Economics Bulletin

Volume 38, Issue 1

Envy-free allocation of indivisible goods with money and externalities

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Abstract

This paper considers an envy-free allocation of a single indivisible good in the quasi-linear utility environment where a monetary transfer is allowed and externalities among agents exist. We show that an envy-free allocation does not exist if a degree of the externalities is high enough and there are two groups of agents: one group of agents can but the other group of agents cannot enjoy externalities. We also show that both efficient allocations and envy-free allocations are generally incompatible whereas they are always compatible without externalities.

I thank Koichi Tadenuma, Norio Takeoka, Takashi Ui, an associate editor and an anonymous referee for valuable comments. The financial support from Japan Society for Promotion of Science (JSPS) is acknowledged. All remaining errors are my own.
Citation: Satoshi Nakada, (2018) "Envy-free allocation of indivisible goods with money and externalities", *Economics Bulletin*, Volume 38, Issue 1, pages 52-59
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Submitted: April 07, 2017. **Published:** January 21, 2018.

1 Introduction

We consider a fair allocation of a indivisible good under the quasi-linear utility environment with monetary transfers and externalities. Central fairness criterion of such problem is called *envy-freeness* (Foley, 1967; Kolm, 1971). An allocation is envy-free if no one wants to swap any other agent's allocation with his/her own. We show that an envy-free allocation does not exist if a degree of externalities is high enough. Moreover, we show that both efficient allocations and envy-free allocations are incompatible in general.

To understand our motivation, consider the following example. Suppose that a public facility (e.g, a library, a sports center, and a park etc.) is to be located in some city. Each city has or does not have transportation services to each other. If they exist, even people who do not live in the city where the facility is located can also use it. Otherwise, the facility is only used by people who live in the city. In this sense, accessibility to a facility will induce network externalities to users. Another example is an allocation of patent licenses of medical products. If a license is allocated to a firm which is willing to offer an open access or access with low cost, other firms can also use the product's information. As these examples suggest, we have to take externalities into account in some indivisible goods allocation problems.

Without externalities, Svensson (1983) shows that envy-free allocations are always efficient under quasi-linear utility environment. Moreover, envy-free allocations always exist and the set of them is characterized by Fujinaka and Sakai (2009). In contrast, we show that an envy-free allocation does not exist if a degree of the externalities is high enough and there are two groups of agents, that is, one group of agents can but the other group of agents cannot enjoy externalities. An intuition behind this result is based on the following observation. To allocate a good to agents in a fair way, each agent who does not obtain the good must be monetary compensated by the consumer. If there are agents who cannot enjoy externality (i.e, there is no connection with the consumer), their compensation must be higher than that of other agents who can enjoy externality. However, the upper bound for the compensation is limited and depends on the degree of externalities because of the fairness for the consumer: the upper bound is lower if the degree of externalities is high enough. Therefore, if a degree of externalities is high enough, such an agreeable monetary compensation is impossible. We also show that envy-free allocations are not necessarily efficient even if the degree of externalities is low. This trade-off arises because, for envy-free allocations, the highest valued agent must obtain the good, whereas, for efficient allocations, the most influential agent, measured by how he can induce externalities among agents, must obtain the good.

In the literature on fair allocation problem of indivisible goods, although the existence of envy-free allocations and their properties are intensively considered,[1] the case of externalities are not well considered. One exception is Velez (2016), who considers envy-free allocations of multiple indivisible goods with externalities. He mainly considers that externalities are symmetric and each agent's monetary transfer is positive. In contrast, in our paper, externalities are asymmetric depending on the network structure and a positive transfer among all agents is impossible because there is no extra money in the environment. He also considers that a negative transfer is allowed in a special class of payoff function. Especially, he considers that there are externalities from both indivisible goods consumption and monetary transfer. However, we consider that the source of externalities is only the consumption of an indivisible good via network structures.

The rest of this paper is organized as follows. Section 2 introduces the model. Section 3 provides our results. Section 4 is the conclusion.

