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The economics of production planning

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Abstract

This note scrutinizes the role of share parameters in CES production functions. It shows that a firm, when planning production or selecting a production process, aims at maximizing one share parameter at the expense of another in a CES environment. Interior solutions can exist only if factor prices are uncertain to begin with.

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1. Introduction

Production theory relies to a large extent on the constant elasticity of substitution (CES) production function. This production function received its first attention when used in a growth context by Solow (1956), and was developed by Arrow *et al* (1961). A helpful feature of this functional form is that it includes the Cobb-Douglas form that assumes a unity elasticity of substitution. The CES production function can also easily accommodate technological progress, whether in the form of factor-augmenting technological progress or raising the total factor productivity. It is thus fair to say that the CES production function has become a major workhorse in both theoretical and empirical economics.

The specification of the CES function features a share (or distribution) parameter that is taken as given. This parameter specifies how important a certain factor is within the production process. What happens if a firm can alter this share parameter? In particular, what happens if an increase in one share parameter has the cost of reducing the share parameter of another factor?

The purpose of this note is to scrutinize this question: suppose that a team of engineers has to plan a production process, starting from a natural well-known plan with given share parameters. We call this process production planning, and we ask whether the production planner would like to change the plan and to what extent. This problem is equivalent to the problem of technology choice in which the firm has to select a production function out of a range of possible CES functions. The surprising result is that this problem has no interior solution if factor costs are given: ignoring production planning costs, the production planner will always want to maximize one share parameter at the expense of the other. An interior solution is possible only if at least one factor price is uncertain when planning the production process.

2. Production plans

We consider a two-stage process: in the first stage, the production planner sets the share parameters, in the second stage the management aims at minimizing the unit cost. We assume that the technology is such that the production planner's only freedom is to choose the share parameterα for the firm's production function, which must belong to the set of CES production functions that take the form:

$$q = A\left[\alpha x_1^{\gamma} + (1-\alpha)x_2^{\gamma}\right]^{h/\gamma} \text{ where } h > 0,\ \gamma \in (-\infty, 1) \text{ and } \gamma \neq 0. \tag{1}$$

Production according to production function (1) uses two inputs x_1 and x_2. The degree of homogeneity is given by $h > 0$, and the elasticity of substitution σ is related to γ by $\gamma = (\sigma - 1)/\sigma$. As Arrow *et al.* (1961) show, the production function (1) can be regarded as a transformation of the production function

$$q = [\beta_1 x_1^\gamma + \beta_2 x_2^\gamma]^{h/\gamma}\,, \tag{2}$$

where $\beta_i > 0$ denotes the weight production factor i has in production. Production function (1) follows from production function (2) by normalizing such that

$$\beta_1 + \beta_2 = A^{\gamma/h} \text{ and } \beta_1 = \alpha A^{\gamma/h} \tag{3}$$

holds. From (3), we observe that

$$\alpha = \frac{\beta_1}{\beta_1 + \beta_2},$$

which proves that variations in the share parameter α, leaving $A = (\beta_1 + \beta_2)^{-\gamma/h}$ constant, are equivalent to a change in the relative weights of the production factors, keeping $\beta_1 + \beta_2$ constant. We set $A = 1$ for simplicity. The objective of the production planner is to find the optimal share parameter α. In doing so, the production planner faces a natural $\bar{\alpha}$, and we ask whether and how she would like to change it. We suppose that the set of feasible changes is restricted, such that $\alpha \geq \varepsilon$ and $(1-\alpha) \geq \varepsilon$ must holds, where ε is a given small positive number.

