



Volume 37, Issue 3

Closed spaces induced by deviation measures

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Abstract

We show that a generalized deviation measure can induce a seminorm, which is a norm in important cases. The generalized deviation measure is finite and continuous with respect to the introduced norm. From the norm, we extend to a closed space, which can be understood as a natural domain for generalized deviation measures.

1 Introduction

Since the work of Artzner et al. (1999), the risk of financial positions has been investigated from both theoretical and axiomatic bases. This stream of literature has grown significantly, focusing on losses and tail risks. In this regard, Rockafellar et al. (2006a) introduced the axiomatic concept of generalized deviation measures, as generalizations of the standard deviation and similar measures, in order to develop axioms and build a theoretical body of results. These measures have been used successfully in financial and engineering problems (e.g., Rockafellar et al. (2006b), Rockafellar et al. (2007), Grechuk et al. (2009) and Rockafellar and Uryasev (2013), Righi (2016)), as well as other fields.

Both risk and deviation measures are functions defined on some vector space. The choice of a domain leads to distinct representation results. In most applied problems, such as optimization, aspects such as finiteness and continuity of functions are crucially important. These properties are linked directly to the domain on which the functions are defined. Hence, establishing a natural space for the domain is a relevant task. In the case of risk measures, this is well discussed in Delbaen (2002) for L^∞ , Inoue (2003) for L^p , and Cheridito and Li (2009) for Orlicz spaces. With regard to generalized deviation measures, Rockafellar et al. (2006a) consider L^2 , Grechuk et al. (2009) focus on L^p , while Kountzakis (2013) considers partially ordered linear spaces and some closed subspaces. Thus, a “general” space for the domain is being pursued in order to unify the theory. In this sense, Delbaen (2009) and Filipovic and Svindland (2012) have shown that L^1 is the canonical domain for risk measures, while Pichler (2013) has considered a space induced by a norm obtained from spectral risk measures (coherent risk measures that also respect law invariance and comonotonic additivity).

However, such a natural space for generalized deviation measures has yet to be identified. Thus, we show that a generalized deviation measure can induce a seminorm, which is a norm in important cases. This seminorm is the maximum of a generalized deviation measure for a random variable and its opposite (say X and $-X$), and such a measure is finite and continuous with respect to the introduced norm. This solves the lack of symmetry that prevents generalized deviation measures from being norms. Furthermore, we extend this to a closed space, which can be understood as a natural domain for generalized deviation measures.

2 Main results

Consider the random result X of any asset ($X \geq 0$ is a gain; $X < 0$ is a loss) defined in an atom-less probability space, $\mathcal{X}(\Omega, \mathcal{F}, \mathbb{P})$. Here, $E[X]$ is the expected value of X under \mathbb{P} . All equalities and inequalities are considered to be almost surely in \mathbb{P} . Let $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$, with $1 \leq p \leq \infty$, be the space of random variables defined by the norm $\|X\|_p = (E[|X|^p])^{\frac{1}{p}}$, with finite p and $\|X\|_\infty = \inf\{k : |X| \leq k\}$. Then, $X \in L^p$ indicates that $\|X\|_p < \infty$. L^0 is the space of all random variables, endowed with convergence in probability.

We begin by defining a generalized deviation measure, as proposed by Rockafellar et al. (2006a), exposing the desired axioms. We begin by defining the seminorm induced by a generalized deviation measure. Basically, the idea is to derive the maximum of $\mathcal{D}(X)$ and $\mathcal{D}(-X)$ in order to obtain a “symmetric” effect, which is crucial for norms. Based on such seminorm we can define the space induced by any generalized deviation measure

\mathcal{D} .

