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Economic Growth and the CES Production Function with Human Capital

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Abstract

Employing a neoclassical growth model with a constant elasticity of substitution production function, we develop a comparative static and dynamic analysis of the effects of the elasticity of substitution between inputs on the steady state growth path, growth threshold, speed of convergence and savings rates. Unlike earlier studies along these lines, we incorporate human capital, along with physical capital and raw labor, as a third input in the production function. We prove that a higher elasticity of substitution between inputs can lead to a higher steady state level for income per effective unit of labor. To illustrate the quantitative significance of the elasticity of substitution, we consider two ways in which human capital enters the production function. Employing cross country data, we find estimates for the normalized CES production functions with human capital to be significantly below unity.

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1 Introduction

The importance of human capital in economic growth has long been emphasized in the economic growth literature. Two classic works, Lucas (1988) and Mankiw *et al.* (1992) emphasize human capital and its accumulation as important feature of economic development. These works construct neoclassical growth models that employ a Cobb–Douglas aggregate production function, for the goods market, and assume accumulation of human capital over time, while incorporating human capital in different fashions: one as labor augmenting and other as a separate factor of production. Although, there are many other differences in these two works, like many papers that include human capital, the use of the Cobb–Douglas aggregate production function is common to both.[1] A drawback to relying on the Cobb–Douglas production function is that the elasticity of substitution between a pair of inputs is unity, inherently, and this does not emphasize the importance of the level of elasticity of substitution for economic growth. Theoretical works of De La Grandville (1989), Klump & Grandville (2000), and Klump & Preissler (2000) have emphasized that the level of elasticity of substitution is crucial for economic growth. In addition, recent empirical papers, such as Antràs (2004) and Klump *et al.* (2007), have emphasized that the aggregate production function is not Cobb–Douglas and the assumption of a unitary elasticity of substitution is dubious, at least in the context of the United States.

In this paper, we relax the assumption of a unitary elasticity of substitution and construct a neoclassical growth model employing an aggregate constant elasticity of substitution (CES, henceforth) production function that incorporates three factors of production: physical capital, raw labor, and human capital. We consider two cases: 1. where human capital is a separate factor of production eschewing the "perfect substitutability" between human capital and labor inputs, following Mankiw *et al.* (1992), and 2. where human capital is labor augmenting to allow for the assumption of labor and human capital being "perfect substitutes" in production like most other studies.[2] Next, we examine the importance of the elasticity of substitution on economic growth using the normalization approach, De La Grandville (1989). This approach facilitates the identification of the effects of elasticity of substitution on economic growth and convergence by defining several model parameters as functions of the elasticity of substitution. Finally, we estimate the elasticity of substitution using cross-country data for both models.

This study contributes both, theoretically and empirically, to the importance of the elasticity of substitution for economic growth models with human capital. In our theoretical exercise, we demonstrate that a higher elasticity of substitution between inputs can lead to a higher steady state level for income per effective unit of labor (economic growth). The long-run levels for physical capital and human capital per effective unit of labor are both influenced by the elasticity of substitution. Our empirical exercise illustrates the quantitative significance of the elasticity of substitution, while allowing for human capital to enter the production function. Employing cross country data, we find estimates for the elasticity of

[1] Mankiw *et al.* (1992) explains that properly accounting for human capital may alter one's view of the economic growth process.

[2] Labor and human capital being "perfect substitutes" in production imply that increasing labor by one unit has the same effect on production as increasing human capital by one unit, see Barro & Sala-I-Martin (2004, p. 240).

substitution to be significantly below unity.

2 Related Literature

The use of the Cobb–Douglas aggregate production function has long been debated in the dynamic macroeconomic literature. Berndt (1976) has supported the use of the Cobb–Douglas specification as an aggregate production function, for U.S. data. While others, such as Antràs (2004), have argued against the use of the Cobb–Douglas production function, in the same context. Antràs (2004) suggested that the U.S. aggregate production function can be misleading by assuming Hicks neutral technology and that this assumption biases the functional form of the aggregate production function in favor of the Cobb–Douglas suggesting that the production function should assume CES with biased technology. De La Grandville (1989) and Klump & Preissler (2000) corroborated Antràs's analysis and found that a higher level of elasticity of substitution leads to a higher steady state level of capital per worker.

De La Grandville (1989) emphasized the importance of the elasticity of substitution by suggesting that the parameters of the CES could be endogenously influenced by the elasticity of substitution. This is achieved by writing the model parameters as a function of the elasticity of substitution and arbitrary baseline values for capital and labor—or the capital-labor ratio—and the marginal rate of technical substitution for the inputs. This technique is referred to as normalizing the CES production function. Klump & Preissler (2000) and Klump *et al.* (2007) have investigated the application of the normalization approach developed by De La Grandville (1989). They found that for a neoclassical growth model a higher elasticity of substitution implies a higher steady state level of income per capita. They further developed this methodology for two factors of production to reinforce the importance of the elasticity of substitution for economic growth.

Estimates of the level of elasticity of substitution between physical capital and labor have been found to be significantly less than unity for the US. Antràs (2004) estimated a range between .407 to .948 and Klump *et al.* (2007) estimated a value around 0.5. More recently, Mallick (2012) provided country specific estimates for the elasticity of substitution and suggested that the elasticity of substitution for most countries, significantly differ from unity. León-Ledesma *et al.* (2010) identified the conditions under which identification of the elasticity of substitution and biased technological change are feasible and robust.[3]

3 Model and Normalization

Solow (1956) examined long-run growth in relation to numerous specifications of the aggregate production function. One well known specification is the CES production function. This specification includes two factors of production: physical capital and labor, and does not include technological progress.

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[3] Masanjala & Papageorgiou (2004) find the estimate to be greater than one in a non normalized CES production function. Klump *et al.* (2012) discuss various benefits that normalization brings for empirical estimation and empirical growth research. They suggest that neglecting normalization techniques can significantly bias results. Normalization under a CES production function (which is highly non-linear) removes the problem that arises from the fact that labor, capital and human capital are measured in different units.

We modify the CES aggregate production function to include three factors of production, and incorporate human capital as a third input:

$$Y = F(K, H, L) = [\alpha K^{\psi} + \beta H^{\psi} + (1 - \alpha - \beta)(AL)^{\psi}]^{\frac{1}{\psi}}, \tag{1}$$

where K denotes physical capital, H denotes human capital, and L is labor.[4] The model parameters are α, β, and ψ, where $\psi = \psi(\sigma) = \frac{\sigma-1}{\sigma}$ is the substitution parameter and σ is the partial elasticity of substitution (elasticity of substitution, hereafter).[5] We follow Mankiw *et al.* (1992) and assume the level of labor and technology at time t are determined by $L = L_0 e^{nt}$ and $A = A_0 e^{gt}$, respectively, where the initial level of labor and technology are given by L_0 and A_0, respectively. The growth rates of labor and technology are assumed to be exogenous at population growth rate, n, and growth rate, g, respectively. We assume only labor augmenting technological change, for two purposes. First, this assumption is consistent with the three factor aggregate production of Mankiw *et al.* (1992), which is a limiting case of (1) when $\sigma \to 1$ ($\psi \to 0$).[6] Second, it is necessary for technology to be only labor augmenting for a steady state with constant growth rates.[7] We denote the factors of production and output in per effective worker terms by K/AL, H/AL and Y/AL by lower case letters k, h, and y, respectively.

3.1 CES Production Function with Additive Human Capital

We apply the normalization technique to the CES production functions that incorporate human capital to identify of the effects of the elasticity of substitution on economic growth. Following Klump & Preissler (2000), we assume initial values, as baseline values, for physical capital, human capital, and labor denoted by K_0, H_0, and L_0, respectively. We assume initial values for the marginal rate of technical substitution for labor and physical capital, labor and human capital, and physical and human capital, denoted by μ_0, γ_0, ρ_0, respectively.