Solving backwards, let w_i denote the price of factor i. Consider the problem of finding the least-cost way of producing one unit of output subject to the constraint $[\alpha(x_1)^\gamma + (1-\alpha)(x_2)^\gamma]^{h/\gamma} = 1$. This constraint is equivalent to $\alpha(x_1)^\gamma + (1-\alpha)(x_2)^\gamma = 1$. Using standard techniques, optimization implies that the cost-minimizing factor demands, denoted by x_i^* , are given by

$$\begin{aligned} w_1 x_1^* &= (\alpha)^{\frac{1}{1-\gamma}} (w_1)^{\frac{\gamma}{\gamma-1}} \left[\alpha^{\frac{1}{1-\gamma}} (w_1)^{\frac{\gamma}{\gamma-1}} + (1-\alpha)^{\frac{1}{1-\gamma}} (w_2)^{\frac{\gamma}{\gamma-1}}\right]^{-\frac{1}{\gamma}} \text{ and} \\ w_2 x_2^* &= (1-\alpha)^{\frac{1}{1-\gamma}} (w_2)^{\frac{\gamma}{\gamma-1}} \left[\alpha^{\frac{1}{1-\gamma}} (w_1)^{\frac{\gamma}{\gamma-1}} + (1-\alpha)^{\frac{1}{1-\gamma}} (w_2)^{\frac{\gamma}{\gamma-1}}\right]^{-\frac{1}{\gamma}}. \end{aligned}$$

Consequently, the unit cost function is given by

$$w_1 x_1^* + w_2 x_2^* = \left[\alpha^{\frac{1}{1-\gamma}} (w_1)^{\frac{\gamma}{\gamma-1}} + (1-\alpha)^{\frac{1}{1-\gamma}} (w_2)^{\frac{\gamma}{\gamma-1}}\right]^{\frac{\gamma-1}{\gamma}} = c(w_1, w_2, \alpha).$$

We find:

Proposition 1. *Cost minimization with respect to α leads to corner solutions.*

Proof. See Appendix A.1 □

Proposition 1 does not imply that no critical point exists. But the critical point is given by

$$\frac{\alpha}{1-\alpha} = \frac{w_1}{w_2},$$

and cannot qualify for a cost minimum, but only for a cost maximum, as you will want to put more weight on input 1 with an increase in w_1 only in case of a cost maximum. The CES specification obviously leads to an incentive to specialize on one input as much as you can: while there is substitution in terms of factor inputs for given share parameters and factor prices, the CES specification does not allow for substitution of share parameters for given factor prices.

We now show that uncertainty is the key for a production plan that implies diversification. We present this result by assuming that one out of the two factor prices is not known at the stage of production planning. In particular, suppose that w_2 is known with certainty, but w_1 is random: it can take one of two possible values, w_L and w_H where $w_L < w_H$, with probabilities ρ and $1-\rho$, respectively. The objective is to minimize, with respect to α, the expected cost of producing one unit of output:

$$\min_{\alpha} E\left[\alpha^{\frac{1}{1-\gamma}}(w_1)^{\frac{\gamma}{\gamma-1}} + (1-\alpha)^{\frac{1}{1-\gamma}}(w_2)^{\frac{\gamma}{\gamma-1}}\right]^{\frac{\gamma-1}{\gamma}}$$

subject to $\alpha \geq \varepsilon$ and $1-\alpha \geq \varepsilon$. We find:

Proposition 2. *An interior solution for the cost minimization problem is possible if*

$$w_L < \frac{1-\alpha}{\alpha} w_2 < w_H.$$

Proof. See Appendix A.2 □

Not surprisingly, uncertainty is the key for understanding diversification, but here it applies to factor shares and the planning of the production process. If the factor price that is not subject to uncertainty is in between the low and the high realization of the uncertain factor price, and interior solution may exist. However, Proposition 2 gives only necessary conditions for an interior solution and does not imply that the second-order conditions are fulfilled. Appendix A.3 shows that these results also carry over to Cobb-Douglas production functions, but for the Cobb-Douglas case, the necessary conditions are also sufficient.

3. Concluding remarks

This note is the first to endogenize the role of share parameters in CES production functions. It has shown that if factor prices are not subject to variability, a firm will always wish to maximize one share parameter at the expense of the other. If the share parameters were completely flexible, the firm would prefer to use one factor of production only. These results extend easily to more than one factor, leaving only one factor of production. However, we could also show that uncertain factor prices may imply an interior solution.

When planning production or selecting a production technology, the CES production function makes sense if future factor prices are uncertain to begin with.

