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The effect of externality on the transitional dynamics: the case of Lucas model.

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Abstract

The main aim of this paper was to provide a closed-form solution to the model of Lucas with externalities and to prove that this solution is unique. The method we proposed enables us to determine the unknown starting values of the control variables and to obtain the optimal trajectories of all variables of the system.

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1 Introduction

The paper of Lucas (1988) is obviously one of the most important papers written in the field of economic growth theory. Immediately after its publication, in less than twenty years, a huge number of published papers tried to improve, to complete or to develop the results obtained by Lucas. As an immediate consequence, this paper became one of the papers the most cited in the field of the economic theory. Among all these papers we mention here some of them, more precisely only the papers which we consider to have not only a substantial contribution, but also a connection with our paper.

The first one was written, a few years later by Lucas himself (1993). In that paper Lucas observed that, from 1960 to 1988, GDP per capita in the Philippines grew at about 1.8 percent per year, and in Korea, over the same period, per capita income grew at 6.2 percent per year, a rate consistent with the doubling of living standards every 11 years. Consequently, he tried to answer to the following questions. How did it happen? Why did it happen in Korea and not in the Philippines? Based on some new concepts introduced in his earlier paper, Lucas concludes that the accumulation of human capital is the main engine of growth and the main source of differences in living standards among nations.

Almost at the same moment, Benhabib and Perli (1993) published a very interesting paper where they tried to give some answers to the same question of Lucas. The authors restricted their analysis to the balanced growth path (briefly BGP) and based on the concept of indeterminacy, they claim that: "The implications of indeterminacy in endogenous growth models are as follows: two identically endowed economies with identical initial conditions may consume, and allocate labor between the production of human and physical capital, at completely different rates. Only in the long run will those economies converge to the same growth rate, but not to the same level of output and human and physical capital". This conclusion is obviously true, but only along the BGP.

Many of the other published papers (more or less recently), gave a certain clarification to the same problem, by generalizing some of the above mentioned results, as those of Gomez (2004) and Gupta and Chakraborty (2007). In order to give complete answers to the questions of Lucas, we need to understand what is happening along the transitional growth path. The papers of Benhabib and Perli, Gupta and Chakraborty and Gomez, do not answer to the problem of transitional path.

Some new and very interesting results concerning the trajectories along the transitional path were obtained in the last decade. Using the Gaussian hypergeometric functions, the paper of Boucekkine and Ruiz-Tamarit (2008) is the first one to provide a solution to the Lucas-Uzawa model, able to characterize the dynamics of the original variables along the transitional path. A few years later, Chilarescu (2011) provide a similar result, but the method proposed is, in our opinion, more simple and use only classical mathematical tools. An identical solution to that developed by Chilarescu, was recently obtained by Naz et al. (2016), by using the new technique of partial Hamiltonian operator.

Even if the results obtained by the above mentioned authors are extremely encouraging for the process of understanding of the transitional path, these results are relatively limited and this characteristic is generated by one of the starting hypotheses. The positive externality parameter γ was considered equal to zero, that is the key element of the Lucas approach was eliminated. Of course, as mathematical procedure it was a good idea to simplify certain hypotheses in order to obtain results which allow to describe the trajectories to the balance growth path, but as for its economic consequences, we can claim that this simplification eliminated Lucas's essential idea - the externality effect.

There is a large amount of theoretical literature on the model of endogenous growth developed by Lucas and this literature has continued to expand, on both alternatives of this model: the centralized solution and the market solution. We mention here only some of the most cited papers on this field: Mulligan and Sala-I-Martin (1993), Xie (1994) Mattana and Venturi (1999), Mattana (2004), Nishimura and Shigoka (2006), Mattana, Nishimura and Shigoka (2009), Bethmann and Reiß (2012), Bella and Mattana (2014), Manuelli and Seshadri (2014) and Bella, Mattana and Venturi (2017).

The first attempts to determine close-form solutions to the general model proposed by Lucas, are those of Boucekkine and Ruiz-Tamarit (2004) and Ruiz-Tamarit (2008) but unfortunately, both of them are only particular solutions and this characteristic is generated by the fact that the authors assumed that the inverse of the intertemporal elasticity of substitution equals the elasticity of output with respect to physical capital. This is of course a too strong limitation. Even under this assumption, their results are only partial results and in a recent paper Chilarescu and Viasu (2017), improved and complete the results developed by the mentioned authors.

