Economics Bulletin

Volume 37, Issue 3

On single-peakedness and strategy-proofness: ties between adjacent alternatives

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Abstract

We extend the classical characterizations of social choice rules that satisfy strategy-proofness in the single-peaked domain of preferences by Moulin (Public Choice, 1980) and Barberà, Gul, and Stacchetti (Journal of Economic Theory, 1993) to multivalued social choice rules that admit either the selection of single alternatives or the selection of pairs of adjacent alternatives.

I thank the Associate Editor, Vicki Knoblauch, and an anonymous referee for their useful comments and suggestions. Financial support from Ministerio de Economía y Competitividad through grant ECO2016-76818-C3-3-P and from Fundación Ramón Areces is gratefully acknowledged.

Citation: Carmelo Rodríguez-Álvarez, (2017) "On single-peakedness and strategy-proofness: ties between adjacent alternatives", *Economics Bulletin*, Volume 37, Issue 3, pages 1966-1974
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Submitted: March 13, 2017. **Published:** August 31, 2017.

1 Introduction

We consider the problem of a society of voters selecting from a finite set of alternatives under single-peaked preferences when ties between adjacent alternatives are possible. The classical result by Moulin (1980) shows that, when ties are not allowed and voters' are restricted to report their preferred alternative –their peaks–, the family of voting rules that satisfy strategy-proofness is characterized by a generalization of the selection of the median of the reported peaks. Barberà *et al*. (1993) extend the analysis to multidimensional spaces of alternatives and show that every single-valued voting rule that satisfies strategy-proofness and unanimity only uses the information contained in the peaks reported by the voters.

Despite the natural appeal of median rules, it could be argued that sometimes they may fail to provide an optimal compromise between the conflicting interests of the voters. Think of a society with 100 voters, 50 voters vote for alternative a, 50 voters vote for b. A median voter rule would choose either a or b, but the choice of an alternative in between a or b may seem more adequate. In the case that there is no alternative in between a and b, we could consider the possibility of selecting both alternatives, pending the final selection to an eventual screening process after additional information is obtained.

The previous observation drives our attention to voting rules that take into account voters' preferences to select either a single alternative or a pair of adjacent alternatives. We require that such rules satisfy strategy-proofness and unanimity. Since the rules admit the selection of sets of alternatives, voters' preferences over alternatives do not contain enough information to compare the possible outcomes of the social choice. Thus, voters' preferences over alternatives must be extended to preferences over sets of alternatives. We assume that each voter ranks any pair of adjacent alternatives in between the two alternatives that form the pair. Under this assumption, we define an auxiliary order on the set of possible outcomes and show that voters' preferences over outcomes are single-peaked with respect to that auxiliary order, but the preferences are restricted to have the peak on a single alternative. We show that the main results in Moulin (1980) and Barberà *et al*. (1993) can be extended to this subdomain of single-peaked preferences. The contribution of this note relies on providing the precise additional arguments required in the proofs by Moulin (1980) and Barberà *et al*. (1993) to apply in the subdomain generated when ties between adjacent alternatives are admissible.

There is a bast literature analyzing either multivalued social choice rules or the domain of single-peaked preferences as ways to avoid the negative conclusions of the Gibbard–Satterthwaite Theorem (Gibbard, 1973; Satterthwaite, 1975). We refer the reader to Barberà (2011) for a comprehensive survey. There are a few papers however, that deal with multivalued social choice rules under single-peaked preferences. Miyagawa (1998, 2001), Ehlers and Gordon (2013), and Heo (2013) consider the problem of locating two public facilities in a line when each voter may attend one facility. Klaus and Storcken (2002) consider the choice of intervals of alternatives in multidimensional spaces. All those papers consider different restrictions on feasible sets and on preferences over sets than the restrictions analyzed here. Finally, Massó and Moreno de Barreda (2011) analyze strategy-proofness in the subdomain of symmetric single-peaked preferences. After this brief review of the literature, in Section 2, we present notation and definitions. In Section 3, we state the characterization results and the proofs.

2 Basic Notation and Definitions

Consider a society formed by a finite set of voters $N = \{1, \ldots, n\}$ that select from a finite set of alternatives A. The alternatives are labelled by consecutive integers and ordered in an integer segment. That is, for some $a, b \in \mathbb{Z}$ such that $a \leq b$, $A \equiv \{a, a+1, \ldots, b-1, b\}$.[1] Let $\mathcal{R}$ denote the set of all complete, reflexive, and transitive binary relations on A. We call each element $R_i \in \mathcal{R}$ a ***preference over alternatives***. Each voter is equipped with a preference over alternatives. We denote by R_i, voter i's preference. For each R_i we denote by P_i the associated strict preference relation.