References

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Appendix

A.1 Proof of Proposition 1

We have to distinguish two cases: (i) high substitutability, that is, $\sigma > 1 \Leftrightarrow 0 < \gamma < 1$, and (ii) low substitutability, that is, $\sigma < 1 \Leftrightarrow \gamma < 0$. In the case of high substitutability, minimizing $w_1 x_1^* + w_2 x_2^*$ with respect to α is equivalent to *maximizing*, with respect to α, the term

$$\alpha^{\frac{1}{1-\gamma}} (w_1)^{\frac{\gamma}{\gamma-1}} + (1-\alpha)^{\frac{1}{1-\gamma}} (w_2)^{\frac{\gamma}{\gamma-1}}$$

subject to $\alpha \geq \varepsilon$ and $(1-\alpha) \geq \varepsilon$ because the exponent $(\gamma-1)/\gamma < 0$ in this case. Let $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ be the respective Lagrange multipliers, so the first order condition is given by

$$\frac{1}{1-\gamma}\alpha^{\frac{\gamma}{1-\gamma}} (w_1)^{\frac{\gamma}{\gamma-1}} - \frac{1}{1-\gamma}(1-\alpha)^{\frac{\gamma}{1-\gamma}} (w_2)^{\frac{\gamma}{\gamma-1}} + \lambda_1 - \lambda_2 = 0.$$

At an interior solution, $\lambda_1 = \lambda_2 = 0$, and the second derivative is given by

$$\left(\frac{1}{1-\gamma}\right)\frac{\gamma}{1-\gamma}\alpha^{-\frac{1}{1-\gamma}} (w_1)^{\frac{\gamma}{\gamma-1}} + \left(\frac{1}{1-\gamma}\right)\frac{\gamma}{1-\gamma}(1-\alpha)^{-\frac{1}{1-\gamma}} (w_2)^{\frac{\gamma}{\gamma-1}} > 0.$$

Thus any interior solution does not satisfy the second-order condition.

In case of low substitutability, minimizing $w_1 x_1^* + w_2 x_2^*$ with respect to α is the same as maximizing, with respect to α, the term

$$-\alpha^{\frac{1}{1-\gamma}} (w_1)^{\frac{\gamma}{\gamma-1}} - (1-\alpha)^{\frac{1}{1-\gamma}} (w_2)^{\frac{\gamma}{\gamma-1}}$$

subject to $\alpha \geq \varepsilon$ and $(1-\alpha) \geq \varepsilon$. Let $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ again denote the Lagrange multipliers. The first-order condition reads:

$$-\left(\frac{1}{1-\gamma}\right)\alpha^{\frac{\gamma}{1-\gamma}}(w_1)^{\frac{\gamma}{\gamma-1}}+\left(\frac{1}{1-\gamma}\right)(1-\alpha)^{\frac{\gamma}{1-\gamma}}(w_2)^{\frac{\gamma}{\gamma-1}}+\lambda_1-\lambda_2=0.$$

At an interior solution, $\lambda_1=\lambda_2=0$ and the second derivative is given by

$$-\gamma\left(\frac{1}{1-\gamma}\right)^2\alpha^{\frac{\gamma}{1-\gamma}-1}(w_1)^{\frac{\gamma}{\gamma-1}}-\gamma\left(\frac{1}{1-\gamma}\right)^2(1-\alpha)^{\frac{\gamma}{1-\gamma}-1}(w_2)^{\frac{\gamma}{\gamma-1}}.$$

This is positive (because here $\gamma<0$), and thus any interior solution does not satisfy the second-order condition.

A.2 Proof of Proposition 2

The problem is equivalent to choosing α to maximize

$$\begin{aligned}&-\rho\left[\alpha^{\frac{1}{1-\gamma}}(w_L)^{\frac{\gamma}{\gamma-1}}+(1-\alpha)^{\frac{1}{1-\gamma}}(w_2)^{\frac{\gamma}{\gamma-1}}\right]^{\frac{\gamma-1}{\gamma}}\\&-(1-\rho)\left[\alpha^{\frac{1}{1-\gamma}}(w_H)^{\frac{\gamma}{\gamma-1}}+(1-\alpha)^{\frac{1}{1-\gamma}}(w_2)^{\frac{\gamma}{\gamma-1}}\right]^{\frac{\gamma-1}{\gamma}}\end{aligned}$$