Normalizing the parameters for the aggregate production function, (1), with additive human capital yields the following, (see Appendix A for the derivation):

$$\begin{aligned}
\alpha(\sigma) &= \frac{\rho_0 K_0^{1-\psi} Y_0^{\psi}}{(\rho_0 K_0 + H_0) + \rho_0 \mu_0 L_0} = \pi_K^0 \left(\frac{Y_0}{K_0}\right)^{\psi}, \\
\beta(\sigma) &= \frac{H_0^{1-\psi} Y_0^{\psi}}{(\rho_0 K_0 + H_0) + \rho_0 \mu_0 L_0} = \pi_H^0 \left(\frac{Y_0}{H_0}\right)^{\psi}, \quad \text{and} \\
A_0(\sigma) &= \left(\frac{Y_0^{\psi} - \alpha K_0^{\psi} - \beta H_0^{\psi}}{L_0^{\psi}(1-\alpha-\beta)}\right)^{\frac{1}{\psi}} = \left(\frac{\left(1-\pi_K^0-\pi_H^0\right)\left(\frac{Y_0}{L_0}\right)^{\psi}}{1-\pi_K^0\left(\frac{Y_0}{K_0}\right)^{\psi}-\pi_H^0\left(\frac{Y_0}{H_0}\right)^{\psi}}\right)^{\frac{1}{\psi}},
\end{aligned} \tag{2}$$

where π_K^0 and π_H^0 are the factor share of physical and human capital at an arbitrary baseline value and the substitution parameter is a function of the elasticity of substitution, $\psi(\sigma)$.

[4] A more general specification for the aggregate production function could include a varying elasticity of substitution between inputs.

[5] Following Allen (1938, p. 503–509), we assume that the partial elasticity of substitution for all factors of production are the same, i.e. $\sigma_{i,j} = \sigma$ for all $i \neq j$ where $i, j \in \{K, H, L\}$.

[6] The Mankiw *et al.* (1992) functional form for the three-factor aggregate production function is given by $Y(t) = K(t)^{\alpha} H(t)^{\beta} [A(t)L(t)]^{1-\alpha-\beta}$.

[7] See the Barro & Sala-I-Martin (2004, p. 78–80) proof that technological progress must be labor augmenting.

Incorporating the normalized parameters, we rewrite the aggregate production function, (1), by replacing the model parameters, α and β, and the initial level of technology, A_0, by their normalized counterparts.[8]

3.2 CES Production Function with Labor Augmenting Human Capital

The seminal work of Lucas (1988) as well as other recent works, introduce human capital as a factor of production that is non-separable from labor. Unlike the previous subsection, human capital is not assumed to have an explicit market. Alternatively, the labor market is affected by the level of human capital per worker. Following this literature, we introduced human capital into the CES production by assuming that human capital can not be separated from labor. The human capital stock, H, is described as each worker in the labor force, L, with human capital $\hat{h}$, H is equal to $\hat{h}L$.[9]

As before, the aggregate production function is described by (1). Under this scenario, hiring one more worker implies hiring an additional unit of labor equipped with $\hat{h}$.

Normalizing the parameters for the aggregate production function, (1), with additive human capital yields the following, (see Appendix A for the derivation):

$$\begin{aligned} \alpha(\sigma) &= \frac{\beta H_0^{\psi}+(1-\beta)(A_0L_0)^{\psi}}{\mu_0 K_0^{\psi-1}L_0+(A_0L_0)^{\psi}} = \pi_0\left(\frac{Y_0}{K_0}\right)^{\psi} \quad \text{and} \\ \beta(\sigma) &= \frac{Y_0^{\psi}(\mu_0K_0^{\psi-1}L_0+(A_0L_0)^{\psi})-(A_0L_0)^{\psi}(\mu_0K_0^{\psi-1}L_0+K_0^{\psi})}{(H_0^{\psi}-(A_0L_0)^{\psi})(\mu_0K_0^{\psi-1}L_0+K_0^{\psi})} = \frac{Y_0^{\psi}(1-\pi_0)-(A_0L_0)^{\psi}\left(1-\pi_0\left(\frac{Y_0}{K_0}\right)^{\psi}\right)}{H_0^{\psi}-(A_0L_0)^{\psi}}. \end{aligned} \tag{3}$$

Since we do not allow for a market for human capital, we cannot identify the initial level of technology and both model parameters α and β as a function of baseline values for the factors of production, factor shares or relative prices, and the elasticity of substitution. Instead, we identify α and β as a function of the elasticity of substitution, initial level of technology, A_0, and baseline for the inputs and output. We represent the normalized parameters in per effective worker terms for convenience.

4 Elasticity of Substitution and Economic Growth

4.1 Additive Human capital

We assume a one sector economy that produces a homogenous good, Y, under perfect competition to examine the impact of the elasticity of substitution on long-run growth. Following Mankiw *et al.* (1992), we assume exogenous savings rates, where the marginal propensity to save for physical capital and human capital are denoted by, s_k and s_h, and the total income saved is given by $(s_k + s_h)Y$. The goods market is assumed to be in equilibrium when investment in each type of capital is equal to the savings for each type of capital ($s_kY = I_K$ and $s_hY = I_H$). Both types of capital are assumed to accumulate in

[8] Klump & Preissler (2000) explain that the normalized CES represents a family of aggregate production functions that share the same baseline values.

[9] See, chapter 7 Sørensen & Whitta-Jacobsen (2005)

a similar fashion and depreciate at the same rate as in Mankiw *et al.* (1992). The capital accumulation equations for physical capital and human capital are determined by:

$$\begin{aligned} \dot{K} &= I_K - \delta K \quad \text{and} \\ \dot{H} &= I_H - \delta H. \end{aligned} \tag{4}$$

In the balanced growth steady state with $\dot{k}(t) = \dot{h}(t) = 0$, break-even investment for each type of capital per effective unit of labor is equal to its respective savings. The balanced growth steady state levels for physical capital, human capital, and output, all in terms of effective unit of labor, are identified by $k^\star$, $h^\star$, and $y^\star$, respectively, and the steady state level for each is determined by:

$$\begin{aligned} k^\star &= \left(\frac{s_k^\psi(1-\alpha-\beta)}{(n+g+\delta)^\psi - \alpha s_k^\psi - \beta s_h^\psi}\right)^{\frac{1}{\psi}}, \\ h^\star &= \left(\frac{s_h^\psi(1-\alpha-\beta)}{(n+g+\delta)^\psi - \alpha s_k^\psi - \beta s_h^\psi}\right)^{\frac{1}{\psi}}, \quad \text{and} \\ y^\star &= \left(\frac{(n+g+\delta)^\psi(1-\alpha-\beta)}{(n+g+\delta)^\psi - \alpha s_k^\psi - \beta s_h^\psi}\right)^{\frac{1}{\psi}}. \end{aligned} \tag{5}$$

The long-run equilibrium factor shares are given by

$$\begin{aligned} \pi_K^\star &= \alpha\left(\frac{k^\star}{y^\star}\right)^\psi = \alpha\left(\frac{s_k}{n+g+\delta}\right)^\psi \quad \text{and} \\ \pi_H^\star &= \beta\left(\frac{h^\star}{y^\star}\right)^\psi = \beta\left(\frac{s_h}{n+g+\delta}\right)^\psi. \end{aligned} \tag{6}$$

The existence and stability of the steady state require that all inputs are essential for production and all factors must receive a positive share of income, similar to Barro & Sala-I-Martin (2004, p. 19). In addition, the concavity of $y \equiv f(k,h)$, and the linearity of $(n+g+\delta)k$ and $(n+g+\delta)h$ ensure that the steady state exists and is unique. When we assume that human capital is augmented by labor, we focus on the capital and labor share of income, since we do not include a market for human capital.

