *Every trim GCLU game where $N$ is the set of players and each player $i$ uses aggregation rules from $\mathfrak{U}_i$ admits an exact potential.*

**1.2.** *Every generalized congestion game where $N$ is the set of players and each player $i$ uses aggregation rules from $\mathfrak{U}_i$ admits an exact potential.*

**1.3.** *Every generalized game with structured utilities where $N$ is the set of players and each player $i$ uses aggregation rules from $\mathfrak{U}_i$ admits an exact potential.*

**1.4.** *There is a mapping $\lambda : R \to \mathbb{R}$ and, for every $U \in \mathfrak{U}$, a constant $C(U) \in \mathbb{R}$ such that*

$$U(v_{\Sigma(U)}) = C(U) + \sum_{s\in\Sigma(U)} \lambda(v_s) \tag{2}$$

*for all $U \in \mathfrak{U}$ and $v_{\Sigma(U)} \in R^{\Sigma(U)}$.*

**Remark.** In contrast to Kukushkin (2007, 2017), the set $R$ here need not be an open interval in $\mathbb{R}$; e.g., the theorem remains valid if the attention is restricted to games with integer values of local utility functions.

In simple words, Theorem 1 delivers on the promise made in Introduction: The existence of an exact potential in every game of this kind is ensured if and only if each player sums up the relevant utilities. To be more precise, (2) also allows a strictly monotone transformation $\lambda$, the same for all strategies of all players, and adding a constant $C(U)$, which may depend on the player and the strategy.

## 4 Proof

The implications Condition 1.1 $\Rightarrow$ Condition 1.2 and Condition 1.1 $\Rightarrow$ Condition 1.3 are trivial.

Assuming Condition 1.4 to hold, let us prove Condition 1.1. Let $\Gamma$ be a trim GCLU game where $N$ is the set of players and each player $i$ uses aggregation rules from $\mathfrak{U}_i$. We have to show that $\Gamma$ admits an exact potential. We define a CLU game $\Gamma^*$ by: $N^* := N$; $\mathrm{A}^* := \mathrm{A} \cup \bigcup_{i \in N}(\{i\} \times X_i)$; $X_i^* := X_i$ for each $i \in N$; $\mathrm{B}_i^*(x_i) := \mathrm{B}_i(x_i) \cup \{(i, x_i)\}$ for each $i \in N$ and every $x_i \in X_i$; $\varphi_\alpha^*(I, x_I) := \lambda \circ \varphi_\alpha(I, x_I)$ for all $\alpha \in \mathrm{A}$, $I \in \mathcal{I}_\alpha$, and $x_I \in X_I^\alpha$; $\varphi^*_{(i,x_i)}(\{i\}, x_i) := C(U_i^{x_i})$ for each $i \in N$ and every $x_i \in X_i$. These two facts are easy to check: first, $\Gamma^*$ is trim, and hence admits an exact potential by Theorem 1 of Kukushkin (forthcoming); second, $u_i^*(x_N) = u_i(x_N)$ for all $x_N \in X_N$, and hence every exact potential of $\Gamma^*$ is an exact potential of $\Gamma$ as well.

Now let us turn to the implications Condition 1.2 $\Rightarrow$ Condition 1.4 and Condition 1.3 $\Rightarrow$ Condition 1.4; the two proofs only differ in the first step.

**Claim 1.** *Let Condition 1.2 hold. Let $i, j \in N$, $i \neq j$, $U \in \mathfrak{U}_i$, $V \in \mathfrak{U}_j$, $s \in \Sigma(U)$, $t \in \Sigma(V)$, $u, u' \in R$, $v_{\Sigma(U)}, v'_{\Sigma(U)} \in R^{\Sigma(U)}$, $v_s = u$, $v'_s = u'$, $v_{-s} = v'_{-s}$, $w_{\Sigma(V)}, w'_{\Sigma(V)} \in R^{\Sigma(V)}$, $w_t = u$, $w'_t = u'$, $w_{-t} = w'_{-t}$. Then*