To our knowledge, a solution procedure of the general model of Lucas with externalities is still unknown. In this paper, we propose a method for solving this model, having as a starting point the method developed by Chilarescu. One of the most significant results we find is the confirmation of Lucas intuition. "The main engine of growth is the accumulation of human capital and the main source of differences in living standards among nations is differences in human capital". This result is obviously generated by the presence of externalities.

The paper is organised as follows. In the second section we present the model of Lucas with externalities, determine the differential equations that drives the economy over time and provide the relations that characterize the balanced growth path. In the third section we completely describe the method to determine the solution of the differential system presented in the previous section and identify the initial values for the control variables. In the last section, we present some numerical simulations that will confirm the essential role of the accumulation of human capital, and finally we present some conclusions.

2 The Model of Lucas with externality

The two sectors considered in this paper are the good sector that produces consumable and gross investment in physical capital, and the education sector that produces human capital, both of them under conditions of constant returns to scale. Without loss of generality, we suppose that the economy is populated by a large and constant number of identical agents, normalized to one, so that all the variables can be interpreted as per capita quantities. The set of paths $\{k, h, c, u\}$ is called an optimal solution if it solves the following optimization problem:

$$V_0 = \max_{u,c} \int_0^{\infty} \frac{[c(t)]^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt, \tag{1}$$

subject to

$$\begin{cases} \dot{k}(t) = A\,[k(t)]^{\beta}\,[u(t)h(t)]^{1-\beta}\,[h_a(t)]^{\gamma} - \pi k(t) - c(t), \\ \\ \dot{h}(t) = \delta[1-u(t)]h(t), \\ \\ k_0 = k(0),\ h_0 = h(0), \end{cases} \tag{2}$$

where $k_0 > 0$ and $h_0 > 0$ are given, β is the elasticity of output with respect to physical capital, ρ is a positive discount factor, the efficiency parameters $A > 0$ and $\delta > 0$ represent the constant technological levels in the good sector and, respectively in the education sector, γ is a positive externality parameter, k is physical capital, h is human capital, c is the real per-capita consumption and u is the fraction of labor allocated to the production of physical capital and the term h_a^γ is intended to capture the external effects of human capital. σ^{-1} represents the constant elasticity of intertemporal substitution, and throughout this paper we suppose that $\sigma \neq 1$ and $\sigma \neq \beta$. The equations (2) give the resources constraints and initial values for the state variables k and h. Of course, the two state variables and the two control variables as well as the variable h_a, are all functions of times, but when no confusions are possible, we simply write k, h, c and u. To solve the problem (1) subject to (2), we define the Hamiltonian function:

$$H = \frac{c^{1-\sigma} - 1}{1-\sigma} + \left[Ak^\beta \left(uh\right)^{1-\beta} h_a^\gamma - \pi k - c \right] \lambda + \delta(1-u)h\mu.$$

The boundary conditions include initial values (k_0, h_0), and the transversality conditions:

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t)k(t) = 0 \ \text{ and } \ \lim_{t\to\infty} e^{-\rho t}\mu(t)h(t) = 0.$$

According to Lucas, the system is in equilibrium if the solution path $h(t)$ for the problem above coincides with the given path $h_a(t)$. Differentiating the Hamiltonian with respect to c and then with respect to u, we obtain the preliminary solutions of the optimal controls:

$$c^{-\sigma} = \lambda \tag{3}$$

$$A(1-\beta)k^\beta h^{\gamma-\beta} u^{-\beta} \lambda = \delta\mu. \tag{4}$$

After some algebraic manipulations, we can write down the dynamical system that drives the economy over time.

$$\begin{cases} \dot{k} = \left[A \left(\frac{h^{\frac{1-\beta+\gamma}{1-\beta}} u}{k} \right)^{1-\beta} - \pi \right] k - c, \\ \dot{h} = \delta [1-u] h, \\ \dot{c} = \left[-\frac{\rho+\pi}{\sigma} + \frac{A\beta}{\sigma} \left(\frac{h^{\frac{1-\beta+\gamma}{1-\beta}} u}{k} \right)^{1-\beta} \right] c, \\ \dot{u} = \left[\frac{(\delta+\pi)(1-\beta)+\gamma\delta}{\beta} - \frac{c}{k} + \frac{\delta(1-\beta+\gamma)}{1-\beta} u \right] u, \\ \dot{\lambda} = \left[\rho + \pi - A\beta \left(\frac{h^{\frac{1-\beta+\gamma}{1-\beta}} u}{k} \right)^{1-\beta} \right] \lambda \\ \dot{\mu} = \left[\rho - \delta - \frac{\gamma\delta}{1-\beta} u \right] \mu. \end{cases} \tag{5}$$

In order to prove the next proposition (for proof see Appendix), we need the following preliminary result, which use the definition of the finite limit of a function at infinity.