When we examine the effects of the elasticity of substitution on the normalized CES production function, for both (2) and (3), our main result is that for both physical and human capital per effective unit of labor, an increase in the elasticity of substitution has positive effects on their long-run levels, $\frac{\partial k^\star}{\partial \sigma} > 0$ and $\frac{\partial h^\star}{\partial \sigma} > 0$, see Appendix B for proof.

The production function in Mankiw *et al.* (1992) is a limiting case of ours when the elasticity of substitution is unitary. Mankiw *et al.* (1992) guarantee the existence of a balanced growth steady state, for the three input Cobb–Douglas production function, by assuming $\alpha + \beta < 1$, which implies decreasing returns to all capital. Similarly, in our model the existence of a balanced growth steady state is also guaranteed by $\alpha + \beta < 1$, for unitary elasticity of substitution. Assuming a unitary elasticity of substitution and using the capital accumulation equations we find that long-run equilibrium levels of the capital shares of income are given by $\alpha = \pi_K^\star$ and $\beta = \pi_H^\star$. Thus, the existence of the steady state, for elasticity of substitution equal to unity, implies $\pi_K^\star + \pi_H^\star < 1$. For a non-unitary elasticity of substitution, the existence and stability of the steady state will hinge upon the parameters of the production function: depreciation of capital per effective unit, $n + g + \delta$, saving rates for both types of capital, and elasticity of substitution. Thus, for suitable values for the elasticity of substitution, it is possible to remain or leave the domain of steady states, which we will refer to as the growth threshold, following Klump & Preissler (2000).

One criticism of Mankiw *et al.* (1992) is that their model does not equate the net returns to human and physical capital. It has been argued by Barro & Sala-I-Martin (2004) that it is reasonable to think that households will invest in the capital that has the highest net return, therefore, the returns between the two types of capital should be equated. The result of this assumption is the following

$$h(t) = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{1-\psi}} k(t), \tag{7}$$

for perfectly competitive firms. After incorporating this restriction into the model, for a high elasticity of substitution, defined by $\sigma > 1$, the existence condition can be determined from (4) and (7) and is given by:

$$\frac{(n+g+\delta)}{s_k} > \left\{\alpha(\sigma) + \beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}\right\}^{1/\psi} = \lim_{k\to\infty} f'(k) = \lim_{k\to\infty} f(k)/k. \tag{8}$$

Figures 1 and 2 in Appendix B.2 represent conditions for existence under high and low elasticities of substitution.

4.2 Labor augmenting Human capital

We find similar results for labor augmented human capital, see proof in Appendix B.3. Further, the existence for the steady for the model with labor augmented human capital is guaranteed by:

$$\begin{aligned} \frac{\dot{k}}{k} &= \lim_{k\to\infty} s_k f(k)/k - (n+g+\delta) \\ &= s_k \alpha(\sigma)^{1/\psi} - (n+g+\delta) > 0, \quad \text{and} \qquad (9) \\ \frac{\dot{h}}{h} &= \lim_{h\to\infty} s_h f(h)/h - (n+g+\delta) \\ &= s_h \beta(\sigma)^{1/\psi} - (n+g+\delta) > 0. \qquad (10) \end{aligned}$$

To examine the speed with which an economy moves toward its steady state, we focus on conditional convergence, for details see Appendix C. When the baseline level of capital per effective unit of labor is greater than the steady state level, an increase in the elasticity of substitution leads to an increase in the speed of convergence. An alternative explanation is that when capital is relatively less scarce than effective labor when compared to the state steady levels then a higher elasticity of substitution leads to a higher speed of convergence.

5 Estimation of the Elasticity of Substitution

The discussion on the functional form of the aggregate production function continues towards functions with nonunitary elasticities of substitution. We estimate the normalized constant elasticity of substitution production functions with additive human capital and labor augmenting human capital to test whether the elasticity of substitution is significantly different than unity. Our estimation is done for cross-country data consistent with Mankiw *et al.* (1992).

5.1 Data

The data are from the following two sources: Penn World Table version 9.0 and Mankiw *et al.* (1992), data period is from 1960 to 1985. We consider two samples of countries: Non-oil producing countries (the most comprehensive sample, 98 countries for the Cobb–Douglas case and 84 countries for the CES case) and intermediate countries, a subset of the former whose populations were less than a million in 1960, see Mankiw *et al.* (1992).[10] We use series for labor, output, physical capital investment, and physical capital, and human capital index. These are from the Penn World Table, version 9.0. The number of workers engaged in the labor force is denoted by L. The level of output, Y, Real GDP at constant 2011 prices. The human capital index, H, is based on years of schooling and returns to education. Capital, K, is determined by capital stock at constant 2011 prices. The savings rate for physical capital is given by dividing physical capital investment by output. We follow Mankiw *et al.* (1992), for the average physical capital savings rates, and proxy for the average savings rate for human capital that are assumed to be exogenous. The proxy is calculated as the percentage of working-age population that is in secondary school.[11] We also assume that g and δ are constant across time, at 5%, to match available data. The parameter, g, represents the advancement of knowledge that is not country specific. The initial level of technology reflects not only technology, but resource endowments, climate, institutions, and so on, which may differ across countries.

5.2 Empirical Model

The normalized aggregate per capita production function, with additive human capital at the steady state is estimated using nonlinear least squares and is determined by the following technology:

$$\begin{aligned}\log\left(\frac{Y_{it}}{L_{it}}\right) &= \log A_0(\pi_k^0,\pi_h^0,\sigma)+gt+\left(\frac{1}{\psi}\right)\log[\alpha(\pi_k^0,\sigma)(k(\pi_k^0,\pi_h^0,\sigma,s_{k,i},s_{h,i},n_i))^{\psi}+\\&\quad \beta(\pi_h^0,\sigma)(h(\pi_k^0,\pi_h^0,\sigma,s_{k,i},s_{h,i},n_i))^{\psi}+\\&\quad (1-\alpha(\pi_k^0,\sigma)-\beta(\pi_h^0,\sigma))], \qquad (11)\end{aligned}$$

[10] Our sample is reduced from 98 to 84 countries for the CES case since data on initial values for some countries were not available, something essential for the estimation of the CES production functions.

[11] For this, we utilize data obtained from the United Nations Educational, Scientific and Cultural Organisation (UNESCO), Institute for Statistics. We require the fraction of the population that is enrolled in secondary school, which is expressed as a percentage of the official school-aged population between 12 to 17 years, and the fraction of the working-age population, between age 15 to 19. The school age population is multiplied by the fraction of the working age population yield the proxy for the savings rate for human capital. As explained in Mankiw *et al.* (1992), this is an imperfect, yet suitable, proxy due to the age ranges and ignoring teachers input, primary education, and higher education.

where i denotes the country and t denotes the time period. For the production function that includes labor augmented human capital, the normalized aggregate production function at the steady state is given by the following technology:

$$\begin{aligned}\log\left(\frac{Y_{it}}{L_{it}}\right) &= \log A_0 + gt + \left(\frac{1}{\psi}\right)\log[\alpha(\pi_0,\sigma)(k(\pi_0,\sigma,s_{k,i},s_{h,i},n_i))^{\psi} + \\ &\quad \beta(\pi_0,\sigma)(h(\pi_0,\sigma,s_{k,i},s_{h,i},n_i))^{\psi} + \\ &\quad (1-\alpha(\pi_0,\sigma)-\beta(\pi_0,\sigma))],\end{aligned} \tag{12}$$

where π is the capital share of income. Additionally, the normalized parameters are country specific and determined by the average values for output, physical capital, human capital, and labor between 1960 and 1985.[12] Following Mankiw *et al.* (1992), we estimate log income per capita at a given time, time zero for simplicity.