subject to $\alpha\geq\varepsilon$ and $1-\alpha\geq\varepsilon$. Let $\mu_1\geq0$ and $\mu_2\geq0$ be the associated Lagrange multipliers. The first-order condition is given by

$$\frac{\rho}{\gamma}L^{\frac{-1}{\gamma}}\left[\left(\frac{\alpha}{w_L}\right)^{\frac{\gamma}{1-\gamma}}-\left(\frac{1-\alpha}{w_2}\right)^{\frac{\gamma}{1-\gamma}}\right]+\frac{1-\rho}{\gamma}H^{\frac{-1}{\gamma}}\left[\left(\frac{\alpha}{w_H}\right)^{\frac{\gamma}{1-\gamma}}-\left(\frac{1-\alpha}{w_2}\right)^{\frac{\gamma}{1-\gamma}}\right]+\mu_1-\mu_2=0,$$

where

$$\begin{aligned}L&\equiv\alpha^{\frac{1}{1-\gamma}}(w_L)^{\frac{\gamma}{\gamma-1}}+(1-\alpha)^{\frac{1}{1-\gamma}}(w_2)^{\frac{\gamma}{\gamma-1}},\\H&\equiv\alpha^{\frac{1}{1-\gamma}}(w_H)^{\frac{\gamma}{\gamma-1}}+(1-\alpha)^{\frac{1}{1-\gamma}}(w_2)^{\frac{\gamma}{\gamma-1}}.\end{aligned}$$

Then an interior solution is possible if $\mu_1=\mu_2=0$ so that

$$\frac{\rho}{\gamma}L^{\frac{-1}{\gamma}}\left[\left(\frac{\alpha}{w_L}\right)^{\frac{\gamma}{1-\gamma}}-\left(\frac{1-\alpha}{w_2}\right)^{\frac{\gamma}{1-\gamma}}\right]+\frac{1-\rho}{\gamma}H^{\frac{-1}{\gamma}}\left[\left(\frac{\alpha}{w_H}\right)^{\frac{\gamma}{1-\gamma}}-\left(\frac{1-\alpha}{w_2}\right)^{\frac{\gamma}{1-\gamma}}\right]=0,$$

Consequently, both terms can add up to zero if

$$\left(\frac{\alpha}{w_L}\right)^{\frac{\gamma}{1-\gamma}} - \left(\frac{1-\alpha}{w_2}\right)^{\frac{\gamma}{1-\gamma}} \tag{A.1}$$

has the opposite sign of

$$\left(\frac{\alpha}{w_H}\right)^{\frac{\gamma}{1-\gamma}} - \left(\frac{1-\alpha}{w_2}\right)^{\frac{\gamma}{1-\gamma}}. \tag{A.2}$$

Consider the case where $0 < \gamma < 1$. If (A.1) is positive and (A.2) is negative, we find that $w_L < (1-\alpha)w_2/\alpha$ and $w_H > (1-\alpha)w_2/\alpha$, leading to Proposition 2. The opposite case, that is, a negative (A.1) and a positive (A.2) is impossible as it implied $w_L > w_H$. Using $\sigma = 1/(1-\gamma) > 0$, the second order condition requires

$$\begin{aligned}
&-\frac{\rho}{\gamma^2}L^{-\frac{1+\gamma}{\gamma}}\sigma\left[\left(\frac{\alpha}{w_L}\right)^{\frac{\gamma}{1-\gamma}} - \left(\frac{1-\alpha}{w_2}\right)^{\frac{\gamma}{1-\gamma}}\right]^2 \\
&-\frac{1-\rho}{\gamma^2}H^{-\frac{1+\gamma}{\gamma}}\sigma\left[\left(\frac{\alpha}{w_H}\right)^{\frac{\gamma}{1-\gamma}} - \left(\frac{1-\alpha}{w_2}\right)^{\frac{\gamma}{1-\gamma}}\right]^2 \\
&+\rho L^{\frac{-1}{\gamma}}\sigma\left[\frac{1}{\alpha}\left(\frac{\alpha}{w_L}\right)^{\frac{\gamma}{1-\gamma}} + \frac{1}{1-\alpha}\left(\frac{1-\alpha}{w_2}\right)^{\frac{\gamma}{1-\gamma}}\right] \\
&+(1-\rho)\,H^{\frac{-1}{\gamma}}\sigma\left[\frac{1}{\alpha}\left(\frac{\alpha}{w_H}\right)^{\frac{\gamma}{1-\gamma}} + \frac{1}{1-\alpha}\left(\frac{1-\alpha}{w_2}\right)^{\frac{\gamma}{1-\gamma}}\right] \leq 0
\end{aligned}$$

which may or may not be satisfied, because the first two terms are negative, but the last two terms are positive.

