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Extending the state-space representation of the judgement-augmented Hodrick-Prescott filter

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Abstract

Introducing judgement, or restrictions, in the analytical form of the Hodrick-Prescott (HP) filter involves setting a parameter for how tightly the restrictions should hold. The current paper suggests an interpretation of this parameter and suggests a way to extend the state-space form of the judgement-augmented filter to include a corresponding parameter. The paper thereby bridges the remaining gap between the analytical form and the state-space form of the restricted HP filter.

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1 Introduction

The filter of Hodrick and Prescott (1997) is extensively used across the field of economics as a tool for decomposing an aggregate series into a trend and a cyclical component. When using the standard HP filter, a smoothing parameter has to be set. The interpretation of this parameter can be that of the relative variance of the cyclical component and the trend innovations.

When introducing judgement, or restrictions, in the HP filter additional information can be brought into the filter, and thereby contribute to better identification of the components of an aggregate time series. The judgement-augmented filter suggested by Jönsson (2010) introduces judgement regarding the components in the analytical computation of the filter. This is done by adding a weighted sum of squared deviations between the imposed restrictions and the extracted components to the loss function that is minimized in the computation of the components. The higher the weight on the sum of squared deviations, the closer the extracted components will be to the judgement imposed.

Recently, Jönsson (2017) showed how the restrictions in the analytical form of the filter can be imposed in the state-space representation of the HP filter under the assumption that the restrictions are imposed to hold with equality. This is tantamount to letting the weight on the sum of squared deviations of the estimated component from the imposed restrictions in the analytical form of the filter tend to infinity.

The current paper makes two contributions to the previous results on the judgement-augmented, or restricted, HP filter. First, it is shown how to impose restrictions in the state-space framework without the restrictions having to hold with equality in the estimated components. More specifically, it is shown how uncertainty regarding the judgement, which translates to allowing a certain degree of deviation between the imposed restrictions and the extracted components, can be allowed for in the state-space framework. The second contribution of the paper is that it provides an illustrative interpretation of the weight that is attached to the sum of squared deviations of the estimated components from the imposed restrictions in the analytical formulation of the filter.

The rest of the paper is organized as follows. In Section 2, an interpretation of the judgement-related parameter of the analytical form of the restricted HP filter in Jönsson (2010) is suggested. In order to see if this interpretation is viable, Section 3 first extends the results of Jönsson (2017) to allow for uncertainty regarding the imposed restrictions in the state-space form of the restricted HP filter. After that, Section 3 gives a comparison of the parameters of the different forms for the restricted filter and explore to what extent their suggested interpretations support each other. Finally, Section 4 concludes the paper.

2 Analytical forms for the HP filter

The HP filter decomposes an aggregate time series, y_t for $t \in \{1, \ldots, T\}$, additively into a trend component, τ_t, and a cycle component, c_t, as in (1).

$$y_t = \tau_t + c_t \tag{1}$$

The trend component is calculated by minimizing the loss function, L_{HP}, in (2) (see e.g. Hodrick and Prescott, 1997).

$$L_{HP} = \sum_{t=1}^{T} (y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2 \tag{2}$$

Hodrick and Prescott (1997) note that if the cyclical component and the second difference of the trend component were both iid normal with mean zero and variances σ_c^2 and σ_τ^2, then the expectation of τ_t conditional on the observations, y_t, would be obtained by minimizing (2) while letting $\lambda = \sigma_c^2/\sigma_\tau^2$. Hence, under some specific assumptions, λ could be interpreted as the relative variance of the components' innovations. This feature of the smoothing parameter, and also discussions regarding how to set the parameter, have been treated by e.g. Ravn and Uhlig (2002) and Hamilton (2017). An intuition for interpreting λ as a relative variance is that if the cyclical component of a series is assumed to be more volatile relative to the second difference of the trend component, then the second difference of the trend component should be emphasized more, and obtain a higher weight, in the loss function. This will yield an extracted trend component that would exhibit less variation in its growth, i.e. becoming closer to a linear trend. Correspondingly, if the variation in the growth of the trend component is assumed to larger relative to the variation in the cyclical components, then the term with cyclical component should receive a relatively high weight in the loss function, yielding an extracted trend that is closer to the observed series. The reasoning behind this interpretation of λ can be illustrated more clearly if one disregards the normalization on the first term of the loss function in (2) and instead weight each of the terms in the loss function in (2) above with the inverse of the variances of the respective components as in (3).

$$L_{HP} = \frac{1}{\sigma_c^2} \sum_{t=1}^{T} (y_t - \tau_t)^2 + \frac{1}{\sigma_\tau^2} \sum_{t=2}^{T-1} [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2 \tag{3}$$