Table 1: Estimates using Nonlinear Least Squares for Non-oil producing countries

Dependent Variable: log of GDP per working age person in 1985, Non-oil producing countries.				
	Cobb-Douglas	CES$_H$	Cobb-Douglas$_{Imposed}$	CES$_{lh}$
ψ		-0.94*		-0.27
$\log(A_0)$	7.85***			10.45
π_k	0.31***	0.10***	0.08***	0.58***
π_h	0.27***	0.03	0.00	
Observations	98	84	84	84
AIC	131.59	-94.89	-91.04	237.05

*** $p < .001$, ** $p < .01$, and * $p < .05$

Tables 1 and 2 report our main estimation results for non-oil producing and intermediate countries. In each table, column 1 represents estimation results from a Cobb-Douglas production function for comparison, following Mankiw *et al.* (1992). Columns 2 and 4 represent estimates from the CES production function with additive human capital, CES$_H$, and from labor augmenting human capital, CES$_{lh}$. Column 3 represents estimates for the normalized CES production function with additive human capital, when unitary elasticity is imposed, referred to as the normalized Cobb-Douglas, or Cobb-Douglas$_{Imposed}$.

The results for the Cobb–Douglas production function show that shares of physical capital, π_k, and human capital, π_h, in output are highly significant and closely match results in Mankiw *et al.* (1992). For the model with CES and additive human capital, CES$_H$, we find the composite parameter, ψ, to be significant. A value of -0.94 for ψ implies a value of 0.5 for the elasticity of substitution, σ, which is significantly below unity. For the model with labor augmenting human capital, CES$_{lh}$, we find the composite parameter, ψ, to be weakly significant with a p-value of 11%. A value of -0.27 for ψ implies a value of 0.78 for the elastic-

[12] We follow Klump *et al.* (2012) for the "appropriate" selection of baseline values for the normalized production function.

ity of substitution, σ, which is also weakly significant and below unity. We find comparable estimates for the factor shares between the normalized CES and the imposed Cobb-Douglas for the production function with additive human capital. When estimating the CES with additive human capital, we find relatively low significant physical capital share of income, as compared to the non-normalized Cobb-Douglas and the normalized CES production function with labor augmented human capital. Further, we find that the model with additive human capital fits the data best given its lowest Akaike Information Criterion (AIC).

It can be noted that the sum of shares of both types of capital in output in the Cobb-Douglas case, $(0.31 + 0.27 = 0.58)$. The share of capital in the CES_{lh} case (0.58) is equal to the sum of shares of both types of capital under the Cobb-Douglas case. Despite the differences in model specification, the Cobb-Douglas case and the CES_{lh} are very similar in spirit. Both these specifications allow human capital to be labor augmenting. It is intuitive to think about human capital as inseparable from labor since its services have to be sold together with labor. In both cases, the services traded on the labor market are not units of "raw labor" but units of raw labor endowed with a certain level of education or human capital.[13]

Table 2: Estimates using Nonlinear Least Squares for Intermediate countries

Dependent Variable: log of GDP per working age person in 1985, Intermediate countries.				
	Cobb-Douglas	CES_H	$\text{Cobb-Douglas}_{Imposed}$	CES_{lh}
ψ		-1.13		-0.23
$\log(A_0)$	7.97***			11.04
π_k	0.29***	0.08*	.05*	0.64***
π_h	0.30***	0.05	.00**	
Observations	75	67	67	67
AIC	149.93	-82.97	-81.92	173.34

*** $p < .001$, ** $p < .01$, and * $p < .05$

Our results for intermediate countries are similar and suggest that the elasticity of substitution is weakly significant and below unity for both CES cases. Overall, our estimates provide evidence that the elasticity of substitution is significantly below unity and ranges between .5 and .78.

6 Conclusions

Although the classic work of Solow (1956) and many others examine various functional forms for the aggregate production function consistent with the neoclassical theory of economic growth, the workhorse for the dynamic macroeconomics literature continues to be the Cobb-Douglas production function for which the elasticity of substitution among factors of

[13]Further, previous research has suggested that there might be a strong complementarity between the level of skilled labor (HL) and the level of capital, whereas unskilled labor (raw labor input, L) and capital are more likely to exhibit substitutability, see Duffy & Papageorgiou (2000).

production is exactly unity. The longstanding debate on the functional form of the aggregate production function with two factors of production, physical capital and labor, strongly suggests that the elasticity of substitution between these factors significantly differs from unity for most countries. While the macroeconomic literature has emphasized the importance of human capital as a third input in production, this too has relied on the Cobb-Douglas production function to describe a country's technology for producing goods.

This paper employs a neoclassical growth model and assumes that aggregate income is determined by a normalized CES production function with three factors of production: physical capital, human capital, and labor. Further, we allow human capital to enter as a perfect substitute to labor but also as a separate factor in production.

In the comparative static analysis, we examine the impact of a change in the elasticity of substitution for steady state variables, growth thresholds, and speed of convergence. We find that a higher elasticity of substitution can lead to a higher income per effective worker. We also find that a higher elasticity of substitution will lead to a higher speed of convergence when the baseline level of capital per effective unit of labor is greater than the steady state level. Our findings emphasize that the elasticity of substitution is an important determinant of economic growth, as the parameter affects the per effective unit of labor steady state as well as the rate of convergence to said steady state. These results are also in line with Klump & Preissler (2000, p. 49) who conclude "that within one family of (normalized) CES functions, an increase in the elasticity of substitution has a positive effect on the level of the steady state". Our empirical estimates for the normalized aggregate CES production functions with human capital suggest that the elasticity of substitution is significantly below unity.

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Appendices

A Normalizing the CES Production Functions

A.1 CES with Additive Human Capital

$$\frac{F_L(K_0,H_0,L_0)}{F_K(K_0,H_0,L_0)} = \frac{1-\alpha-\beta}{\alpha}A_0^{\psi}\left(\frac{K_0}{L_0}\right)^{1-\psi} = \mu_0, \tag{13}$$

$$\Leftrightarrow \alpha = (1-\beta)\frac{A_0^{\psi}K_0^{1-\psi}}{A_0^{\psi}K_0^{1-\psi}+\mu_0 L_0^{1-\psi}}, \tag{14}$$

$$\frac{F_L(K_0,H_0,L_0)}{F_H(K_0,H_0,L_0)} = \frac{1-\alpha-\beta}{\beta}A_0^{\psi}\left(\frac{H_0}{L_0}\right)^{1-\psi} = \gamma_0 \tag{15}$$

$$\Leftrightarrow \beta = (1-\alpha)\frac{A_0^{\psi}H_0^{1-\psi}}{A_0^{\psi}H_0^{1-\psi}+\gamma_0 L_0^{1-\psi}}, \quad \text{and} \tag{16}$$

$$\frac{F_K(K_0,H_0,L_0)}{F_H(K_0,H_0,L_0)} = \frac{\alpha}{\beta}\left(\frac{H_0}{K_0}\right)^{1-\psi} = \rho_0 \tag{17}$$

$$\Leftrightarrow \beta = \frac{\alpha}{\rho_0}\left(\frac{H_0}{K_0}\right)^{1-\psi}. \tag{18}$$

Plugging (18) into (14),

$$\alpha = \frac{\rho_0 A_0^{\psi}K_0^{1-\psi}}{\rho_0(A_0^{\psi}K_0^{1-\psi}+\mu_0 L_0^{1-\psi})+A_0^{\psi}H_0^{1-\psi}}. \tag{19}$$

From (18)

$$\beta = \frac{A_0^{\psi}H_0^{1-\psi}}{\rho_0(A_0^{\psi}K_0^{1-\psi}+\mu_0 L_0^{1-\psi})+A_0^{\psi}H_0^{1-\psi}}. \tag{20}$$

Thus,

$$1-\alpha-\beta = \frac{\rho_0\mu_0 L_0^{1-\psi}}{\rho_0(A_0^{\psi}K_0^{1-\psi}+\mu_0 L_0^{1-\psi})+A_0^{\psi}H_0^{1-\psi}} \tag{21}$$