Each voter's preference over alternatives is restricted to be ***single-peaked***. That is, for each R_i, there is a unique alternative $p(R_i) \in A$ –the ***peak of*** R_i– such that (i) for each $c \in A \setminus \{p(R_i)\}$, $p(R_i)\ P_i\ c$, and (ii) for each $c', c'' \in A$, $c' < c'' < p(R_i)$ or $p(R_i) < c'' < c'$ imply $c''\ P_i\ c'$. Let $\mathcal{S}$ denote the set of all single-peaked preferences over A.

Let $\overline{A}$ be the set of pairs of adjacent alternatives, $\overline{A} \equiv \{\{c, c'\} \in A \setminus \{b\} \times A \mid c' = c + 1\}$. The ***extended set of alternatives***, denoted by $\mathcal{A} \subset 2^A \setminus \{\varnothing\}$, consists of all of singleton sets of alternatives and all pairs of adjacent alternatives. That is, $\mathcal{A} \equiv A \cup \overline{A}$.

Let the order $\lesssim$ defined on $\mathcal{A}$ be such that (i) for each $c, c' \in A$, $c \leq c'$ iff $\{c\} \lesssim \{c'\}$, and (ii) for each $c, (c+1) \in A$, $\{c\} \lesssim \{c, c+1\} \lesssim \{c+1\}$.[2] For each pair $\alpha, \beta \in \mathcal{A}$ with $\alpha \lesssim \beta$, we denote by $[\alpha, \beta]$ the ***minimal interval*** containing α and β, $[\alpha, \beta] \equiv \left\{\gamma \in \mathcal{A} \mid \alpha \lesssim \gamma \lesssim \beta\right\}$. For each set $C \subseteq \mathcal{A}$, we denote respectively by $\overline{\inf}\{C\}$ and $\overline{\sup}\{C\}$ the ***infimum*** and the ***supremum*** of C according to the order $\lesssim$,

$$\overline{\inf}\{C\} \equiv \left\{\alpha \in C \mid \text{ for each } \alpha' \in C,\ \alpha \lesssim \alpha'\right\}, \quad \overline{\sup}\{C\} \equiv \left\{\beta \in C \mid \text{ for each } \beta' \in C,\ \beta' \lesssim \beta\right\}.$$

Voters' preferences over the extended set of alternatives are naturally obtained from voters' preferences over alternatives. Let $\mathcal{D}$ be the set of all complete, reflexive, and transitive binary relations over $\mathcal{A}$. We call each element $\succsim_i \in \mathcal{D}$ a ***preference over sets***. For each $\succsim_i \in \mathcal{D}$, $\succ_i$ refers to the strict component of $\succsim_i$. We denote by $\succsim = (\succsim_1, \ldots, \succsim_n) \in \mathcal{D}^N$ a profile of voters' preferences over sets. For each voter i and each profile of preferences over sets $\succsim$, we denote the restriction of $\succsim$ to the voters $N \setminus \{i\}$ by $\succsim_{-i}$.

For each R_i, we say that $\succsim_i$ is R_i***-consistent*** if:

(i) for each $c, c' \in A$, $c\ R_i\ c'$ iff $\{c\} \succsim_i \{c'\}$,

(ii) for each $\{c, c'\} \in \overline{A}$, $c\ P_i\ c'$ iff $\{c\} \succ_i \{c, c'\} \succ_i \{c'\}$.

For each $R_i \in \mathcal{S}$, we denote by $\mathcal{D}(R_i)$ the set of all R_i-consistent preferences over sets.[3] Finally, we denote by $\overline{\mathcal{S}} \equiv \cup_{R_i \in \mathcal{S}} \mathcal{D}(R_i)$ the domain of all consistent preferences over sets, and $\overline{\mathcal{S}}^N$ the set of all profiles of consistent preferences over sets.

[1] For each $c \in \mathbb{Z}$ such that $a \leq c \leq b$, $c \in A$.

[2] An order on a set X is a complete, reflexive, transitive, and antisymmetric binary relation on X.

[3] Both conditions on consistent preferences over sets are satisfied by preferences over sets based on Expected Utility Maximization (Barberà *et al*., 2001; Ching and Zhou, 2002; Duggan and Schwartz, 2000; Rodríguez-Álvarez, 2007), leximin preferences (Campbell and Kelly, 2000; Özyurt and Sanver, 2009), and leximax preferences (Ehlers and Gordon, 2013). That is not the case for maximin and maximax preferences (Heo, 2013). It's worth to note that for each R_i, $\mathcal{D}(R_i)$ is not single–valued.