Lemma 1. *If the function $f = f(x)$ has a finite limit l, when $x \to \infty$, then there exists a finite $x_* > 0$ such that for all $x > x_*$, $|f(x) - l| < \varepsilon$*[1]*.*

Proposition 1. *If the parameters lie in the set Ω,*

$$\Omega = \left\{ \delta > \frac{\rho(1-\beta)}{1-\beta+\gamma}, \sigma > \max\{\sigma_1, \sigma_2\} \right\},$$

where

$$\sigma_1 = 1 - \frac{\rho(1-\beta)}{\delta(1-\beta+\gamma)}, \ \sigma_2 = \frac{\beta\left[(1-\beta)(\delta-\rho)+\gamma\delta\right]}{(1-\beta)\left[\delta+\pi(1-\beta)\right]+\gamma\delta},$$

then for all $t \geq t_*$, the system reaches the balanced growth path and the following statements are valid:

[1] see James Stewart (2008), page 138

a. $r_u = 0$, $r_{k*} = r_{c*} = r_* \neq r_{h*}$ *with*

$$r_{h*} = \frac{\delta(1-\beta+\gamma) - \rho(1-\beta)}{\sigma(1-\beta+\gamma)},\ r_* = \theta r_{h_*},\ \theta = \frac{1-\beta+\gamma}{1-\beta}, \qquad (6)$$

b. $u_* \in (0,1)$ *and is given by,*

$$u_* = \frac{\delta(1-\beta+\gamma)(\sigma-1) + \rho(1-\beta)}{\delta\sigma(1-\beta+\gamma)}, \qquad (7)$$

c. the ratio c_*/k_* *is given by*

$$\frac{c_*}{k_*} = \xi > 0, \qquad (8)$$

with

$$\xi = \phi - \chi,\ \phi = \frac{(1-\beta)\left[\delta + \pi(1-\beta)\right] + \gamma\delta}{\beta(1-\beta)},\ \chi = \frac{(1-\beta)(\delta-\rho) + \gamma\delta}{\sigma(1-\beta)}.$$

The next section of our paper is dedicated to study the transitional growth path. To find closed-form solutions to the optimal problem (1) subject to (2), we need starting values for the control variables. These starting values obviously depend on the starting values of state variables and therefore should be computed as part of the closed-form solution.

3 The Transitional Growth Path Solution

In order to simplify the computation procedure, we introduce a new variable. To do this we turn back to system (5) and denote by

$$z = \frac{h^{\frac{1-\beta+\gamma}{1-\beta}} u}{k}. \qquad (9)$$

Obviously, z is a positive function of time. Differentiating (9) with respect to time we arrive to the following differential equation

$$\dot{z} = \left[\tau - Az^{1-\beta}\right] z,\ \tau = \frac{(\delta+\pi)(1-\beta) + \gamma\delta}{\beta(1-\beta)}. \qquad (10)$$

A non-constant admissible solution of equation (10) is given by

$$z^{1-\beta} = \frac{z_*^{1-\beta} z_0^{1-\beta}}{\left(z_*^{1-\beta} - z_0^{1-\beta}\right) e^{-\varphi t} + z_0^{1-\beta}}, \tag{11}$$

with

$$z_0 = \frac{h_0^{\frac{1-\beta+\gamma}{1-\beta}} u_0}{k_0},\ z_*^{1-\beta} = \frac{(\delta+\pi)(1-\beta)+\gamma\delta}{A\beta(1-\beta)},\ \varphi = \frac{(\delta+\pi)(1-\beta)+\gamma\delta}{\beta}.$$

In the proving process of our main result we need the following lemma (for proof see Appendix).

Lemma 2. *z is a bounded positive function having the following properties:*

a. If $z_0 < z_*$, then z is an increasing function of time.

b. If $z_0 > z_*$ then z is a decreasing function of time.

c. z is an increasing function of u_0.

The solution (11) is the key tool in the solving process of the system (5) and the results are presented in the next theorem.