Solving for the baseline level of technology, A_0, as a function of the elasticity of substitution, we find that:

$$\begin{aligned} Y_0 &= [\alpha K_0^{\psi}+\beta H_0^{\psi}+(1-\alpha-\beta)(A_0L_0)^{\psi}]^{\frac{1}{\psi}} \\ \Rightarrow \quad A_0(\sigma) &= \left(\frac{\rho_0\mu_0 L_0^{1-\psi}Y_0^{\psi}}{(\rho_0K_0+H_0)-(\rho_0K_0^{1-\psi}+H_0^{1-\psi})Y_0^{\psi}+\rho_0\mu_0L_0}\right)^{\frac{1}{\psi}}. \end{aligned} \tag{22}$$

From (22) and (19), $\alpha(\sigma)$ can be written as

$$\alpha(\sigma) = \frac{\rho_0 K_0^{1-\psi} Y_0^{\psi}}{(\rho_0 K_0 + H_0) + \rho_0 \mu_0 L_0}. \tag{23}$$

From (22) and (20), $\beta(\sigma)$ can be written as

$$\beta(\sigma) = \frac{H_0^{1-\psi} Y_0^{\psi}}{(\rho_0 K_0 + H_0) + \rho_0 \mu_0 L_0}. \tag{24}$$

A.2 CES with Labor Augmented Human Capital

$$\begin{aligned} F_K(K_0, H_0, L_0) &= \alpha \left(\frac{K_0}{Y_0}\right)^{\psi-1} \quad \text{and} \\ F_L(K_0, H_0, L_0) &= (\beta H_0^{\psi} L_0^{-1} + (1-\alpha-\beta) A_0^{\psi} L_0^{\psi-1}) Y_0^{1-\psi} \end{aligned}$$

Equating the marginal product of capitals to their rental rates yield:

$$\begin{aligned} \alpha(\sigma) &= \pi_0 \left(\frac{Y_0}{K_0}\right)^{\psi} \quad \text{and} \\ \beta(\sigma) &= \frac{Y_0^{\psi}(1-\pi_0) - (A_0 L_0)^{\psi}\left(1 - \pi_0 \left(\frac{Y_0}{K_0}\right)^{\psi}\right)}{H_0^{\psi} - (A_0 L_0)^{\psi}}. \end{aligned}$$

B Elasticity of Substitution and the Steady States

B.1 CES with Additive Human Capital

$$\frac{d\alpha}{d\sigma} = \frac{-\alpha}{\sigma^2 \psi} \ln \left(\frac{k_0}{y_0}\right)^{\psi} \tag{25}$$

$$\frac{d\beta}{d\sigma} = \frac{-\beta}{\sigma^2 \psi} \ln \left(\frac{h_0}{y_0}\right)^{\psi} \tag{26}$$

$$\begin{aligned} \frac{d\pi_k^{\star}}{d\sigma} &= \left(\frac{s_k}{n+g+\delta}\right)^{\psi} \frac{d\alpha(\sigma)}{d\sigma} + \frac{\alpha(\sigma)}{\sigma^2}\left(\frac{s_k}{n+g+\delta}\right)^{\psi} \ln\left(\frac{s_k}{n+g+\delta}\right) \\ &= \frac{1}{\sigma^2\psi}\alpha(\sigma)\left(\frac{s_k}{n+g+\delta}\right)^{\psi}\left\{-\ln\left(\frac{k_0}{y_0}\right)^{\psi} + \ln\left(\frac{s_k}{n+g+\delta}\right)^{\psi}\right\} \\ &= \frac{1}{\sigma^2\psi}\pi_k^{\star}\ln\left(\frac{\pi_k^{\star}}{\pi_k^0}\right) \end{aligned} \tag{27}$$

$$
\begin{aligned}
\frac{d\pi_h^\star}{d\sigma} &= \left(\frac{s_h}{n+g+\delta}\right)^\psi \frac{d\beta(\sigma)}{d\sigma} + \frac{\beta(\sigma)}{\sigma^2}\left(\frac{s_h}{n+g+\delta}\right)^\psi \ln\left(\frac{s_h}{n+g+\delta}\right) \\
&= \frac{1}{\sigma^2\psi}\beta(\sigma)\left(\frac{s_h}{n+g+\delta}\right)^\psi \left\{-\ln\left(\frac{h_0}{y_0}\right)^\psi + \ln\left(\frac{s_h}{n+g+\delta}\right)^\psi\right\} \\
&= \frac{1}{\sigma^2\psi}\pi_h^\star \ln\left(\frac{\pi_h^\star}{\pi_h^0}\right). \qquad (28)
\end{aligned}
$$

The derivative of $k^\star$ with respect to the elasticity of substitution is given by

$$
\begin{aligned}
\frac{dk^\star}{d\sigma} =& k^\star \left\{\frac{-1}{\sigma^2\psi^2}\left(\ln(1-\alpha(\sigma)-\beta(\sigma)) - \ln\alpha(\sigma) + \ln\pi_K^\star - \ln(1-\pi_K^\star-\pi_H^\star)\right)\right. \\
&\left. + \frac{1}{\psi}\left(\frac{-\alpha'(\sigma)-\beta'(\sigma)}{1-\alpha(\sigma)-\beta(\sigma)} - \frac{\alpha'(\sigma)}{\alpha(\sigma)} + \frac{\pi_K^{\star}{}'(\sigma)}{\pi_K^\star} + \frac{\pi_K^{\star}{}'(\sigma)+\pi_H^{\star}{}'(\sigma)}{1-\pi_K^\star-\pi_H^\star}\right)\right\}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{dk^\star}{d\sigma} =& \frac{-1}{\sigma^2\psi^2}\left(\frac{k^\star}{1-\pi_K^\star-\pi_H^\star}\right)\left\{(1-\pi_K^\star-\pi_H^\star)\ln\left(\frac{1-\alpha(\sigma)-\beta(\sigma)}{1-\pi_K^\star-\pi_H^\star}\right)\right. \\
&\left. - \left(\frac{1-\pi_K^\star-\pi_H^\star}{1-\alpha(\sigma)-\beta(\sigma)}\right)\left(\alpha(\sigma)\ln\left(\frac{k_0}{y_0}\right)^\psi + \beta(\sigma)\ln\left(\frac{h_0}{y_0}\right)^\psi\right) + \pi_K^\star \ln\frac{\pi_K^0}{\pi_K^\star} + \pi_H^\star\ln\frac{\pi_H^0}{\pi_H^\star}\right\}.
\end{aligned}
$$

From the production function and (2), it is clear that

$$
\begin{aligned}
1-\alpha(\sigma)-\beta(\sigma) &= y_0^\psi - \alpha(\sigma)k_0^\psi - \beta(\sigma)h_0^\psi \\
&= y_0^\psi(1-\pi_K^0-\pi_H^0).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{dk^\star}{d\sigma} =& \frac{-1}{\sigma^2\psi^2}\left(\frac{k^\star}{1-\pi_K^\star-\pi_H^\star}\right)\left\{(1-\pi_K^\star-\pi_H^\star)\ln\left(\frac{1-\pi_K^0-\pi_H^0}{1-\pi_K^\star-\pi_H^\star}\right)\right. \\
&\left. + \left(\frac{1-\pi_K^\star-\pi_H^\star}{1-\alpha(\sigma)-\beta(\sigma)}\right)\left(\ln y_0^\psi - \alpha(\sigma)\ln k_0^\psi - \beta(\sigma)\ln h_0^\psi\right) + \pi_K^\star\ln\frac{\pi_K^0}{\pi_K^\star} + \pi_H^\star\ln\frac{\pi_H^0}{\pi_H^\star}\right\}.
\end{aligned}
$$