2 Model

Let $N = \{1, \cdots, n\}$ be the set of agents. In this environment, there are two kinds of goods: an indivisible good and money. We denote by $a_i \in \{0, 1\}$ an agent i's assignment of the indivisible good and by $a = (a_1, \cdots, a_n) \in \{0, 1\}^N$, with $\sum_{i \in N} a_i = 1$, a profile of assignments. Let A be the set of all profiles of assignments. We also denote by $t_i \in \mathbb{R}$ an agent i's monetary transfer and by $t = (t_1, \cdots, t_n) \in \mathbb{R}^N$ a profile of monetary transfers. Let $X = \{(a, t) \in A \times \mathbb{R}^N | a \in A, \sum_{i \in N} t_i = 0\}$ be the set of all feasible allocations. Each agent i has a valuation for the indivisible goods $v_i \in \mathbb{R}_{++}$. Let $V_i \subset \mathbb{R}_{++}$ be the set of i's valuations. We denote by $V = V_1 \times \cdots \times V_n$ the set of agents' valuation profiles. For $v \in V$, let $M(v) = \max_{i \in N} v_i$ be the maximum value in the valuation profile v.

To describe how externalities exist, we introduce a network among agents. Let $g^N = \{ij | i, j \in N, i \neq j\}$ be the set of all possible links. Then, a network g is a symmetric subset of g^N. We denote by $\mathbb{G}^N = \{g | g \subset g^N\}$ the set of all networks. For each network $g \in \mathbb{G}^N$ and player $i \in N$, let $N_i(g) = \{j \in N | ij \in g\}$ be the set of i's neighborhoods in g. For each network $g \in \mathbb{G}^N$ and for each distinct $i, j \in N$, let

$$g_{ij} = \begin{cases} 1 & \text{if } ij \in g, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\delta \in (0, 1)$ be a parameter of the degree of externalities. We call $\mathcal{E} = (v, g, \delta) \in V \times \mathbb{G}^N \times (0, 1)$ an environment.

Each agent i's utility function does not only depend on his/her own allocation but

[1]See, Thomson (1996), Tadenuma (1996) and Tadenuma and Thomson (1991, 1993, 1995).

also his/her neighbor's consumption of indivisible goods because of externalities. For each environment $\mathcal{E}$, let $u_i : X \to \mathbb{R}$ be a utility function defined as a quasi-linear form $u_i(x) = v_i\big(a_i + \delta \sum_{j \neq i} a_j g_{ij}\big) + t_i$.[2]

The central fairness criterion of allocations is called *envy-free* by Foley (1967) and Kolm (1971). We introduce envy-freeness regarding externalities following Velez (2016). To understand the idea, suppose that agent 1 and agent 2 are connected, but agent 3 is isolated. Suppose also that agent 1 has an indivisible good, but agent 1 prefers agent 2's allocation to his own because agent 2's transfer is high enough and he can still enjoy externalities. In this sense, agent 1 may want to "swap" his allocation with that of agent 2. Regarding this point, we say that an allocation is envy-free if no one wants to swap other agent's allocation with his/her own.

Definition 1 (Velez, 2016)**.** *An allocation $x \in X$ is envy-free in $\mathcal{E}$ if for any $i, j \in N$, $u_i(x) \geq u_i(y)$ where $y_i = x_j, y_j = x_i$ and $y_k = x_k$ for any $k \neq i, j$.*

Let $F(\mathcal{E})$ be the set of all envy-free allocations in an environment $\mathcal{E}$.

3 Results

3.1 Existence and non-existence of envy-free allocations

We first show the following characterization of envy-free allocations, which is a generalization of Fujinaka and Sakai (2009).

Lemma 1. *Fix an environment $\mathcal{E}$ and an allocation $x \in X$. Let i^* be the agent such that $a_{i^*} = 1$. Then, $x \in F(\mathcal{E})$ if and only if there is $\tilde{t} \in \mathbb{R}$ such that*

(1) $v_{i^*} = M(v)$,

(2) $\tilde{t} = t_i = t_j$ *for all* $i, j \neq i^*$,

(3)

$$\tilde{t} \in \begin{cases} \left[\frac{\max_{j \neq i^*} v_j}{n}, \frac{M(v)}{n}\right] & \text{if } N_{i^*}(g) = \emptyset, \\ \left[\frac{\max_{j \neq i^*}(1-\delta)v_j}{n}, \frac{(1-\delta)M(v)}{n}\right] & \text{if } N \setminus (N_{i^*}(g) \cup \{i^*\}) = \emptyset, \\ \left[\max\left\{\frac{\max_{j \in N_{i^*}(g)}(1-\delta)v_j}{n}, \frac{\max_{j \notin N_{i^*}(g) \cup \{i^*\}} v_j}{n}\right\}, \frac{(1-\delta)M(v)}{n}\right] & \text{otherwise.} \end{cases}$$

Proof. We only show the necessity part. The sufficiency part can be shown by the same way of necessity.