Definition 1. A function $\mathcal{D} : \mathcal{X} \rightarrow \mathbb{R}_+ \cup \infty$ is a generalized deviation measure if it fulfills the following axioms:

$$(TI): \mathcal{D}(X + C) = \mathcal{D}(X), \forall X \in \mathcal{X}, C \in \mathbb{R}.$$

$$(PH): \mathcal{D}(\lambda X) = \lambda \mathcal{D}(X), \forall X \in \mathcal{X}, \lambda \in \mathbb{R}_+.$$

$$(SA): \mathcal{D}(X + Y) \leq \mathcal{D}(X) + \mathcal{D}(Y), \forall X, Y \in \mathcal{X}.$$

$$(NC): \mathcal{D}(X) > 0, \forall \text{ non constant } X \in \mathcal{X}.$$

Moreover, it can fulfill the following additional axiom:

$$(LRD): \mathcal{D}(X) \leq E[X] - \inf X, \forall X \in \mathcal{X}.$$

Here, (TI), (PH), (SA), (NC) and (LRD) denote Translation Insensitivity, Positive Homogeneity, Sub-additivity, Non-constancy and Lower Range Dominance, respectively.

Remark 2. The first axiom, (TI), indicates that the deviation in relation to an expected value does not change if a constant value is added. The second axiom, (PH), indicates that the deviation proportionally increases with the position size. PH alone implies that $\mathcal{D}(0) = 0$, and, together with TI, implies that $\mathcal{D}(C) = 0, C \in \mathbb{R}$. (SA), the third axiom, implies that the deviation of a combined position is less than the sum of the individual positions. (SA) and (PH) together are known as sub-linearity, and imply convexity. (NC) ensures there is uncertainty for non-constant positions. (LRD) restricts the measure to a range that is lower than the maximum possible variation from the expected value to the infimum.

Definition 3. Let $\mathcal{D} : \mathcal{X} \rightarrow \mathbb{R}_+ \cup \infty$ be a generalized deviation measure. Then, we have the following induced (semi)norm $\|X\|_{\mathcal{D}} = \mathcal{D}(X) \vee \mathcal{D}(-X)$. Moreover, let $L := \{X \in L^0 : \|X\|_{\mathcal{D}} < \infty\}$, and $N := \{X \in L : \|X\|_{\mathcal{D}} = 0\}$. Then, the quotient space $L^{\mathcal{D}} := L/N$ is the space induced by \mathcal{D} . We represent this space as $(L^{\mathcal{D}}, \|\cdot\|_{\mathcal{D}})$.

Remark 4. Note that $\|\cdot\|_{\mathcal{D}}$ is itself a generalized deviation measure, conform Proposition 4 in Rockafellar et al. (2006a). A simpler functional form such as $\|X\|_{\mathcal{D}} = \mathcal{D}(|X|)$, which is the choice of Pichler (2013), fails to be a norm because it violates triangle inequality. Moreover, $\|\cdot\|_{\mathcal{D}}$ is a norm on $L^{\mathcal{D}}$ because $\|Y\|_{\mathcal{D}} = 0$ implies $Y = 0$ since elements of N behave similarly to constants. Under this framework, it is direct that $d(X, Y) = \|X - Y\|_{\mathcal{D}}$ defines a metric on $L^{\mathcal{D}}$.

Proposition 5. Let $\mathcal{D} : \mathcal{X} \rightarrow \mathbb{R}_+ \cup \infty$ be a generalized deviation measure. Then:

(i) $\|\cdot\|_{\mathcal{D}}$ is a seminorm on L .

(ii) \mathcal{D} is finite valued on L .

(iii) \mathcal{D} is Lipschitz-continuous with respect to $\|\cdot\|_{\mathcal{D}}$.

(iv) $(L^{\mathcal{D}}, \|\cdot\|_{\mathcal{D}})$ is a closed space.

(v) If the domain of \mathcal{D} is L^p and it is lower range dominated, then $\|X\|_{\mathcal{D}} \leq \|X - E[X]\|_p$.

Proof. We begin with (i). We have that $\|\cdot\|_{\mathcal{D}}$ has positive homogeneity and sub-additivity, in addition to non-negativity, by definition. Let $\lambda \in \mathbb{R}$. If $\lambda \geq 0$, we have $\|\lambda X\|_{\mathcal{D}} = \lambda\|X\|_{\mathcal{D}}$. On the other hand, if $\lambda < 0$, then $\lambda^* = -\lambda > 0$. Thus, $\|\lambda X\|_{\mathcal{D}} = \mathcal{D}(-\lambda^* X) \vee \mathcal{D}(\lambda^* X) = \lambda^* (\mathcal{D}(X) \vee \mathcal{D}(-X)) = -\lambda\|X\|_{\mathcal{D}}$. We then obtain $\|\lambda X\|_{\mathcal{D}} = |\lambda|\|X\|_{\mathcal{D}}$. Furthermore, we have that $\|X+Y\|_{\mathcal{D}} \leq \|X\|_{\mathcal{D}} + \|Y\|_{\mathcal{D}}$, which is triangle inequality for norms. Hence, $\|\cdot\|_{\mathcal{D}}$ is a seminorm on L .