Following Klump & Preissler (2000), we make use of the concavity of the natural logarithm and assume that $k^\star \neq k_0$, $h^\star \neq h_0$, $\pi_K^\star \neq \pi_K^0$, and $\pi_H^\star \neq \pi_H^0$. Therefore,

$$
\begin{aligned}
ln\left(\frac{1-\pi_K^0-\pi_H^0}{1-\pi_K^\star-\pi_H^\star}\right) &< \frac{1-\pi_K^0-\pi_H^0}{1-\pi_K^\star-\pi_H^\star}-1=\frac{(1-\pi_K^0-\pi_H^0)-(1-\pi_K^\star-\pi_H^\star)}{1-\pi_K^\star-\pi_H^\star},\\
\ln y_0^\psi &< y_0^\psi-1,\\
\ln k_0^\psi &< k_0^\psi-1,\\
\ln h_0^\psi &< h_0^\psi-1,\\
\ln\frac{\pi_K^0}{\pi_K^\star} &< \frac{\pi_K^0}{\pi_K^\star}-1=\frac{\pi_K^0-\pi_K^\star}{\pi_K^\star},\quad \text{and}\\
\ln\frac{\pi_H^0}{\pi_H^\star} &< \frac{\pi_H^0}{\pi_H^\star}-1=\frac{\pi_H^0-\pi_H^\star}{\pi_H^\star}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(1-\pi_K^\star-\pi_H^\star)\ln\left(\frac{1-\pi_K^0-\pi_H^0}{1-\pi_K^\star-\pi_H^\star}\right)+\pi_K^\star\ln\frac{\pi_K^0}{\pi_K^\star}+\pi_H^\star\ln\frac{\pi_H^0}{\pi_H^\star} &< (1-\pi_K^0-\pi_H^0)-(1-\pi_K^\star-\pi_H^\star)\\
&\quad +\pi_K^0-\pi_K^\star+\pi_H^0-\pi_H^\star=0\quad\text{and}\\
\ln y_0^\psi-\alpha(\sigma)\ln k_0^\psi-\beta(\sigma)\ln h_0^\psi &< y_0^\psi-\alpha(\sigma)k_0^\psi-\beta(\sigma)h_0^\psi\\
&\quad -(1-\alpha(\sigma)-\beta(\sigma))=0.
\end{aligned}
$$

Therefore, for $k^\star \neq k_0$ and $h^\star \neq h_0$,

$$
\begin{aligned}
\frac{dk^\star}{d\sigma} =& \frac{-1}{\sigma^2}\frac{1}{\psi^2}\left(\frac{k^\star}{1-\pi_K^\star-\pi_H^\star}\right)\Bigg\{(1-\pi_K^\star-\pi_H^\star)\ln\left(\frac{1-\pi_K^0-\pi_H^0}{1-\pi_K^\star-\pi_H^\star}\right)\\
&+\left(\frac{1-\pi_K^\star-\pi_H^\star}{1-\alpha(\sigma)-\beta(\sigma)}\right)\left(\ln y_0^\psi-\alpha(\sigma)\ln k_0^\psi-\beta(\sigma)\ln h_0^\psi\right)+\pi_K^\star\ln\frac{\pi_K^0}{\pi_K^\star}+\pi_H^\star\ln\frac{\pi_H^0}{\pi_H^\star}\Bigg\}>0.
\end{aligned}
$$

Taking the derivative of $h^\star$ with respect to the elasticity of substitution, for $k^\star \neq k_0$ and $h^\star \neq h_0$, leads to:

$$
\begin{aligned}
\frac{dh^\star}{d\sigma} =& \frac{-1}{\sigma^2}\frac{1}{\psi^2}\left(\frac{h^\star}{1-\pi_K^\star-\pi_H^\star}\right)\Bigg\{(1-\pi_K^\star-\pi_H^\star)\ln\left(\frac{1-\pi_K^0-\pi_H^0}{1-\pi_K^\star-\pi_H^\star}\right)\\
&+\left(\frac{1-\pi_K^\star-\pi_H^\star}{1-\alpha(\sigma)-\beta(\sigma)}\right)\left(\ln y_0^\psi-\alpha(\sigma)\ln k_0^\psi-\beta(\sigma)\ln h_0^\psi\right)+\pi_K^\star\ln\frac{\pi_K^0}{\pi_K^\star}+\pi_H^\star\ln\frac{\pi_H^0}{\pi_H^\star}\Bigg\}>0.
\end{aligned}
$$

Lastly, taking the derivative of $y^\star$ with respect to the elasticity of substitution, for $k^\star \neq k_0$ and $h^\star \neq h_0$, leads to:

$$
\begin{aligned}
\frac{dy^\star}{d\sigma} =& \frac{-1}{\sigma^2}\frac{1}{\psi^2}\left(\frac{y^\star}{1-\pi_K^\star-\pi_H^\star}\right)\Bigg\{(1-\pi_K^\star-\pi_H^\star)\ln\left(\frac{1-\pi_K^0-\pi_H^0}{1-\pi_K^\star-\pi_H^\star}\right)\\
&+\left(\frac{1-\pi_K^\star-\pi_H^\star}{1-\alpha(\sigma)-\beta(\sigma)}\right)\left(\ln y_0^\psi-\alpha(\sigma)\ln k_0^\psi-\beta(\sigma)\ln h_0^\psi\right)+\pi_K^\star\ln\frac{\pi_K^0}{\pi_K^\star}+\pi_H^\star\ln\frac{\pi_H^0}{\pi_H^\star}\Bigg\}>0.
\end{aligned}
$$

B.2 Existence

Figure 1: $\sigma > 1$: threshold for the existence of a steady state with zero growth for human capital and physical capital both per effective unit of labor

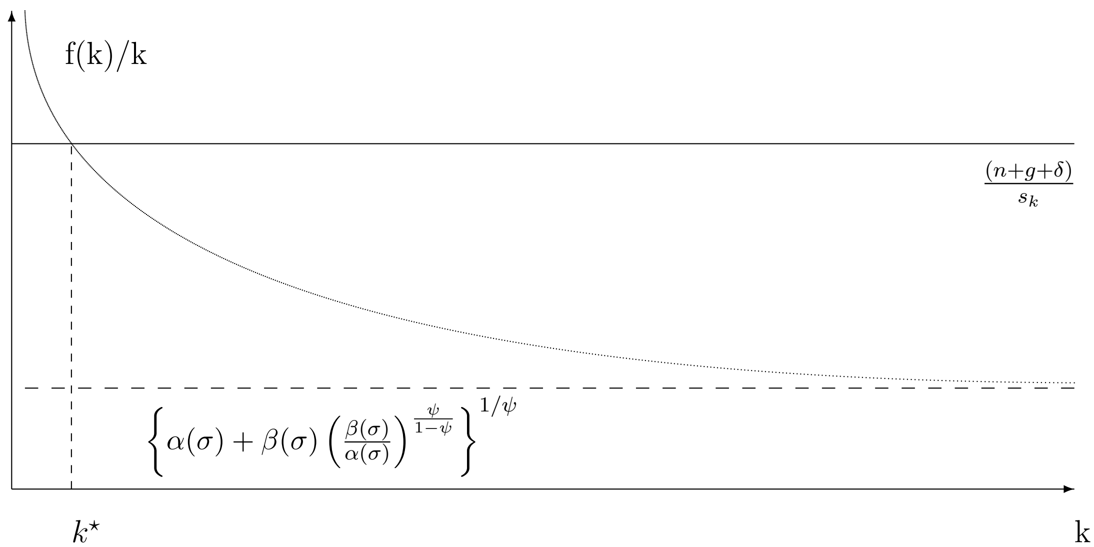

Figure 1 depicts the condition for the existence of a steady state for a high elasticity of substitution. The existence condition, (8), is determined by the asymptote of $f(k)/k$ and the depreciation curve over the savings rate for physical capital, $(n+g+\delta)/s_k$, see Appendix B.2. The existence of the steady state for a high elasticity of substitution will not depend on very low capital per effective unit of labor since the $\lim_{k\to 0} f(k)/k = \infty$. If the condition for the existence of the steady state, (8), is violated under high elasticity of substitution, perpetual growth will occur. Thus, when $\lim_{k\to\infty} f(k)/k$ is greater than the depreciation curve over the savings rate for physical capital, the long-term growth rates for physical and human capital per effective unit of labor are given by

$$\begin{aligned}
\frac{\dot{k}}{k} &= \lim_{k\to\infty} s_k f(k)/k - (n+g+\delta) \\
&= s_k \left\{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}\right\}^{1/\psi} - (n+g+\delta) > 0, \quad \text{and} \qquad (29)\\
\frac{\dot{h}}{h} &= \lim_{h\to\infty} s_h f(h)/h - (n+g+\delta) \\
&= s_h \left\{\alpha(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}+\beta(\sigma)\right\}^{1/\psi} - (n+g+\delta) > 0. \qquad (30)
\end{aligned}$$