[2]This kind of utility function, where the effect from network structure is incorporated linearly, is assumed naturally in the context of network games. See, for example, Jackson (2008).

(1): Suppose that $a_{i^*} = 1$. Then, by envy-freeness, for all $j \neq i^*$,

$$v_{i^*} + t_{i^*} \geq v_{i^*}\delta g_{i^*j} + t_j \Leftrightarrow (1 - \delta g_{i^*j})v_{i^*} \geq (t_j - t_{i^*}).$$

Also, by envy-freeness, for all $j \neq i^*$,

$$v_j\delta g_{ji^*} + t_j \geq v_j + t_{i^*} \Leftrightarrow (t_j - t_{i^*}) \geq (1 - \delta g_{i^*j})v_j.$$

Combining these inequalities, we have $(v_{i^*} - v_j)(1 - \delta g_{i^*j}) \geq 0$. Since $\delta \in (0, 1)$, $v_{i^*} \geq v_j$ for all $j \neq i^*$, which means that $v_{i^*} = M(v)$.

(2): This is immediate from the definition of envy-free allocations for $i, j \neq i^*$.

(3): Since the arguments are same, it is enough to show the case of $N_{i^*}(g) \neq \emptyset$ and $N_{i^*}(g) \neq \emptyset$. By (2), let $\tilde{t} = t_j$ for all $j \neq i^*$. Then, by feasibility, $t_{i^*} = -(n-1)\tilde{t}$. For any $j \in N_{i^*}(g)$, by the same argument of (1), we have $(1 - \delta)v_j \leq n\tilde{t} \leq (1 - \delta)M(v)$, which means that

$$\max_{j \in N_{i^*}(g)}(1 - \delta)v_j \leq n\tilde{t} \leq (1 - \delta)M(v).$$

Also, for any $j \notin N_{i^*}(g)$, $v_j \leq n\tilde{t} \leq M(v)$, which means that

$$\max_{j \notin N_{i^*}(g)} v_j \leq n\tilde{t} \leq M(v).$$

Combining these inequalities, we obtain the condition. □

Hereafter, we consider environments with generic valuations, i.e, $v \in V$ such that $|\text{argmax}_{i \in N} v_i| = 1$. Then, for envy-free allocations, we can focus on the allocations such that $a_{i^*} = 1$ for $M(v) = v_{i^*}$ by Lemma 1. The following main result of the paper shows how degree of externalities and network structure affect the existence of envy-free allocations.

Theorem 1. *For each environment $\mathcal{E}$, let $\delta(\mathcal{E}) \equiv 1 - \frac{n\bar{t}}{M(v)} \in (0, 1)$ where $\bar{t} = \max_{j \in N \setminus (N_{i^*}(g) \cup \{i^*\})} \frac{v_j}{n}$. Then, $F(\mathcal{E}) \neq \emptyset$ if and only if one of the following conditions holds:*

(1) $|N_{i^*}(g)| = 0$,

(2) $|N_{i^*}(g)| = n - 1$,

(3) $1 \leq |N_{i^*}(g)| \leq (n-2)$ and $\delta \leq \delta(\mathcal{E})$.

Proof. By Lemma 1, in all cases, it must be that $a_{i^*} = 1$. Then, we want to show that the allocations $x = (a, t)$ with following transfer profiles t constitute envy-free allocations.
(1): By Lemma 1, $F(\mathcal{E})$ must be the set of allocations with following transfer profiles:

$$t_{i^*} = -(n-1)\tilde{t}, t_j = \tilde{t} \text{ for any } j \neq i^* \text{ where } \tilde{t} \in \big[\frac{\max_{j \neq i^*} v_j}{n}, \frac{M(v)}{n}\big].$$