Consider next the case where $\gamma < 0$ so that $\gamma/(1-\gamma) < 0$. If (A.1) is negative and (A.2) is positive, we find that $w_L < (1-\alpha)w_2/\alpha$ and $w_H > (1-\alpha)w_2/\alpha$, leading to Proposition 2. The opposite case, that is, a positive (A.1) and a negative (A.2) is impossible as it implied $w_L > w_H$. Again, the second order condition may or may not be satisfied, because the first two terms of the second derivative are negative, but the last two terms are positive.

A.3 Cobb-Douglas production function

A firm uses two inputs, x and y, to produce an output q. The production function is of the form

$$q = x^\alpha y^\beta.$$

Suppose the firm can choose α and β subject to the constraints that (i) $\alpha + \beta = \gamma$ (where γ is an exogenously given constant, $\gamma \leq 1$), (ii) α must lie inside an interval $[\underline{\alpha}, \overline{\alpha}]$, where $0 < \underline{\alpha} < \overline{\alpha} < \gamma$ and (iii) β must lie inside an interval $\left[\underline{\beta}, \overline{\beta}\right]$, where $0 < \underline{\beta} < \overline{\beta} < \gamma$. Since

$\alpha+\beta=\gamma$, it is natural to assume that $\underline{\beta}=\gamma-\overline{\alpha}$ and $\overline{\beta}=\gamma-\underline{\alpha}$, so that when β is equal to its maximum permissible value, then α is equal to its minimum permissible value (and vice versa).

For given α and β, and given factor prices v and w, the firm chooses x and y to minimize the cost of producing q units of output, $vx+wy$ subject to $x^{\alpha}y^{\beta}=q$.

This minimization problem is standard and leads to the unit cost function

$$C(v,w)=\frac{\alpha+\beta}{\alpha^{\frac{\alpha}{\alpha+\beta}}\beta^{\frac{\beta}{\alpha+\beta}}}v^{\frac{\alpha}{\alpha+\beta}}w^{\frac{\beta}{\alpha+\beta}}.$$

Using $\alpha+\beta=\gamma$, this function can be rewritten as

$$C(v,w)=\frac{\gamma}{(\gamma-\beta)^{\frac{\gamma-\beta}{\gamma}}\beta^{\frac{\beta}{\gamma}}}v^{\frac{\gamma-\beta}{\gamma}}w^{\frac{\beta}{\gamma}}=\gamma\left[\left(\frac{v}{\gamma-\beta}\right)^{\gamma-\beta}\left(\frac{w}{\beta}\right)^{\beta}\right]^{\frac{1}{\gamma}}.$$

Assume the firm can choose any β such that $\underline{\beta}\leq\beta\leq\overline{\beta}$, where $\underline{\beta}$ and $\overline{\beta}$ are exogenous upper and lower bounds, with $\gamma-\overline{\beta}>0$ (and thus $\gamma-\underline{\beta}>0$). The problem is equivalent to choosing β to maximize

$$\Omega(\beta)\equiv\left(\frac{\gamma-\beta}{v}\right)^{\gamma-\beta}\left(\frac{\beta}{w}\right)^{\beta}$$

or to maximize $\ln\Omega(\beta)$ subject to $\overline{\beta}-\beta\geq0$ and $\beta-\underline{\beta}\geq0$. Let $\lambda_1\geq0$ and $\lambda_2\geq0$ be Lagrange multipliers associated with these constraints. The first-order condition of the Lagragian function $L=(\gamma-\beta)\ln(\gamma-\beta)-(\gamma-\beta)\ln v+\beta\ln\beta-\beta\ln w+\lambda_1\left(\overline{\beta}-\beta\right)+\lambda_2\left(\beta-\underline{\beta}\right)$ is