Lemma 1. *For each $R_i \in \mathcal{S}$, if $\succsim_i \in \overline{\mathcal{S}}$ is R_i-consistent, then $\succsim_i$ is single-peaked with respect to the order $\overline{\leq}$.*

Proof. Let $R_i \in \mathcal{S}$, and let $\succsim_i \in \overline{\mathcal{S}}$ be such that $\succsim_i$ is R_i–consistent. Consider the peak of R_i, $p(R_i)$. For each $c \in A$, $p(R_i)\ P_i\ c$ implies that $\{p(R_i)\} \succ_i \{c\}$. Since for each $\{c, c'\} \in \overline{A}$, $c\ P_i\ c'$ implies $\{c\} \succ_i \{c, c'\} \succ_i \{c'\}$, by transitivity, for each $\alpha \in \mathcal{A} \setminus \{p(R_i)\}$, $\{p(R_i)\} \succ_i \alpha$. Finally, let $\alpha, \beta \in \mathcal{A} \setminus \{p(R_i)\}$ be such that $\alpha \overline{\leq} \beta \overline{\leq} \{p(R_i)\}$ and $\alpha \neq \beta$. By $\succsim_i$'s transitivity, $\{p(R_i)\} \succ_i \beta \succ_i \alpha$. The same argument applies to $\{p(R_i)\} \overline{\leq} \beta \overline{\leq} \alpha$. $\square$

Abusing notation, for each $\succsim_i \in \overline{\mathcal{S}}$, we denote by $p(\succsim_i)$ the ***peak of*** $\succsim_i$. The set $\overline{\mathcal{S}}$ consists of all single-peaked preferences over $\mathcal{A}$ with peak belonging to the set A.

A ***minimally extended social choice function*** is a mapping $\varphi : \overline{\mathcal{S}}^N \rightarrow \mathcal{A}$ such that for each profile of voters' consistent preferences over sets selects an element of the extended set of alternatives.

Since the outcome of a minimally extended social choice function may consist of a set of two adjacent alternatives, a minimally extended social choice function is a social choice correspondence with a constrained range. Furthermore, since voters express their preferences over the outcomes of the social choice, a minimally extended social choice function is a standard social choice function operating on a restricted domain of preferences.

We conclude this section with two axioms for minimally extended social choice functions.

Strategy-proofness. For each $i \in N$, each $\succsim \in \overline{\mathcal{S}}^N$, and each $\succsim_i' \in \overline{\mathcal{S}}$, $\varphi(\succsim) \succsim_i \varphi(\succsim_i', \succsim_{-i})$.

Strategy-proofness implies that a voter cannot obtain a better outcome from the social choice function by misreporting her preferences. At every preference profile, truth-telling is a weakly dominant strategy in the induced revelation game.

Unanimity. For each $\succsim \in \overline{\mathcal{S}}^N$, $p(\succsim_i) = p(\succsim_j) = \{p^*\}$ for each $i, j \in N$ implies $\varphi(\succsim) = \{p^*\}$.

Unanimity implies that if voters' preferences agree on the same peak, then the social choice results in the selection of that peak.

3 The Results

We start this section by analyzing an interesting property that simplifies the analysis and it is widely used in real-life applications. We are interested in minimally extended social choice functions that only use the information contained in the peaks of voters' preferences.

Peak-onliness. For each $\succsim, \succsim' \in \overline{\mathcal{S}}^N$, $p(\succsim_i) = p(\succsim_i')$ for each $i \in N$ implies $\varphi(\succsim) = \varphi(\succsim')$.

Barberà *et al.* (1993, Theorem 1) prove that in the domain of single-peaked preferences social choice functions that satisfy *strategy-proofness* and *unanimity* also satisfy *peak-onliness*. Our first result extends the result to the subdomain $\overline{\mathcal{S}}$.

Theorem 1. *If a minimally extended social choice function φ satisfies* strategy-proofness *and* unanimity*, then φ satisfies* peak-onliness.

Proof. To prove the result, we show that for each $i \in N$ and each pair of profiles $\succsim, \succsim' \in \overline{\mathcal{S}}^N$ such that (i) $p(\succsim_i) = p(\succsim'_i)$ and (ii) for each $j \neq i$, $\succsim_j = \succsim'_j$, *strategy-proofness* and *unanimity* imply $\varphi(\succsim) = \varphi(\succsim')$. Assume to the contrary that there are $i \in N$ and $\succsim, \succsim' \in \overline{\mathcal{S}}^N$ such that $p(\succsim_i) = p(\succsim'_i)$, $\succsim_j = \succsim'_j$ for each $j \in N \setminus \{i\}$, and $\varphi(\succsim) \neq \varphi(\succsim')$. Let $\mathcal{X} \equiv \left[\overline{\inf}_{j\in N}\{p(\succsim_j)\}, \overline{\sup}_{j\in N}\{p(\succsim_j)\}\right]$. That is, $\mathcal{X}$ is the minimal interval defined by the infimum and the supremum of the voters' peaks. In addition, suppose that the profiles $\succsim, \succsim'$ are such that $\mathcal{X}$ is minimal in the sense there are no two pair of profiles that generate a violation of *peak-onliness* and such that the minimal interval defined by voters' peaks is strictly contained in $\mathcal{X}$.[4] We prove the result by a series of steps that follow the arguments in the proof of Barberà *et al*. (1993, Theorem 1). Step 2 is new and required in Step 3 to solve the cases involving pairs of adjacent alternatives.

Step 1. $\varphi(\succsim) \in \mathcal{X}$.