$$U(v'_{\Sigma(U)}) - U(v_{\Sigma(U)}) = V(w'_{\Sigma(V)}) - V(w_{\Sigma(V)}).$$

*Proof of Claim 1.* For each $h \in N \setminus \{i, j\}$, we fix a $U_h \in \mathfrak{U}_h$ and denote $U_i := U$ and $U_j := V$. Then we consider a generalized congestion game where: $\mathrm{A} := \{\alpha, \beta\} \cup \bigcup_{h \in N}[\{h\} \times \Sigma(U_h)]$, assuming $\{\alpha, \beta\} \cap \bigcup_{h \in N}[\{h\} \times \Sigma(U_h)] = \emptyset$; $X_i := \{x_i, y_i\}$ with $x_i := \{\alpha\} \cup (\{i\} \times [\Sigma(U) \setminus \{s\}])$ and $y_i := \{\beta\} \cup (\{i\} \times [\Sigma(U) \setminus \{s\}])$, $X_j := \{x_j, y_j\}$ with $x_j := \{\alpha\} \cup (\{j\} \times [\Sigma(V) \setminus \{t\}])$ and $y_j := \{\beta\} \cup (\{j\} \times [\Sigma(V) \setminus \{t\}])$, and $X_h := \{\{h\} \times \Sigma(U_h)\}$ for each $h \in N \setminus \{i, j\}$ (having singleton strategy sets, players $h \neq i, j$ participate in the game only in a purely technical sense); player $i$ uses aggregation rule $U$ with both strategies, player $j$ uses aggregation rule $V$ with both strategies, each player $h \neq i, j$ uses aggregation rule $U_h$; $\mu_i^{x_i}(s) := \mu_j^{x_j}(t) := \alpha$, $\mu_i^{y_i}(s) := \mu_j^{y_j}(t) := \beta$, $\mu_i^{x_i}(s') := \mu_i^{y_i}(s') := (i, s')$ for all $s' \in \Sigma(U) \setminus \{s\}$, $\mu_j^{x_j}(t') := \mu_j^{y_j}(t') := (j, t')$ for all $t' \in \Sigma(V) \setminus \{t\}$, $\mu_h^{x_h}(s') := (h, s')$ for all $h \neq i, j$ and $s' \in \Sigma(U_h)$; $\psi_\alpha(2) := \psi_\beta(2) := u$, $\psi_\alpha(1) := \psi_\beta(1) := u'$, $\psi_{(i,s')}(1) := v_{s'}\, [= v'_{s'}]$ for all $s' \in \Sigma(U) \setminus \{s\}$, $\psi_{(j,t')}(1) := w_{t'}\, [= w'_{t'}]$ for all $t' \in \Sigma(V) \setminus \{t\}$.

Since only players $i$ and $j$ matter, the game is adequately described by the following $2 \times 2$ matrix (player $i$ chooses rows, player $j$ columns):

$$\begin{array}{ccc} & x_j & y_j \\ x_i & \langle U(v_{\Sigma(U)}), V(w_{\Sigma(V)})\rangle & \langle U(v'_{\Sigma(U)}), V(w'_{\Sigma(V)})\rangle \\ y_i & \langle U(v'_{\Sigma(U)}), V(w'_{\Sigma(V)})\rangle & \langle U(v_{\Sigma(U)}), V(w_{\Sigma(V)})\rangle. \end{array}$$

Straightforward calculations show that $0 = [P(y_i, x_j) - P(x_i, x_j)] + [P(y_i, y_j) - P(y_i, x_j)] + [P(x_i, y_j) - P(y_i, y_j)] + [P(x_i, x_j) - P(x_i, y_j)] = [U(v'_{\Sigma(U)}) - U(v_{\Sigma(U)})] + [V(w_{\Sigma(V)}) - V(w'_{\Sigma(V)})] + [U(v'_{\Sigma(U)}) - U(v_{\Sigma(U)})] + [V(w_{\Sigma(V)}) - V(w'_{\Sigma(V)})]$; therefore, $U(v'_{\Sigma(U)}) - U(v_{\Sigma(U)}) = V(w'_{\Sigma(V)}) - V(w_{\Sigma(V)})$ indeed. $\square$

**Claim 2.** *Let Condition 1.3 hold. Let $i, j \in N$, $i \neq j$, $U \in \mathfrak{U}_i$, $V \in \mathfrak{U}_j$, $s \in \Sigma(U)$, $t \in \Sigma(V)$, $u, u' \in R$, $v_{\Sigma(U)}, v'_{\Sigma(U)} \in R^{\Sigma(U)}$, $v_s = u$, $v'_s = u'$, $v_{-s} = v'_{-s}$, $w_{\Sigma(V)}, w'_{\Sigma(V)} \in R^{\Sigma(V)}$, $w_t = u$, $w'_t = u'$, $w_{-t} = w'_{-t}$. Then*