(2): By Lemma 1, $F(\mathcal{E})$ must be the set of allocations with following transfer profiles:

$$t_{i^*} = -(n-1)\tilde{t}, t_j = \tilde{t} \text{ for any } j \neq i^* \text{ where } \tilde{t} \in \big[\frac{\max_{j \neq i^*} (1-\delta) v_j}{n}, \frac{(1-\delta)M(v)}{n}\big].$$

(3): Fix an environment with $1 \leq |N_{i^*}(g)| \leq (n-2)$. Let $\hat{t}(\delta) = \max_{j \in N_{i^*}(g)} \frac{(1-\delta)}{n} v_j$. By Lemma 1, $F(\mathcal{E})$ must be the set of allocations with following transfer profiles:

$$t_{i^*} = -(n-1)\tilde{t}, t_j = \tilde{t} \text{ for all } j \neq i^* \text{ where } \tilde{t} \in \big[\max\{\hat{t}(\delta), \bar{t}\}, \frac{(1-\delta)M(v)}{n}\big].$$

Note that $\hat{t}(\delta) < \frac{(1-\delta)M(v)}{n}$ for any $\delta \in (0,1)$ and $\max\{\hat{t}(\delta), \bar{t}\} = \bar{t}$ for some $\delta \in (0,1)$. Therefore, $F(\mathcal{E}) \neq \emptyset$ if and only if

$$\begin{aligned} & \bar{t} \leq \frac{(1-\delta)M(v)}{n} \\ \Leftrightarrow \quad & \delta \leq \delta(\mathcal{E}). \end{aligned}$$

□

We first clarify the difference between our result and that of Velez (2016). He considers general class of preferences and shows an existence result when externalities are symmetric. Also, he assumes that agent's monetary transfer is positive, i.e, some amount of money is given in his environment. In contrast, in our paper, externalities are asymmetric depending on the network structure and a positive transfer among all agents is impossible because there is no extra money in the environment. He also considers that a negative transfer is allowed in a special class of payoff function, which is different from ours. In particular, he considers that there are externalities from both indivisible goods consumption and monetary transfer. However, we consider that the source of externalities is only the consumption of an indivisible good via network structures.

The first and second parts in Theorem 1 state that an envy-free allocation always exists regardless of externality if i^* is isolated or centered. The reasons behind these parts are that these environments are almost identical to the case of no externality, where the existence of envy-free allocations are guaranteed. In the former case, no agent enjoys externalities, so the environment is virtually identical as the case of no externality, i.e, $\delta = 0$. In the latter case, only difference from the no externality case is the set of feasible transfers. Since every agents can enjoy externalities, in view of i^*, a transfer to other agents must be small. Also, in view of $j \neq i^*$, a small transfer is enough to have no envy to i^*. In particular, as $\delta \to 1$, $\tilde{t} \to 0$, so that only no transfer is feasible.

The third part clarifies how the externality affects the existence of envy-free allocations in the broad class of networks. More precisely, if there are two groups of agents who can and cannot enjoy positive externalities and network externalities are high enough, then there is no envy-free allocation. The intuition is as follows. To allocate a good to agents in a fair way, each agent who does not obtain the good must be monetary compensated by the consumer. If there are agents who cannot enjoy externality (i.e, there is no connection with the consumer), their compensation must be higher than that of other agents who can enjoy externalities. However, the upper bound for the compensation is limited and depends on the degree of externalities because of the fairness for the consumer: the upper bound is lower if the degree of externalities is high enough. Therefore, if a degree of externalities is high enough, such an agreeable monetary compensation is impossible.

We stress that this result is not solely an impossibility result. This result states that an envy-free allocation does not exist if externalities are strong enough. In contrast, even if there are weak externalities, we can guarantee the existence of an envy-free allocation. Hence, in such cases, the usual results in the literature can be applied. We will illustrate this trade-off by an example later.

3.2 Efficient allocations vs envy-free allocations

Definition 2. *An allocation $x \in X$ is Pareto efficient in $\mathcal{E}$ if there is no $y \in X$ such that (1) $u_i(y) \geq u_i(x)$ for any $i \in N$ and (2) $u_i(y) > u_i(x)$ for some $i \in N$.*

Let $P(\mathcal{E})$ be the set of all Pareto-efficient allocations for an environment $\mathcal{E}$. Since each agent's utility function is quasi-linear, efficient allocations are characterized by the budget balance $\sum_{i \in N} t_i = 0$ and the solution of following maximization problem:

$$\max_{a \in A} \quad \sum_{i \in N} v_i\big(a_i + \delta \sum_{j \neq i} a_j g_{ij}\big).$$

Since the objective function can be rewritten by $\sum_{i \in N}(v_i + \delta \sum_{j \neq i} v_j g_{ij}) a_i$ and $\sum_{i \in N} a_i = 1, a_i \in \{0, 1\}$ for all $i \in N$, we must have $a_{i^{**}} = 1$ if and only if $i^{**} \in \text{argmax}_{i \in N}(v_i + \delta \sum_{j \neq i} v_j g_{ij})$.