Assume to the contrary that $\varphi(\succsim) \notin \mathcal{X}$. In addition, assume that for each $j \in N$, $\varphi(\succsim) \overline{\leq} \{p(\succsim_j)\}$. (The symmetric argument applies to the case $\{p(\succsim_j)\} \overline{\leq} \varphi(\succsim)$.)

Let $\succsim^* \in \overline{\mathcal{S}}^N$ be such that for each $j \in N$, $p(\succsim^*_j) = \overline{\inf}_{k\in N}\{p(\succsim_k)\}$ and for each $\alpha \in [\varphi(\succsim), p(\succsim^*_j)]$ and each $\beta \in \mathcal{A} \setminus [\varphi(\succsim), p_j(\succsim^*_j))]$, $\alpha \succ^*_j \beta$. Let $j \in N$. For each $\alpha' \in \left[\varphi(\succsim), p(\succsim^*_j)\right] \setminus \{\varphi(\succsim)\}, \alpha' \succ_j \varphi(\succsim)$. Hence, by *strategy-proofness*, either $\varphi(\succsim^*_j, \succsim_{-j}) = \varphi(\succsim)$ or $\varphi(\succsim^*_j, \succsim_{-j}) \notin \left[\varphi(\succsim), p(\succsim^*_j)\right]$. Furthermore, for each $\beta' \notin \left[\varphi(\succsim), p(\succsim^*_j)\right]$, $\varphi(\succsim) \succ^*_j \beta'$. Hence, by *strategy-proofness*, $\varphi(\succsim^*_j, \succsim_{-j}) \succsim^*_j \varphi(\succsim)$ and $\varphi(\succsim^*_j, \succsim_{-j}) \in \left[\varphi(\succsim), p_j(\succsim^*_j)\right]$. Thus, $\varphi(\succsim^*_j, \succsim_{-j}) = \varphi(\succsim)$. Repeating the argument changing voters' preferences one at a time, we conclude that $\varphi(\succsim^*) = \varphi(\succsim)$ which contradicts *unanimity*.

Without loss of generality, assume that $\varphi(\succsim) \overline{\leq} \varphi(\succsim')$. Denote by $\mathcal{Y} \equiv [\varphi(\succsim), \varphi(\succsim')]$. That is, $\mathcal{Y}$ is the minimal interval containing both $\varphi(\succsim)$ and $\varphi(\succsim')$.

Step 2. $p(\succsim_i) \in \mathcal{Y} \setminus \{\varphi(\succsim), \varphi(\succsim')\}$.

By *strategy-proofness*, $\varphi(\succsim) \succsim_i \varphi(\succsim')$ and $\varphi(\succsim') \succsim'_i \varphi(\succsim)$. Since $\varphi(\succsim) \neq \varphi(\succsim')$, $\varphi(\succsim) \neq \{p(\succsim_i)\} = \{p(\succsim'_i)\}$ and $\varphi(\succsim') \neq \{p(\succsim_i)\} = \{p(\succsim'_i)\}$. Since $\succsim'_i$ is single-peaked, if $\{p(\succsim'_i)\} \overline{\leq} \varphi(\succsim)$, then $\varphi(\succsim) \succ'_i \varphi(\succsim')$, which contradicts *strategy-proofness*. The symmetric argument applies in the case $\varphi(\succsim') \overline{\leq} \{p(\succsim_i)\}$. Hence, $\varphi(\succsim) \overline{\leq} \{p(\succsim_i)\} \overline{\leq} \varphi(\succsim')$.

Step 3. $\mathcal{X} = \mathcal{Y}$.

Assume to the contrary that there is $j \in N$ with $\{p(\succsim_j)\} \overline{\leq} \varphi(\succsim)$ and $\{p(\succsim_j)\} \neq \varphi(\succsim)$. Let $J \subset N$ be the set of voters with infimum peaks, $J \equiv \left\{k \in N \mid \{p(\succsim_k)\} = \overline{\inf}_{k\in N}\{p(\succsim_k)\}\right\}$. By Step 2, $i \notin J$. We consider two cases: (i) $\varphi(\succsim) \in A$, and (ii) $\varphi(\succsim) \in \overline{A}$

[4] Formally, there is no pair of profiles $\overline{\succsim}, \overline{\succsim}' \in \overline{\mathcal{S}}^N$ such that for each $j \in N, p(\overline{\succsim}_j) = p(\overline{\succsim}'_j), \varphi(\overline{\succsim}) \neq \varphi(\overline{\succsim}')$, and $[\overline{\inf}_{j\in N}\{p(\overline{\succsim}_j)\}, \overline{\sup}_{j\in N}\{p(\overline{\succsim}_j)\}] \subsetneq \mathcal{X}$.