Turning to the judgement-augmented, or restricted HP filter, Jönsson (2010) imposes restrictions by minimizing the loss function L_{RHP} in (4).

$$L_{RHP} = \sum_{t=1}^{T} (y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2 + \gamma \sum_{t \in \iota} (c_t - \tilde{c}_t)^2 \tag{4}$$

In (4), the first two terms are the same as in the original HP filter. In the third term, γ is a weighting parameter, ι is an index set that contains the periods for which restrictions are imposed and $\tilde{c}_t$ are the imposed values for the cyclical components for $t \in \iota$.[1] Hence, the third term in (4) penalizes the loss function by adding the weighted squared deviations between the extracted cyclical component and the restrictions imposed on the cyclical components. A larger value for γ would imply that the third term in (4) would receive a higher weight, leading to an extracted cyclical component that would respect the imposed judgement to a higher degree. Similarly, a lower value for γ would allow for larger deviations between the extracted cyclical components and the imposed judgement.

Under the additional assumption that $c_t - \tilde{c}_t$ is a iid normal with zero mean and variance σ_r^2, looking at the interpretation of λ above, a tentative interpretation of γ would be that it indicates the variability of the cyclical component relative to the variability in, or uncertainty of, the imposed judgement, i.e. $\gamma = \sigma_c^2/\sigma_r^2$. Under this interpretation, a high value of γ would indicate more variability in the cyclical component compared to the uncertainty of the judgement. The high value would stress the third term in more in the loss function L_{RHP} and hence extract a cyclical component that would be closer to the imposed judgement. Similarly, a smaller value for γ would indicate relatively more uncertainty regarding the judgement and would place less emphasis on the third term of L_{RHP}. As a consequence, the restrictions imposed would have less influence on the cyclical components that are extracted by the judgement-augmented filter.

Under the tentative interpretation that $\gamma = \sigma_c^2/\sigma_r^2$ one could rewrite the loss function L_{RHP}, in a way analogous to (3), using the inverses of σ_c^2, σ_τ^2 and σ_r^2 as weights, to obtain the sum in (5).

$$L_{RHP} = \frac{1}{\sigma_c^2}\sum_{t=1}^{T}(y_t - \tau_t)^2 + \frac{1}{\sigma_\tau^2}\sum_{t=2}^{T-1}\left[(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})\right]^2 + \frac{1}{\sigma_r^2}\sum_{t\in\iota}(c_t - \tilde{c}_t)^2 \tag{5}$$

The state-space framework of Jönsson (2017) imposes restrictions without allowing deviations, i.e. there is no uncertainty regarding the imposed judgement. This means implicitly letting $\gamma \to \infty$, which for a fixed σ_c^2 would imply $\sigma_r^2 \to 0$. A key question now becomes how to extend this state-space framework to allow for a situation that corresponds to setting arbitrary values for γ in the analytical framework. It also becomes interesting to see to what degree such an extension to the state-space formulation supports the interpretation $\gamma = \frac{\sigma_c^2}{\sigma_r^2}$.

3 State-space forms for the HP filter

Besides using the analytical formulas for finding the minimum of the loss functions in the previous section, it is possible to extract the HP components by setting up a state-space model. This has been discussed e.g. in Harvey and Jaeger (1993) and more recently in Grant and Chan

[1] Imposing restrictions on the cyclical component also amounts to setting restrictions on the trend component. Hence, restrictions on the trend can be imposed implicitly through the cyclical component. Also, extensions to general forms of linear restrictions can be considered, see e.g. Julio (2011).

(2017), while Gómez (1999) shows formally that the minimization of the loss function and the state-space approaches to HP filtering are equivalent. One possible state-space formulation of the HP filter is given in (6) and (7) below.

$$Y_t = H_t \alpha_t + \eta_t \tag{6}$$

$$\alpha_t = F \alpha_{t-1} + \varepsilon_t \tag{7}$$

In order to obtain the HP filter in this state-space system, the system matrices have to be specified in such a way that the state vector contains a trend component that is integrated of order two, with iid disturbances driving the trend, and a cyclical component that is iid. One way to obtain this is to set $\eta_t = 0$ for all t, let ε_t be iid with $E(\varepsilon_t) = 0$ and $Var(\varepsilon_t) = Q$ and set the system matrices of the state-space model as in (8).

$$H_t = H = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_\tau^2 & 0 \\ 0 & 0 & \sigma_c^2 \end{bmatrix} \tag{8}$$

From the matrices in (8), it can be seen that the observation equation adds the first and the third state-vector components. Letting $Y_t = y_t$, these two components should add up to the observed series. Looking closer at the state vector, it can be noted that the second state variable is an cumulative sum of iid innovations which have variance σ_τ^2. The first state variable is a cumulative sum with the innovation being the second state variable. Hence, the first state variable is a component for which the second difference is an iid innovation. Finally, the third component of the state vector is an iid disturbance with variance σ_c^2. Normalizing the variances on σ_τ^2 and letting $\lambda = \sigma_c^2/\sigma_\tau^2$, as in (9), gives a formulation of the state-space model that supports the intuition of the loss-function-form of the HP filter.