For a low elasticity of substitution $\sigma < 1$ the existence condition is determined by

$$\frac{(n+g+\delta)}{s_k} < \left\{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}\right\}^{1/\psi} = \lim_{k\to 0} f'(k) = \lim_{k\to 0} f(k)/k. \qquad (31)$$

Figure 2: $\sigma < 1$: threshold for the existence of steady state with zero growth for human capital and physical capital both per effective unit of labor

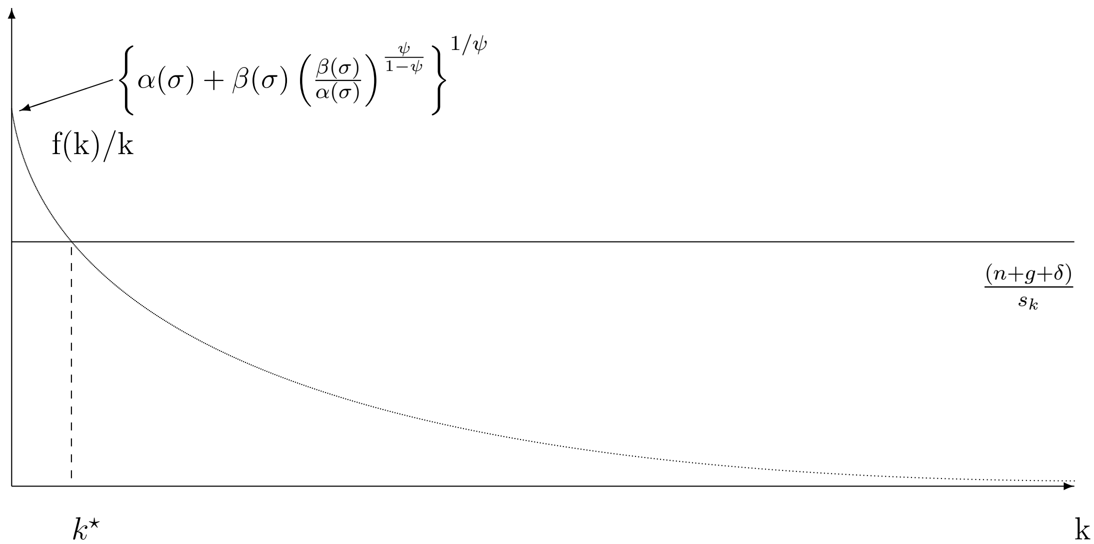

Figure 2 depicts the condition for the existence of the steady state for a low elasticity of substitution; this condition depends on the intercept $f(k)/k$ and the break-even investment over capital, $(n+g+\delta)$. If condition (31) does not hold for a low elasticity of substitution, then a permanent decline will occur until the trivial steady state of $k^\star = h^\star = 0$ is reached.

B.3 CES with Labor Augmented Human Capital

$$\frac{d\alpha}{d\sigma} = \frac{-\alpha}{\sigma^2\psi}\ln\left(\frac{K_0}{Y_0}\right)^{\psi} \tag{32}$$

$$\begin{aligned}\frac{d\beta}{d\sigma} &= -\left(\frac{1}{\sigma^2\psi}\right)\frac{(1-\alpha(\sigma)-\beta(\sigma))(A_0L_0)^{\psi}\ln\left(\frac{A_0L_0}{Y_0}\right)^{\psi}}{H_0^{\psi}-(A_0L_0)^{\psi}} \\ &\quad -\left(\frac{1}{\sigma^2\psi}\right)\frac{\alpha(\sigma)(A_0L_0)^{\psi}\ln\left(\frac{K_0}{Y_0}\right)^{\psi}+\beta(\sigma)(H_0)^{\psi}\ln\left(\frac{H_0}{Y_0}\right)^{\psi}}{H_0^{\psi}-(A_0L_0)^{\psi}}\end{aligned} \tag{33}$$

$$\frac{d\pi^\star}{d\sigma} = \frac{1}{\sigma^2\psi}\pi^\star\ln\left(\frac{\pi^\star}{\pi^0}\right) \tag{34}$$

The derivative of $k^\star$ with respect to the elasticity of substitution is given by

$$\begin{aligned}\frac{dk^\star}{d\sigma} =& k^\star \Bigg\{ \frac{-1}{\sigma^2\psi^2} \left(\ln(1-\alpha(\sigma)-\beta(\sigma)) - \ln\alpha(\sigma) + \ln\pi^\star - \ln\left(1-\pi^\star-\beta\left(\frac{s_H}{n+g+\delta}\right)^\psi\right)\right) \\ &+ \frac{1}{\psi}\left(\frac{-\alpha'(\sigma)-\beta'(\sigma)}{1-\alpha(\sigma)-\beta(\sigma)} - \frac{\alpha'(\sigma)}{\alpha(\sigma)} + \frac{\pi^{\star\prime}(\sigma)}{\pi^\star}\right. \\ &\left. + \frac{\pi^{\star\prime}(\sigma) + \beta'(\sigma)\left(\frac{s_H}{n+g+\delta}\right)^\psi + \frac{\beta}{\sigma^2\psi}\left(\frac{s_H}{n+g+\delta}\right)^\psi \ln\left(\frac{s_H}{n+g+\delta}\right)^\psi}{1-\pi^\star-\left(\frac{s_H}{n+g+\delta}\right)^\psi}\right)\Bigg\}.\end{aligned}$$

Plugging in the derivatives for α and π result in

$$\begin{aligned}\frac{dk^\star}{d\sigma} =& \frac{-1}{\sigma^2\psi^2}\left(\frac{k^\star}{1-\pi^\star-\left(\frac{s_H}{n+g+\delta}\right)^\psi}\right)\Bigg\{\left(1-\pi^\star-\left(\frac{s_H}{n+g+\delta}\right)^\psi\right)\ln\left(\frac{1-\pi^0-\beta\left(\frac{h_0}{y_0}\right)^\psi}{1-\pi^\star-\left(\frac{s_H}{n+g+\delta}\right)^\psi}\right) \\ &+ \left(\frac{1-\pi^\star-\left(\frac{s_H}{n+g+\delta}\right)^\psi}{1-\alpha(\sigma)-\beta(\sigma)}\right)\left((1-\alpha(\sigma)-\beta(\sigma))\ln y_0^\psi - \alpha(\sigma)\ln\left(\frac{k_0}{y_0}\right)^\psi + \sigma^2\psi\beta'(\sigma)\right) \\ &+ \pi_K^\star \ln\frac{\pi_K^0}{\pi_K^\star} - \sigma^2\psi\beta'(\sigma)\left(\frac{s_H}{n+g+\delta}\right)^\psi - \beta(\sigma)\left(\frac{s_H}{n+g+\delta}\right)^\psi \ln\left(\frac{s_H}{n+g+\delta}\right)^\psi\Bigg\}.\end{aligned}$$

Using the concavity of the natural logarithm,

$$\frac{dk^\star}{d\sigma} \;>\; 0,$$

for $k^\star \neq k_0$, $h^\star \neq h_0$, $\pi_K^\star \neq \pi_K^0$, and $\pi_H^\star \neq \pi_H^0$, and $h_0 < y_0$. Similarly, $\frac{dh^\star}{d\sigma} > 0$, under the same restrictions.