Assume first that $\varphi(\succsim) \in A$. Let $\alpha^* = \varphi(\succsim)$. Define $\overline{\succsim} \in \overline{\mathcal{S}}^N$ be such that (i) for each $j \in J$, $p(\overline{\succsim}_j) = \alpha^*$, for each $\beta, \gamma \in \mathcal{Y}$, $\delta \notin \mathcal{Y}$, $\beta \overline{\succsim}_j \gamma$ if and only if $\beta \succsim_j \gamma$, $\beta \overline{\succ}_j \delta$, and (ii) for each $k \in N \setminus J$, $\overline{\succsim}_k = \succsim_k$. Let $j \in J$. By *strategy-proofness*, $\varphi(\overline{\succsim}_j, \succsim_{-j}) \overline{\succsim}_j \varphi(\succsim) = \{p(\overline{\succsim}_j)\}$. Thus, $\varphi(\overline{\succsim}_j, \succsim_{-j}) = \varphi(\succsim)$. Repeating the argument changing preferences of voters in J one at a time, we obtain $\varphi(\overline{\succsim}) = \varphi(\succsim)$. Analogously, let $\overline{\succsim}' \in \overline{\mathcal{S}}^N$ be such that (i) for each $j \in J$, $p(\overline{\succsim}'_j) = \alpha^*$, for each $\beta, \gamma \in \mathcal{Y}$, $\delta \notin \mathcal{Y}$, $\beta \overline{\succsim}'_j \gamma$ if and only if $\beta \succsim'_j \gamma$, $\beta \overline{\succ}_j \delta$, and (ii) for each $k \in N \setminus J$, $\overline{\succsim}'_k = \succsim'_k$. Let $j \in J$. By *strategy-proofness*, $\varphi(\overline{\succsim}'_j, \succsim'_{-j}) \overline{\succsim}'_j \varphi(\succsim')$. Hence, $\varphi(\overline{\succsim}'_j, \succsim'_{-j}) \in \mathcal{Y}$. Since $\succsim'_j$ is single-peaked and $\{p(\succsim'_j)\} \leqq \varphi(\succsim) \leqq \varphi(\succsim')$, for each $\alpha \in \mathcal{Y} \setminus \{\varphi(\succsim')\}$, $\alpha \succ'_j \varphi(\succsim')$. Thus, by *strategy-proofness*, $\varphi(\overline{\succsim}'_j, \succsim'_{-j}) = \varphi(\succsim')$. Repeating the argument, we obtain $\varphi(\overline{\succsim}') = \varphi(\succsim')$. Thus, $\varphi(\overline{\succsim}) \neq \varphi(\overline{\succsim}')$. However, for each $j \in N$, $p(\overline{\succsim}_j) = p(\overline{\succsim}'_j)$, which contradicts the minimality of $\mathcal{X}$.

Finally, assume that $\varphi(\succsim) \in \overline{A}$. Then, $\varphi(\succsim) = \{c, c+1\}$ for some $c \in A \setminus \{b\}$. Let $\alpha^* = \{c+1\}$. Let $\overline{\succsim} \in \overline{\mathcal{S}}^N$ be such that (i) for each $j \in J$, $p(\overline{\succsim}_j) = \alpha^*$, for each $\alpha \in \mathcal{Y} \setminus \{\alpha^*\}$, $\varphi(\succsim) \overline{\succ}_j \alpha$, for each $\beta, \gamma \in \mathcal{Y} \setminus \{\varphi(\succsim), \alpha^*\}$, $\delta \notin \mathcal{Y}$, $\beta \overline{\succsim}_j \gamma$ if and only if $\beta \succsim_j \gamma$, $\beta \overline{\succ}_j \delta$, and (ii) for each $k \in N \setminus J$, $\overline{\succsim}_k = \succsim_k$. By *strategy-proofness*, $\varphi(\overline{\succsim}_j, \succsim_{-j}) \overline{\succsim}_j \varphi(\succsim) = \{p(\overline{\succsim}_j)\}$. Thus, $\varphi(\overline{\succsim}_j, \succsim_{-j}) \in \{\varphi(\succsim), \alpha^*\}$. Repeating the argument, we obtain $\varphi(\overline{\succsim}) \in \{\varphi(\succsim), \alpha^*\}$. Analogously, let $\overline{\succsim}' \in \overline{\mathcal{S}}^N$ be such that (i) for each $j \in J$, $\overline{\succsim}'_j = \overline{\succsim}_j$, and (ii) for each $k \in N \setminus J$, $\overline{\succsim}'_k = \succsim'_k$. By the same argument of the previous paragraph, $\varphi(\overline{\succsim}') = \varphi(\succsim')$. By Step 2, $\varphi(\succsim') \neq \alpha^*$. Thus, $\varphi(\overline{\succsim}) \in \{\varphi(\succsim), \alpha^*\}$ and $\varphi(\overline{\succsim}') = \varphi(\succsim')$, which contradicts the minimality of $\mathcal{X}$ and concludes the proof of Step 3.