For (ii), we have that $\mathcal{D}(X) \leq \mathcal{D}(X) \vee \mathcal{D}(-X) = \|X\|_{\mathcal{D}} < \infty, \forall X \in L$. Hence, \mathcal{D} is finite valued on L .

With regard to (iii), we have, by sub-additivity, that $\mathcal{D}(X) = \mathcal{D}(X + Y - Y) \leq \mathcal{D}(Y) + \mathcal{D}(X - Y) \leq \mathcal{D}(Y) + \mathcal{D}(X - Y) \vee \mathcal{D}(Y - X) = \mathcal{D}(Y) + \|X - Y\|_{\mathcal{D}}$. From this, it follows that $\mathcal{D}(X) - \mathcal{D}(Y) \leq \|X - Y\|_{\mathcal{D}}$. By changing the roles of X and Y , we obtain that $|\mathcal{D}(X) - \mathcal{D}(Y)| \leq \|X - Y\|_{\mathcal{D}}$. Hence, \mathcal{D} is Lipschitz-continuous with respect to $\|\cdot\|_{\mathcal{D}}$. This concludes the proof.

For (iv), consider a sequence $X_n \in L^{\mathcal{D}}$, with limit X . Because $\|X_n - X\|_{\mathcal{D}} \rightarrow 0$, we have $\|X_n - X\|_{\mathcal{D}} < \epsilon, \epsilon > 0$. Thus, we obtain $\|X\|_{\mathcal{D}} = \|X - X_n + X_n\|_{\mathcal{D}} \leq \|X - X_n\|_{\mathcal{D}} + \|X_n\|_{\mathcal{D}} \leq \epsilon + \|X_n\|_{\mathcal{D}}$. Because ϵ is an arbitrary choice, X_n converges, and $\|X_n\|_{\mathcal{D}} < \infty$, we have that $\|X\|_{\mathcal{D}} < \infty$. Hence, $X \in L^{\mathcal{D}}$.

Regarding (v), Rockafellar et al. (2006a) prove that there is a direct relation to some coherent risk measure $\rho : L^p \rightarrow \mathbb{R} \cup \infty$, in the sense of Artzner et al. (1999), conform $\mathcal{D}(X) = \rho(X - E[X])$. It is well known that coherent risk measures are Lipschitz continuous, i.e., $|\rho(X) - \rho(Y)| \leq \|X - Y\|_p$. Choosing $Y = E[X]$ we get

$$\begin{aligned} \|X - E[X]\|_p &\geq |\rho(X) - \rho(E[X])| \\ &= |\rho(X - E[X])| \\ &= \mathcal{D}(X). \end{aligned}$$

Since $\|X - E[X]\|_p = \|-X - E[-X]\|_p = \|E[X] - X\|_p$, we obtain

$$\begin{aligned} \|X - E[X]\|_p &= \|X - E[X]\|_p \vee \|E[X] - X\|_p \\ &\geq \mathcal{D}(X) \vee \mathcal{D}(-X) \\ &= \|X\|_{\mathcal{D}}. \end{aligned}$$

This concludes the proof. □

Remark 6. Other compositions of $\mathcal{D}(X)$ and $\mathcal{D}(-X)$, such as the minimum or average, despite also having the properties of a seminorm, do not guarantee finiteness and/or continuity. This, together with the protective feature of the maximum, justifies our choice of this functional form for the induced norm. Moreover, for $L^p, p > 2$, the open-mapping theorem (Aliprantis and Border (2006)) ensures that $\|\cdot\|_{\mathcal{D}}$ and $\|\cdot\|_p$ are equivalent. This result does not hold for $X \in L^{\mathcal{D}}$.

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