$$U(v'_{\Sigma(U)}) - U(v_{\Sigma(U)}) = V(w'_{\Sigma(V)}) - V(w_{\Sigma(V)}).$$

*Proof of Claim 2.* There is much similarity with the proof of Claim 1. For each $h \in N \setminus \{i, j\}$, we fix a $U_h \in \mathfrak{U}_h$, denote $U_i := U$ and $U_j := V$, and consider a generalized game with structured utilities where: $\mathrm{A} := \{\alpha\} \cup \bigcup_{h \in N} [\{h\} \times \Sigma(U_h)]$, assuming $\alpha \notin \bigcup_{h \in N} [\{h\} \times \Sigma(U_h)]$; $X_i := \{x_i, y_i\}$, $X_j := \{x_j, y_j\}$, $X_h := \{x_h\}$ for each $h \in N \setminus \{i, j\}$ (again, players $h \neq i, j$ participate in the game only in a purely technical sense); $\mathrm{B}_i := \{\alpha\} \cup (\{i\} \times [\Sigma(U) \setminus \{s\}])$, $\mathrm{B}_j := \{\alpha\} \cup (\{j\} \times [\Sigma(V) \setminus \{t\}])$, $\mathrm{B}_h := \{h\} \times \Sigma(U_h)$ for each $h \neq i, j$; player $i$ uses aggregation rule $U$ with both strategies, player $j$ uses aggregation rule $V$ with both strategies, each player $h \neq i, j$ uses aggregation rule $U_h$; $\mu_i^{x_i}(s) := \mu_i^{y_i}(s) := \mu_j^{x_j}(t) := \mu_j^{y_j}(t) := \alpha$, $\mu_i^{x_i}(s') := \mu_i^{y_i}(s') := (i, s')$ for all $s' \in \Sigma(U) \setminus \{s\}$, $\mu_j^{x_j}(t') := \mu_j^{y_j}(t') := (j, t')$ for all $t' \in \Sigma(V) \setminus \{t\}$, $\mu_h^{x_h}(s') := (h, s')$ for all $h \neq i, j$ and $s' \in \Sigma(U_h)$; $\varphi_\alpha(x_i, x_j) := \varphi_\alpha(y_i, y_j) := u$, $\varphi_\alpha(x_i, y_j) := \varphi_\alpha(y_i, x_j) := u'$, $\varphi_{(i,s')}(x_i) := \varphi_{(i,s')}(y_i) := v_{s'}$ $[= v'_{s'}]$ for all $s' \in \Sigma(U) \setminus \{s\}$, $\varphi_{(j,t')}(x_j) := \varphi_{(j,t')}(y_j) := w_{t'}$ $[= w'_{t'}]$ for all $t' \in \Sigma(V) \setminus \{t\}$.

Since only players $i$ and $j$ matter, the game is adequately described by the same $2 \times 2$ matrix as in the proof of Claim 1:

$$\begin{array}{ccc} & x_j & y_j \\ x_i & \langle U(v_{\Sigma(U)}), V(w_{\Sigma(V)}) \rangle & \langle U(v'_{\Sigma(U)}), V(w'_{\Sigma(V)}) \rangle \\ y_i & \langle U(v'_{\Sigma(U)}), V(w'_{\Sigma(V)}) \rangle & \langle U(v_{\Sigma(U)}), V(w_{\Sigma(V)}) \rangle. \end{array}$$

The same calculations show that $U(v'_{\Sigma(U)}) - U(v_{\Sigma(U)}) = V(w'_{\Sigma(V)}) - V(w_{\Sigma(V)})$ again. □

The rest of the proof is the same under Condition 1.2 or Condition 1.3.

**Claim 3.** *Let $U, V \in \mathfrak{U}$, $s \in \Sigma(U)$, $t \in \Sigma(V)$, $u, u' \in R$, $v_{\Sigma(U)}, v'_{\Sigma(U)} \in R^{\Sigma(U)}$, $v_s = u$, $v'_s = u'$, $v_{-s} = v'_{-s}$, $w_{\Sigma(V)}, w'_{\Sigma(V)} \in R^{\Sigma(V)}$, $w_t = u$, $w'_t = u'$, $w_{-t} = w'_{-t}$. Then*

$$U(v'_{\Sigma(U)}) - U(v_{\Sigma(U)}) = V(w'_{\Sigma(V)}) - V(w_{\Sigma(V)}).$$

*Proof of Claim 3.* We consider two alternatives. If $i, j \in N$ can be found such that $i \neq j$, $U \in \mathfrak{U}_i$ and $V \in \mathfrak{U}_j$, then we just invoke Claim 1 or Claim 2 and are home immediately.

Suppose the contrary: both $U$ and $V$ only belong to the same $\mathfrak{U}_i$ ($i \in N$). Then we pick $h \in N \setminus \{i\}$, $\tilde{U} \in \mathfrak{U}_h$ and $\tilde{s} \in \Sigma(\tilde{U})$ arbitrarily, and define $\tilde{v}_{\Sigma(\tilde{U})}, \tilde{v}'_{\Sigma(\tilde{U})} \in R^{\Sigma(\tilde{U})}$ in this way: $\tilde{v}_{\tilde{s}'} := u$ for all $\tilde{s}' \in \Sigma(\tilde{U})$; $\tilde{v}'_{\tilde{s}} := u'$; $\tilde{v}'_{\tilde{s}'} := u$ for all $\tilde{s}' \in \Sigma(\tilde{U}) \setminus \tilde{s}$. Applying Claim 1 or Claim 2 to $i$, $h$, $U$, $\tilde{U}$, etc., we obtain $U(v'_{\Sigma(U)}) - U(v_{\Sigma(U)}) = \tilde{U}(\tilde{v}'_{\Sigma(\tilde{U})}) - \tilde{U}(\tilde{v}_{\Sigma(\tilde{U})})$. Applying Claim 1 or Claim 2 to $h$, $i$, $\tilde{U}$, $V$, etc., we obtain $\tilde{U}(\tilde{v}'_{\Sigma(\tilde{U})}) - \tilde{U}(\tilde{v}_{\Sigma(\tilde{U})}) = V(w'_{\Sigma(V)}) - V(w_{\Sigma(V)})$. □