The final argument replicates the argument in Step 3. By Step 3, there is $j \in N$ such that $\varphi(\succsim) = p(\succsim_j)$. Thus, $\varphi(\succsim) \in A$. Let $J \equiv \{j \in N, \{p(\succsim_j)\} = \varphi(\succsim)\}$. Let $\widehat{\succsim} \in \overline{\mathcal{S}}^N$ be such that (i) for each $j \in J$, $p(\widehat{\succsim}_j) = p(\succsim_i)$, for each $\alpha \in [\varphi(\succsim), p(\succsim_i)]$ and each $\beta \notin [\varphi(\succsim), p(\succsim_i)]$, $\alpha \mathrel{\widehat{\succ}_j} \beta$, and for each $\alpha' \in \mathcal{Y}$ and each $\beta' \notin \mathcal{Y}$, $\alpha' \mathrel{\widehat{\succ}_j} \beta'$, and (ii) for each $k \in N \setminus J$, $\widehat{\succsim}_k = \succsim_k$. Let $j \in J$. By *strategy-proofness*, $\varphi(\widehat{\succsim}_j, \succsim_{-j}) \widehat{\succsim}_j \varphi(\succsim)$, which implies $\varphi(\widehat{\succsim}_j, \succsim_{-j}) \in [\varphi(\succsim), p(\succsim_i)]$. Repeating the argument, we obtain that $\varphi(\widehat{\succsim}) \in [\varphi(\succsim), p(\succsim_i)]$. Finally, let $\widehat{\succsim}'$ be such that (i) for each $j \in J$, $\widehat{\succsim}'_j = \widehat{\succsim}_j$, and (ii) for each $k \in N \setminus J$, $\widehat{\succsim}'_k = \succsim'_k$. Repeating the arguments in Step 3, we obtain $\varphi(\widehat{\succsim}') = \varphi(\succsim')$. By Step 2, $\{p(\succsim_i)\} \neq \varphi(\succsim)$. Thus, $\overline{\inf}\{(p(\succsim_k))_{k \in N}\} \neq \overline{\inf}\left\{(p(\widehat{\succsim}_k))_{k \in N}\right\} = \overline{\inf}\left\{(p(\widehat{\succsim}'_k))_{k \in N}\right\}$, $\overline{\inf}\{(p(\succsim_k))_{k \in N}\} \leqq \overline{\inf}\left\{(p(\widehat{\succsim}_k))_{k \in N}\right\}$, and $\varphi(\widehat{\succsim}) \neq \varphi(\widehat{\succsim}')$, which contradicts the minimality of $\mathcal{X}$. $\square$

With Theorem 1 at hand, and without loss of generality, we focus on minimally extended social choice functions that only use the information of the peaks of voters' preferences. For each $\succsim \in \overline{\mathcal{S}}^N$, let $(p(\succsim_1), \ldots, p(\succsim_n)) \in A^N$ be the profile of peaks associated to $\succsim$. We denote by $p = (p_1, \ldots, p_n) \in A^N$ an arbitrary profile of peaks. For each $i \in N$ and each profile of peaks p, p_i refers to the peak of i's preferences, and p_{-i} denotes the restriction of p to the voters in $N \setminus \{i\}$.

An ***extended voting scheme*** is a mapping $\pi : A^N \to \mathcal{A}$ such that for each profile of peaks selects an element of the extended set of alternatives.

An extended voting scheme is more general than the voting schemes in the standard setting. In our framework, the election may result in an outcome that is not admitted as a valid peak. We next provide the crucial definition of a family of extended voting schemes.

An extended voting scheme π is an ***extended median voter scheme (EMVS)*** if there is a family of parameters $\{a_S\}_{S\subseteq N}$, $a_S \in \mathcal{A}$ with $a_S \overline{\leq} a_T$ whenever $T \subseteq S$, such that for each profile of peaks $(p_1, \ldots, p_n)$

$$\pi(p_1, \ldots, p_n) = \overline{\inf}_{S\subseteq N} \left\{\overline{\sup}_{i\in S} \left\{p_i, a_S\right\}\right\}.$$

Moulin (1980) and Barberà *et al*. (1993) characterize the family of (extended) voting schemes that satisfy *strategy-proofness* when only the potential peaks of the voters can be selected. EMVSs allow the parameters $\{a_S\}_{S\subseteq N}$ to be located on elements of the extended set of alternatives that are not admitted as voters' peaks.

Next, we provide the counterpart to Moulin (1980, Proposition 3) in our framework.