$$\frac{dL}{d\beta}=-\ln(\gamma-\beta)+\ln v+\ln\beta-\ln w-\lambda_1+\lambda_2=0,$$

and it is easy to show that any interior solution $\beta\in\left(\underline{\beta},\overline{\beta}\right)$ to this condition would not maximize but minimize $\ln\Omega(\beta)$ because the second derivative of the objective function $\ln\Omega(\beta)$ is positive, i.e., the function $\ln\Omega(\beta)$ is strictly convex in β for all $\beta\in\left[\underline{\beta},\overline{\beta}\right]$:

$$\frac{d^2\ln\Omega(\beta)}{d\beta^2}=(\gamma-\beta)+1>0$$

since $\beta\leq\overline{\beta}<\gamma$. For the optimal production plan, we must distinguish three cases: (i) If $\frac{w}{v}<\frac{\underline{\beta}}{\gamma-\underline{\beta}}$, i.e., factor y is very cheap relative to factor x, and

$$d\ln\Omega(\beta)/d\beta=\ln\left(\frac{\beta}{\gamma-\beta}\right)-\ln\left(\frac{w}{v}\right)>0,\text{ for all }\beta\in\left(\underline{\beta},\overline{\beta}\right)$$

shows that the optimal β is at the highest permissible value, i.e., $\overline{\beta}$. (ii) Similarly, if $\frac{\underline{\beta}}{\gamma-\underline{\beta}} < \frac{w}{v}$, the optimal β is at the lowest permissible value $\underline{\beta}$. (iii) If

$$\frac{\underline{\beta}}{\gamma-\underline{\beta}} < \frac{w}{v} < \frac{\overline{\beta}}{\gamma-\overline{\beta}}$$

a value $\beta^{\#} \in \left(\underline{\beta}, \overline{\beta}\right)$ exists such that $d\ln\Omega(\beta)/d\beta = 0$ at $\beta^{\#}$, where

$$\frac{\underline{\beta}}{\gamma-\underline{\beta}} < \frac{\beta^{\#}}{\gamma-\beta^{\#}} = \frac{w}{v} < \frac{\overline{\beta}}{\gamma-\overline{\beta}}$$

Since $\frac{d^2\ln\Omega(\beta)}{d\beta^2} = (\gamma-\beta)+1 > 0$ for all $\beta \in \left[\underline{\beta}, \overline{\beta}\right]$, we can infer that at $\beta = \beta^{\#}$, the function $\ln\Omega(\beta)$ attains its minimum. Then the maximum of $\ln\Omega(\beta)$ over the interval $\left[\underline{\beta}, \overline{\beta}\right]$ must occur either at $\beta = \underline{\beta}$ or at $\beta = \overline{\beta}$. It occurs at $\underline{\beta}$ iff $(\gamma-\underline{\beta})\ln(\gamma-\underline{\beta}) - (\gamma-\underline{\beta})\ln v + \underline{\beta}\ln\underline{\beta} - \underline{\beta}\ln w > (\gamma-\overline{\beta})\ln(\gamma-\overline{\beta}) - (\gamma-\overline{\beta})\ln v + \overline{\beta}\ln\overline{\beta} - \overline{\beta}\ln w$.

Now suppose that $v = 1$, but w may take on one of two possible values, w_H and w_L, with probability ρ and $1-\rho$ respectively. Then the firm's optimal production planning problem is to choose β to minimize expected cost of producing one unit of output, i.e.,

$$EC(\beta) = \rho\left[\left(\frac{v}{\gamma-\beta}\right)^{\gamma-\beta}\left(\frac{w_H}{\beta}\right)^{\beta}\right]^{\frac{1}{\gamma}} + (1-\rho)\left[\left(\frac{v}{\gamma-\beta}\right)^{\gamma-\beta}\left(\frac{w_L}{\beta}\right)^{\beta}\right]^{\frac{1}{\gamma}}$$