Therefore, efficient allocations are incompatible with envy-free allocations in general if there are externalities among agents.

Proposition 1. *For each $\mathcal{E}$, $F(\mathcal{E}) \cap P(\mathcal{E}) \neq \emptyset$ only if* $\text{argmax}_{i \in N} v_i \cap \text{argmax}_{i \in N}(v_i + \delta \sum_{j \neq i} v_j g_{ij}) \neq \emptyset$.

The special case is $\delta = 0$, which corresponds to no-externality environments.

3.3 An illustrative example

To illustrate our results, consider the following example. Let $N = \{1, \cdots, 5\}$, $v = (10, 2, 1, 4, 5)$ and $g = \{15, 25, 35, 45\}$. A network g corresponds to a core-periphery network where agent 5 is in a center. A graphical representation of the network is given in Figure 1. If $a_1 = 1$, the social surplus is $10 + 5\delta$. On the other hand, if $a_5 = 1$, the social surplus is $5 + 17\delta$. Therefore, $F(\mathcal{E}) \cap P(\mathcal{E}) \neq \emptyset$ only if $10 + 5\delta \geq 5 + 17\delta \Leftrightarrow \delta \leq 5/12$. For envy-free allocations, we must have $a_1 = 1, a_2 = \cdots a_5 = 0$ and $t_1 = -4\tilde{t}, t_2 = \cdots t_5 = \tilde{t}$ where $\max\{\frac{v_2}{5}, \frac{v_3}{5}, \frac{v_4}{5}, \frac{(1-\delta)}{5} v_5\} \leq \tilde{t} \leq \frac{(1-\delta)}{5} v_1 \Leftrightarrow \max\{4/5, 1-\delta\} \leq \tilde{t} \leq 2(1-\delta)$. Hence, if $\delta > \delta(\mathcal{E}) = 3/5$, $F(\mathcal{E}) = \emptyset$ because such $\tilde{t}$ does not exist. The feasible region of $\tilde{t}$ for each δ under envy-free allocations is described in Figure 2. The region surrounded by colored lines is the existence area.

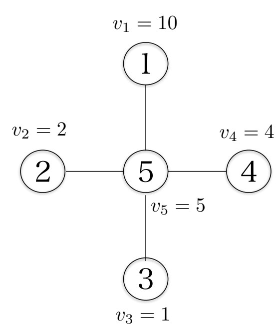

Figure 1: Network structure in $\mathcal{E}$.

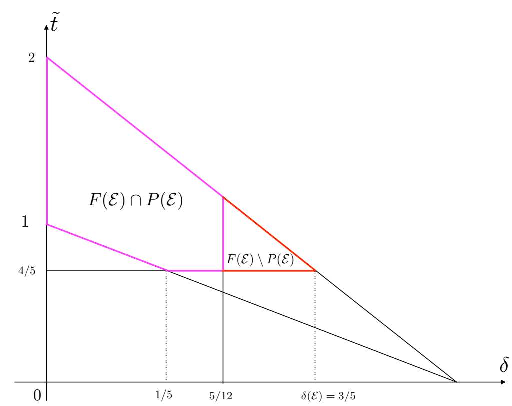

Figure 2: Feasible transfers under no-envy allocations.

4 Concluding remarks

We consider how externality among agents affects the existence of envy-free allocations. Our result shows that, if externality is high enough, envy-free allocation does not exist for broad class of networks, but they still exist otherwise.

We only consider the case of positive externalities, but our model can be rewritten for the case of negative externalities by considering the case $V \subset \mathbb{R}_{--}$. Negative network externalities will be relevant in the NIMBY problem studied by Sakai (2012), especially in the case of severe externalities like a construction of a nuclear plant.

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