Theorem 2. *An extended voting scheme π satisfies* strategy-proofness *if and only if π is an EMVS.*

Proof. We prove sufficiency first. Let π be an EMVS with parameters $\{\alpha_S\}_{S\subseteq N}$. Let $i \in N$, $p \in A^N$, and $\succsim_i \in \overline{\mathcal{S}}$ be such that $p(\succsim_i) = p_i$. Let $\pi(p) = \alpha$. If $\{p_i\} = \alpha$, then for each $p'_i \in A$, $\pi(p) \succsim_i \pi(p'_i, p_{-i})$. Assume now that $\{p_i\} \neq \alpha$. Consider first the case $\{p_i\} \overline{\leq} \alpha$. Since $\pi(p) = \alpha$, for each coalition $S \subseteq N$, $\alpha \overline{\leq} \overline{\sup}_{j\in N}\{p_j, a_S\}$. Specifically, for each $S' \subseteq N$ with $i \in S'$, either there is $j \in S'$ with $\alpha \overline{\leq} \{p_j\}$, or $\alpha \overline{\leq} a_{S'}$. Therefore, for each $p'_i \in A$, $\alpha \overline{\leq} \pi(p'_i, p_{-i})$. Since $\succsim_i$ is single-peaked, $\pi(p) \succsim_i \pi(p'_i, p_{-i})$. Finally, if $\alpha \overline{\leq} \{p_i\}$, then there is $S \subseteq N$ such that $\overline{\sup}_{j\in S}\{p_j, a_S\} = \alpha$, and for each S' with $i \in S'$, $\alpha \overline{\leq} \{p_i\} \overline{\leq} \overline{\sup}_{j\in S'}\{p_j, a_{S'}\}$. Therefore, for each p'_i, $\pi(p'_i, p_{-i}) \overline{\leq} \alpha$ and $\pi(p) \succsim_i \pi(p'_i, p_{-i})$, which concludes the proof of π's *strategy-proofness*.

We prove necessity by induction on the number of voters. The arguments follow the line of the proof of Moulin (1980, Proposition 3). We need an additional argument to prove the result when there is only one voter. Once we prove the simple 1-voter case, the induction argument replicates the arguments of the proof of Moulin (1980, Proposition 3).

Let $N = \{i\}$ and π be an extended voting scheme that satisfies *strategy-proofness*. Let a_π and b_π be, respectively, the infimum and the supremum elements of the range of π.

Assume first that $a_\pi = b_\pi = \alpha$. For each $p_i \in A^N$, $\pi(p_i) = \alpha$. For each $\succsim_i \in \overline{\mathcal{S}}$, $\pi(p_i) = \overline{\inf}\{\alpha, \overline{\sup}\{p_i, \alpha\}\} = \alpha$. Let $a_{\{\varnothing\}} = a_N = \alpha$, and π is an EMVS.

Assume now that $a_\pi \neq b_\pi$. We prove first that the range of π is $[a_\pi, b_\pi]$. Assume to the contrary that there is $c \in A$ such that $\{c\} \in [a_\pi, b_\pi]$, but for each $p_i \in A$, $\pi(p_i) \neq \{c\}$. Let $\succsim_i \in \overline{\mathcal{S}}$ with $p(\succsim_i) = \{c\}$ and assume that $\pi(c) \overline{\leq} \{c\}$. (The symmetric argument applies to the case $\{c\} \overline{\leq} \pi(c)$.) Let $\succsim'_i \in \overline{\mathcal{S}}$ with $p(\succsim'_i) = c$ be such that for each $\alpha, \beta \in \mathcal{A}\backslash\{c\}$, $\{c\} \overline{\leq} \alpha$ and $\beta \overline{\leq} \{c\}$ imply $\alpha \succ'_i \beta$. Since b_π is in the range of π, there is p''_i such that $\pi(p''_i) = b_\pi$. Then, $\pi(p''_i) = b_\pi \succ'_i \pi(c)$, which contradicts *strategy-proofness*. Note that by *strategy-proofness*, for each $\succsim_i \in \overline{\mathcal{S}}$, $\pi(p(\succsim_i)) = \{\alpha \in [a_\pi, b_\pi] \mid \text{for each } \alpha' \in [a_\pi, b_\pi], \alpha \succsim_i \alpha'\}$. Therefore, for each $p_i \in A$, if $\{p_i\} \overline{\leq} a_\pi$, then $\pi(p_i) = a_\pi$. If $\{p_i\} \in [a_\pi, b_\pi]$, then $\pi(p_i) =$

$\{p_i\}$. Finally, if $b_\pi \precsim \{p_i\}$, then $\pi(p_i) = b_\pi$. Therefore, for each $p_i \in A$, $\pi(p_i) = \inf\{b_\pi, \sup\{p_i, a_\pi\}\}$. Just by relabelling $a_{\{\varnothing\}} = b_\pi$ and $a_N = a_\pi$, π is an EMVS.

Induction Basis. If there is a $n \in \mathbb{N}$ such that for each $n' \leq n$, for each society $N = \{1, \ldots, n'\}$ every extended voting scheme $\pi : A^N \to \mathcal{A}$ that satisfies *strategy-proofness* is an EMVS, then for each society $N' = \{1, \ldots, n, n+1\}$, every extended voting scheme $\pi' : A^{N'} \to \mathcal{A}$ that satisfies *strategy-proofness* is an EMVS.