subject to $\overline{\beta} - \beta \geq 0$ and $\beta - \underline{\beta} \geq 0$. To simplify, assume $\gamma = 1$ and turn the problem into a maximization problem where the Lagragian is

$$L = -\rho\left(\frac{v}{1-\beta}\right)^{1-\beta}\left(\frac{w_H}{\beta}\right)^{\beta} - (1-\rho)\left(\frac{v}{1-\beta}\right)^{1-\beta}\left(\frac{w_L}{\beta}\right)^{\beta} + \lambda_1\left(\overline{\beta}-\beta\right) + \lambda_2\left(\beta-\underline{\beta}\right)$$

and $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ are the Lagrange multipliers. Let

$$U(\beta) \equiv \left(\frac{v}{1-\beta}\right)^{1-\beta}\left(\frac{w_H}{\beta}\right)^{\beta}, S(\beta) \equiv \left(\frac{v}{1-\beta}\right)^{1-\beta}\left(\frac{w_L}{\beta}\right)^{\beta}.$$

For any differentiable function $y(x) > 0$, $\frac{1}{y}\frac{dy}{dx} = \frac{d\ln y}{dx}$ and $\frac{dy}{dx} = y(x)\frac{d\ln y}{dx}$ hold, so

$$\frac{dU}{d\beta} = U(\beta)\left[\ln\left(\frac{w_H}{v}\right) - \ln\left(\frac{\beta}{1-\beta}\right)\right], \frac{dS}{d\beta} = S(\beta)\left[\ln\left(\frac{w_L}{v}\right) - \ln\left(\frac{\beta}{1-\beta}\right)\right].$$

If there is an interior solution, it must satisfy

$$-\rho U(\beta)\left[\ln\left(\frac{w_H}{v}\right)-\ln\left(\frac{\beta}{1-\beta}\right)\right]-(1-\rho)S(\beta)\left[\ln\left(\frac{w_L}{v}\right)-\ln\left(\frac{\beta}{1-\beta}\right)\right]=0, \quad \text{(A.3)}$$

which requires

$$\ln\left(\frac{w_H}{v}\right)-\ln\left(\frac{\beta}{1-\beta}\right)=\frac{(1-\rho)}{\rho}\left(\frac{w_L}{w_H}\right)^{\beta}\left[\ln\left(\frac{\beta}{1-\beta}\right)-\ln\left(\frac{w_L}{v}\right)\right]. \quad \text{(A.4)}$$

For condition (A.4) to be satisfied at some $\beta^* \in \left(\underline{\beta},\overline{\beta}\right)$, it is necessary that

$$\ln\left(\frac{w_H}{v}\right)>\ln\left(\frac{\beta^*}{1-\beta^*}\right)>\ln\left(\frac{w_L}{v}\right).$$

This condition is similar to the condition in Proposition 2. Suppose that there exists a value β^* that satisfies eq (A.4), and assume that $\beta^* \in \left(\underline{\beta},\overline{\beta}\right)$. The second-order condition is also satisfied because differentiation of (A.3) yields

$$\begin{aligned}&-\rho U'(\beta)\left[\ln\left(\frac{w_H}{v}\right)-\ln\left(\frac{\beta}{1-\beta}\right)\right]-\rho U(\beta)\left[\frac{\beta}{\beta(1-\beta)}\right]\\&-(1-\rho)S'(\beta)\left[\ln\left(\frac{w_L}{v}\right)-\ln\left(\frac{\beta}{1-\beta}\right)\right]-(1-\rho)S(\beta)\left[\frac{\beta}{\beta(1-\beta)}\right]\end{aligned}$$

This expression is clearly negative, because

$$U'(\beta)\left[\ln\left(\frac{w_H}{v}\right)-\ln\left(\frac{\beta}{1-\beta}\right)\right]=\left[\ln\left(\frac{w_H}{v}\right)-\ln\left(\frac{\beta}{1-\beta}\right)\right]^2>0$$

and

$$S'(\beta)\left[\ln\left(\frac{w_L}{v}\right)-\ln\left(\frac{\beta}{1-\beta}\right)\right]=\left[\ln\left(\frac{w_L}{v}\right)-\ln\left(\frac{\beta}{1-\beta}\right)\right]^2>0.$$