**Remark.** The assumption $\#N \geq 2$ played a crucial role here.

Now we may define a function $\Delta\colon R^2 \to \mathbb{R}$ by $\Delta(u, u') := U(v'_{\Sigma(U)}) - U(v_{\Sigma(U)})$ provided there is $s \in \Sigma(U)$ such that $v_s = u$, $v'_s = u'$, and $v_{-s} = v'_{-s}$. Claim 3 ensures that the choice of $U \in \mathfrak{U}$, $s \in \Sigma(U)$ and $v_{-s} = v'_{-s}$ is irrelevant.

**Claim 4.** *For every $U \in \mathfrak{U}$ and $v_{\Sigma(U)}, v'_{\Sigma(U)} \in R^{\Sigma(U)}$, there holds*

$$U(v'_{\Sigma(U)}) - U(v_{\Sigma(U)}) = \sum_{s \in \Sigma(U)} \Delta(v_s, v'_s). \tag{3}$$

*Proof of Claim 4.* Ordering $\Sigma(U)$ in an arbitrary way, we assume that $\Sigma(U) = \{1, \ldots, m\}$. For each $k \in \{0, \ldots, m\}$, we define $v^k_{\Sigma(U)} \in R^{\Sigma(U)}$ by $v^k_s := v_s$ for $s > k$ and $v^k_s := v'_s$ for $s \leq k$. Obviously, $v^0_{\Sigma(U)} = v_{\Sigma(U)}$, $v^m_{\Sigma(U)} = v'_{\Sigma(U)}$, and $U(v'_{\Sigma(U)}) - U(v_{\Sigma(U)}) = \sum_{k=0}^{m} [U(v^{k+1}_{\Sigma(U)}) - U(v^k_{\Sigma(U)})]$. Since $U(v^{k+1}_{\Sigma(U)}) - U(v^k_{\Sigma(U)}) = \Delta(v_s, v'_s)$ by Claim 3, we have (3). $\square$

To finish the proof, we pick $u^0 \in R$ arbitrarily and set $\lambda(u) := \Delta(u^0, u)$ for every $u \in R$; for every $U \in \mathfrak{U}$, we define $v^0_{\Sigma(U)} \in R^{\Sigma(U)}$ by $v^0_s := u^0$ for all $s \in \Sigma(U)$, and set $C(U) := U(v^0_{\Sigma(U)})$. Now (3) immediately implies (2).

## References


Fotakis, D., Kontogiannis, S., and P. Spirakis (2008) "Atomic congestion games among coalitions" *ACM Transactions on Algorithms* **4** (4), Art. 52

Germeier, Yu.B. and I.A. Vatel' (1974) "On games with a hierarchical vector of interests" *Izvestiya Akademii Nauk SSSR, Tekhnicheskaya Kibernetika*, 3, 54-69 [in Russian; English translation in *Engineering Cybernetics* **12**(3), 25-40 (1974)].

Harks, T., Klimm, M., and R.H. Möhring (2013) "Strong equilibria in games with the lexicographical improvement property" *International Journal of Game Theory* **42**, 461-482.

Kukushkin, N.S. (2007) "Congestion games revisited" *International Journal of Game Theory* **36**, 57-83.

Kukushkin, N.S. (2017) "Strong Nash equilibrium in games with common and complementary local utilities" *Journal of Mathematical Economics* **68**, 1-12.

Kukushkin, N.S. (forthcoming) "A universal construction generating potential games" *Games and Economic Behavior*, accepted in 2017 (doi 10.1016/j.geb.2017.02.012)

Le Breton, M. and S. Weber (2011) "Games of social interactions with local and global externalities" *Economics Letters* **111**, 88-90.

Monderer, D. and L.S. Shapley (1996) "Potential games" *Games and Economic Behavior* **14**, 124-143.

Rosenthal, R.W. (1973) "A class of games possessing pure-strategy Nash equilibria" *International Journal of Game Theory* **2**, 65-67.

Voorneveld, M., Borm, P., van Megen, F., Tijs, S., and G. Facchini (1999) "Congestion games and potentials reconsidered" *International Game Theory Review* **1**, 283-299.