Let $N = \{1, \ldots, n+1\}$ and let $\pi : A^N \to \mathcal{A}$ be an $(n+1)$-voter voting scheme that satisfies *strategy-proofness*. Let $i \in N$ and $p_i \in A$, and define the auxiliary n-voter extended voting scheme $\pi_i : A^{N\setminus\{i\}} \to \mathcal{A}$ in such a way that for each restricted profile of peaks $p_{-i} \in A^{N\setminus\{i\}}$, $\pi_i(p_{-i}) \equiv \pi(p_i, p_{-i})$. Since π satisfies *strategy-proofness*, π_i is a n-voter extended voting scheme that satisfies *strategy-proofness*. By the induction hypothesis, π_i is an n-voter EMVS. Hence, for each $p_{-i} \in A^{N\setminus\{i\}}$,

$$\pi(p_i, p_{-i}) = \pi_i(p_{-i}) = \overline{\inf}_{S \subseteq N\setminus\{i\}} \left\{\overline{\sup}_{j \in S}\{p_j, a_S(p_i)\}\right\}.$$

Let $S_0 \subseteq N \setminus \{i\}$ and consider $\tilde{p}_{-i} \in A^{N\setminus\{i\}}$ such that for each $j \in S_0$, $\tilde{p}_j = a$, and for each $j' \notin S_0$, $\tilde{p}_{j'} = b$. Then, $\pi(p_i, \tilde{p}_{-i}) = \pi_i(\tilde{p}_{-i}) = a_{S_0}(p_i)$.

Let $p'_i \in A$ such that $p'_i \neq p_i$. Let the auxiliary n-voter voting scheme $\pi'_i : A^{N\setminus\{i\}} \to \mathcal{A}$ be such that for each restricted profile of peaks $p_{-i} \in A^{N\setminus\{i\}}$, $\pi_i(p_{-i}) \equiv \pi(p_i, p_{-i})$. By the induction hypothesis, π'_i is an EMVS. Repeating the argument of the previous paragraph, $\pi(p'_i, \tilde{p}_{-i}) = \pi'_i(\tilde{p}_{-i}) = a_{S_0}(p'_i)$.

Since for each $p_i \in A$, $a_{S_0}(p_i) = \pi(p_i, \tilde{p}_{-i})$, we can define $a_{S_0} : A \to \mathcal{A}$ as an 1-voter extended voting scheme. Since π satisfies *strategy-proofness*, a_{S_0} satisfies *strategy-proofness*. By the induction hypothesis, there are two constants, $\alpha_{S_0}, \beta_{S_0}$, with $\alpha_{S_0} \precsim \beta_{S_0}$, such that $a_{S_0}(p_i) = \overline{\inf}\{\beta_{S_0}, \overline{\sup}\{p_i, \alpha_{S_0}\}\}$. Thus,

$$\pi(p_i, p_{-i}) = \pi_i(p_{-i}) = \overline{\inf}_{S_0 \subseteq N\setminus\{i\}} \left\{\overline{\sup}_{j \in S_0} \left\{p_j, \overline{\inf}\{\beta_{S_0}, \overline{\sup}\{p_i, \alpha_{S_0}\}\}\right\}\right\}.$$

For each $S_0 \subseteq N \setminus \{i\}$, let $\bar{p}_{S_0} = \overline{\sup}_{j \in S_0}\{p_j\}$,

$$\begin{aligned}\overline{\sup}_{j \in S_0} \left\{p_j, \overline{\inf}\{\beta_{S_0}, \overline{\sup}\{p_i, \alpha_{S_0}\}\}\right\} &= \overline{\sup}\left\{\bar{p}_{S_0}, \overline{\inf}\{\beta_{S_0}, \overline{\sup}\{p_i, \alpha_{S_0}\}\}\right\} = \\ &= \overline{\inf}\{\overline{\sup}\{\bar{p}_{S_0}, \beta_{S_0}\}, \overline{\sup}\{\bar{p}_{S_0}, p_i, \alpha_{S_0}\}\}.\end{aligned}$$

By relabelling for each $S_0 \subseteq N \setminus \{i\}$, $a_{S_0} = \beta_{S_0}$ and $a_{S_0 \cup \{i\}} = \alpha_{S_0}$, we obtain,

$$\pi(p) = \overline{\inf}_{S_0 \subseteq N\setminus\{i\}} \left\{\overline{\inf}\{\overline{\sup}\{\bar{p}_{S_0}, \beta_{S_0}\}, \overline{\sup}\{\bar{p}_{S_0}, p_i, \alpha_{S_0}\}\}\right\} = \overline{\inf}_{S \subseteq N} \left\{\overline{\sup}_{j \in S}\{p_j, a_S\}\right\}.$$

□

From Theorem 1 and 2, and noting that $\alpha_{\{\varnothing\}} = b$ and $\alpha_N = a$ are necessary and sufficient for an EMVS to satisfy *unanimity*, we obtain our final and main characterization result.

Theorem 3. *A minimally extended social choice function φ satisfies* unanimity *and* strategy-proofness *if and only if φ is an EMVS with $\alpha_{\{\varnothing\}} = b$ and $\alpha_N = a$.*

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