C Elasticity of Substitution and Speed of Convergence

To examine the speed with which an economy moves toward its steady state, we focus on conditional convergence, which assumes that an economy's steady state depends on its own model parameters such as the savings rates for both types of capital. We derive the speed of convergence, λ, which explains the inverse relationship between the growth rate of output per effective unit of labor and its initial level. From the log-linear approximation in the neighborhood of the steady-state, we can examine the path and speed with which the initial level output per effective unit of labor heads to the steady state.

The growth rate of income per efficient unit is given by

$$\frac{\dot{y}(t)}{y(t)} = \pi_K \frac{\dot{k}(t)}{k(t)} + \pi_H \frac{\dot{h}(t)}{h(t)}. \tag{35}$$

Barro & Sala-I-Martin (2004) and other works have criticized Mankiw *et al.* (1992) for not assuming the two rates of returns between the two goods to be equal. This is explained by Barro & Sala-I-Martin (2004, pg. 59), that "it is reasonable to think that households will invest in capital goods that delivers the higher return". Therefore, we will assume $R_K = R_H$ as a result $MP_K = MP_H$. This allows for the determination of output per efficient unit of labor by physical (or human) capital per efficient unit labor at time t. Equating the returns for the two types of capital imply physical and human capital grow at the same rate. Thus,

$$k(t) = \left(\frac{y^{\psi} - (1 - \alpha(\sigma) - \beta(\sigma))}{\alpha(\sigma) + \beta(\sigma) \left(\frac{\beta(\sigma)}{\alpha(\sigma)} \right)^{\frac{\psi}{1-\psi}}} \right)^{\frac{1}{\psi}}. \tag{36}$$

Revisiting the growth rates of physical and human capital per efficient unit,

$$\begin{aligned} \frac{\dot{k}(t)}{k(t)} &= s_k \frac{y(t)}{k(t)} - (n + g + \delta) \\ &= s_k \left(\frac{1 - y(t)^{-\psi}(1 - \alpha(\sigma) - \beta(\sigma))}{\alpha(\sigma) + \beta(\sigma) \left(\frac{\beta(\sigma)}{\alpha(\sigma)} \right)^{\frac{\psi}{1-\psi}}} \right)^{\frac{-1}{\psi}} - (n + g + \delta) \qquad (37) \\ & \qquad (38) \end{aligned}$$

Writing the physical and human capital shares of income as a function of y would result in

$$
\begin{aligned}
\pi_K(y) &= \alpha(\sigma)y(t)^{-\psi}\left(\frac{y(t)^{\psi}-(1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right) \\
&= \alpha(\sigma)\left(\frac{1-y(t)^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right) \qquad (39) \\
\pi_H(y) &= \beta(\sigma)y(t)^{-\psi}\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}\left(\frac{y(t)^{\psi}-(1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right) \\
&= \beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}\left(\frac{1-y(t)^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right). \qquad (40)
\end{aligned}
$$

The growth rate of output (35) can be written as

$$
\begin{aligned}
\frac{\dot{y}(t)}{y(t)} &= \alpha(s_k+s_h)\left(\frac{1-y(t)^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right)^{\frac{\psi-1}{\psi}} \\
&- \left(\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}\right)\left(\frac{1-y(t)^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right)(n+g+\delta). \qquad (41)
\end{aligned}
$$

Taking the log-linear approximation of (41) in the neighborhood of the steady state[14]

$$
\frac{\dot{y}(t)}{y(t)} \cong= -\lambda \log\left(y(t)/y^{\star}\right), \qquad (42)
$$

where

$$
\begin{aligned}
-\lambda &= (\psi-1)\alpha(\sigma)(s_k+s_h)\left(\frac{1-(y^{\star})^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right)^{\frac{-1}{\psi}}\left(\frac{(y^{\star})^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right) \\
&- \psi(y^{\star})^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))(n+g+\delta).
\end{aligned}
$$

[14]The equivalent exercise would be to rewrite the function in terms of logs ie $y = e^{\log(y)}$ and take a first order Taylor approximation

From the capital accumulation equations we replace the savings rates yielding

$$
\begin{aligned}
-\lambda &= (\psi-1)\alpha(\sigma)\left(1+\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{1}{1-\psi}}\right)(n+g+\delta)\left(\frac{(y^\star)^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right) \\
&- \psi(y^\star)^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))(n+g+\delta) \\
&= (y^\star)^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))(n+g+\delta)\left((\psi-1)\left(\frac{\alpha(\sigma)\left(1+\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{1}{1-\psi}}\right)}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right)-\psi\right).
\end{aligned}
$$

Reducing further[15],

$$
\begin{aligned}
-\lambda &= \left(\frac{(n+g+\delta)^\psi-\alpha(\sigma)s_k^\psi-\beta(\sigma)s_h^\psi}{(n+g+\delta)^{\psi-1}}\right)\left((\psi-1)\left(\frac{\alpha(\sigma)\left(1+\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{1}{1-\psi}}\right)}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right)-\psi\right) \\
&= -\left(\frac{(n+g+\delta)^\psi-\alpha(\sigma)s_k^\psi-\beta(\sigma)s_h^\psi}{(n+g+\delta)^{\psi-1}}\right)
\end{aligned}
$$

Therefore,

$$\lambda = (n+g+\delta)(1-\pi_K^\star-\pi_H^\star).$$

For the effects of changing the elasticity of substitution on the speed of convergence, we obtain the following

$$
\begin{aligned}
\frac{d\lambda}{d\sigma} &= \lambda = (n+g+\delta)\left(\frac{d\pi_K^\star}{d\sigma}+\frac{d\pi_H^\star}{d\sigma}\right) \\
&= -\frac{(n+g+\delta)}{\sigma^2}\left(\pi_K^\star+\pi_H^\star\right)\ln\frac{y_0/k_0}{y^\star/k^\star}.
\end{aligned} \tag{43}
$$

The effects of the elasticity of substitution on the speed of convergence for the normalized CES, for additive human capital, is determined as follows:

$$\frac{d\lambda}{d\sigma} = -\frac{(n+g+\delta)}{\sigma^2}\left(\pi_K^\star+\pi_H^\star\right)\ln\frac{y_0/k_0}{y^\star/k^\star}\begin{cases} >0 \iff k^\star<k_0 \\ =0 \iff k^\star=k_0 \\ <0 \iff k^\star>k_0 \end{cases} \tag{44}$$

Due to the concavity of the production function, the baseline capital productivity $y_0 = k_0$ is lower than the steady-state capital productivity $y^\star = k^\star$ if $k^\star < k0$, and vice versa. When the baseline level of capital per effective unit of labor is greater than the steady state level, an increase in the elasticity of substitution leads to an increase in the speed of convergence.

[15]This follows from $\alpha(\sigma)\left(1+\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{1}{1-\psi}}\right) = \alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}$

An alternative explanation is that when capital is relatively less scarce than effective labor when compared to the state steady levels then a higher elasticity of substitution leads to a higher speed of convergence.

Evaluating the log-linear approximation of the growth rate of output per effect unit of labor for $\sigma = 1$ ($\psi = 0$) about the long-run equilibrium, the corresponding speed of convergence for this model, as well as for the Mankiw *et al.* (1992) model, is determined by the depreciation rate of capital per effective unit and the labor share of income. Similarly, Klump & Preissler (2000) determine the speed of convergence for their two factor production function and examine the effects of the elasticity of substitution. Their specification is also special case of (1) for β equal to zero.