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma_c^2/\sigma_\tau^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix} \tag{9}$$

The filter's state-space formulation hence supports the intuition and interpretation from the analytical formulation of the HP filter. Using standard filter and smoothing techniques will, up to differences associated with rounding errors in the numerical algorithms, render the same HP filter components as the analytical HP filter approach.

In the restricted HP-filtering framework, Jönsson (2017) suggests a way to incorporate restrictions in the state-space framework for the case when $\gamma \to \infty$. This is done by introducing time-variation in some of the state-space system matrices as in (10).

$$\begin{aligned} Y_t &= \begin{bmatrix} y_t \end{bmatrix} \quad H_t = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \quad \text{for } t \notin \iota \\ Y_t &= \begin{bmatrix} y_t & \tilde{c}_t \end{bmatrix} \quad H_t = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for } t \in \iota \end{aligned} \tag{10}$$

By now forcing the third state variable to take on the values $\tilde{c}_t$ for $t \in \iota$, restrictions are imposed with equality in the state-space framework. But from the intuition of the analytical form of the restricted HP filter, one interpretation of γ could be that of the uncertainty in the judgement imposed in the HP filter. Incorporating such uncertainty in the state-space form of the restricted HP filter would bridge the remaining gap between the state-space and analytical forms of the HP filter and would allow for a state-space approach for restricted HP filtering for arbitrary values of γ.

From the state-space system in (6) and (7), it can be seen that one way of introducing uncertainty regarding the judgement would be to reintroduce η_t in the restricted state-space formulation and let it be iid with $E(\eta_t) = 0$ and $Var(\eta_t) = R_t$, where R_t is given by (11).

$$\begin{aligned} R_t &= \begin{bmatrix} 0 \end{bmatrix} \quad \text{for } t \notin \iota \\ R_t &= \begin{bmatrix} 0 & 0 \\ 0 & \sigma_r^2 \end{bmatrix} \quad \text{for } t \in \iota \end{aligned} \tag{11}$$

Normalizing also σ_r^2, i.e. normalizing also the variance that is associated with the judgement restrictions, with the variance of the trend component, as in (9), would yield an R_t matrix as in (12) below.

$$R_t = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_r^2/\sigma_\tau^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \delta \end{bmatrix} \quad \text{for } t \in \iota \tag{12}$$

This modification of the state-space system would allow for a degree of uncertainty in the judgement placed on the extracted components. Hence, the extension would bridge the remaining gap between the analytical and the state-space forms when it comes to enforcing judgement, or restrictions, in the extracted components.

Based on the extended state-space framework and the normalization $\delta = \sigma_r^2/\sigma_\tau^2$, it becomes interesting to study to what extent the suggested interpretation of γ in (4) is supported by the parameters in the state-space formulation of the restricted HP filter. In order for this suggested interpretation, i.e. $\gamma = \sigma_c^2/\sigma_r^2$, to be supported it must follow that the variance of the cyclical component relative to the uncertainty regarding the imposed restriction, i.e. λ/δ, should be equal to γ. Based on (9) and (12), one can get (13).

$$\frac{\lambda}{\delta} = \frac{\sigma_c^2/\sigma_\tau^2}{\sigma_r^2/\sigma_\tau^2} = \frac{\sigma_c^2}{\sigma_\tau^2} \cdot \frac{\sigma_\tau^2}{\sigma_r^2} = \frac{\sigma_c^2}{\sigma_r^2} = \gamma \tag{13}$$

From the expression in (13) it becomes evident that the state-space formulation of the restricted HP filter supports the suggested interpretation of γ as the variance of the cyclical component relative to the uncertainty of the imposed judgement.

4 Concluding remarks

The current paper extends previous results on judgement-augmented, or restricted, Hodrick-Prescott filtering to a situation where a state-space formulation of the HP filter allows for deviations between the imposed judgement and the extracted components. Furthermore, the paper suggests an interpretation of a central parameter in the analytical version of the judgement-augmented HP filter. This interpretation is supported by the provided state-space extension. The results of the paper hence close the remaining gap between the analytical form and the state-space form of the judgement-augmented HP filter.

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