Theorem 1. *Let parameters be defined as in the definition* 2*. Then for all $t > 0$ the optimization problem* (1) *subject to* (2) *has the following unique solution:*

$$k(t) = \frac{k_0 z_0}{R_*} [z(t)]^{-1} [R_* - R(t)] e^{\phi t}, \tag{12}$$

$$c(t) = \frac{k_0 z_0}{R_*} [z(t)]^{-\frac{\beta}{\sigma}} e^{\chi t}, \tag{13}$$

$$h(t) = h_0 \left\{ \frac{u_0 e^{\phi t} [R_* - R(t)]}{R_* u(t)} \right\}^{\frac{1-\beta}{1-\beta+\gamma}}, \tag{14}$$

$$u(t) = \frac{\varphi u_0 (R_* - R(t))}{[(\varphi + \delta\theta u_0) R_* - \delta\theta u_0 B(t)] e^{-\varphi t} - \delta\theta u_0 [R_* - R(t)]}, \tag{15}$$

where

$$R(t) = \int_0^t z(s)^{\frac{\sigma-\beta}{\sigma}} e^{-\xi s} ds,\ B(t) = \int_0^t z(s)^{\frac{\sigma-\beta}{\sigma}} e^{-(\xi-\varphi)s} ds,$$

$$R_* = R_*(u_0; k_0, h_0) = \lim_{t\to\infty} R(t),\ B_* = B_*(u_0; k_0, h_0) = \lim_{t\to\infty} B(t),$$

and u_0 and c_0 are solutions of the following equations

$$(\varphi + \delta\theta u_0)\, R_*(u_0; k_0, h_0) - \delta\theta u_0 B_*(u_0; k_0, h_0) = 0,\ c_0 = k_0 R_*^{-1} z_0^{\frac{\sigma-\beta}{\sigma}}. \quad (16)$$

To prove the above theorem we need the following two preliminary results. The first is given below without proof (the proof follows immediately by direct computation).

Proposition 2. *R and B are bounded positive functions having the following properties:*

a. R and B are both increasing functions of time.

b. $R(t) \leq B(t)$ for all $t \geq 0$ and for any $u_0 > 0$.

c. Since z is an increasing function of u_0, it follows that R and B will also be increasing functions of u_0.

The second preliminary result will be necessary in the proving process of the uniqueness of solution (for proof see Appendix).

Proposition 3. *The equation*

$$(\varphi + \delta\theta u_0)\, R_*(u_0; k_0, h_0) - \delta\theta u_0 B_*(u_0; k_0, h_0) = 0$$

has a unique solution in the interval $(0, u_*)$.

Proof of Theorem 1. Substituting the solution (11) into the third equation of the system (5) we obtain the following differential equation

$$\frac{\dot{c}}{c} = -\frac{\rho + \pi}{\sigma} + \frac{A\beta}{\sigma} \frac{z_*^{1-\beta} z_0^{1-\beta}}{\left(z_*^{1-\beta} - z_0^{1-\beta}\right) e^{-\varphi t} + z_0^{1-\beta}},$$

whose solution is given by

$$c(t) = c_0 z_0^{\frac{\beta}{\sigma}} \left[z(t)\right]^{-\frac{\beta}{\sigma}} e^{\chi t}, \quad (17)$$

and from (3) we get

$$\lambda(t) = c_0^{-\sigma} z_0^{-\beta} z(t)^{\beta} e^{-\sigma\chi t}. \quad (18)$$

The substitution of (13) into the first equation of the system (5) will provide the equation

$$\dot{k} = \left(Az^{1-\beta} - \pi\right) k - c_0 z_0^{\frac{\beta}{\sigma}} z^{-\frac{\beta}{\sigma}} e^{\chi t},$$

and therefore the solution for k is given by

$$k(t) = z_0 z(t)^{-1} e^{\phi t} \left[k_0 - c_0 z_0^{\frac{\beta-\sigma}{\sigma}} R(t) \right]. \tag{19}$$

We denote by $R_1(t) = k_0 - c_0 z_0^{\frac{\beta-\sigma}{\sigma}} R(t)$. Since $R(t)$ is a bounded positive function of time, $R_1(0) = k_0 > 0$, $\lim\limits_{t\to\infty} R_1(t) = k_0 - c_0 z_0^{\frac{\beta-\sigma}{\sigma}} R_*$ and $\dot{R}_1(t) = -c_0 z_0^{\frac{\beta-\sigma}{\sigma}} \left[z(t)\right]^{\frac{\sigma-\beta}{\sigma}} e^{-\xi t} < 0$, we deduce that R_1 is a decreasing function of time. Since $k(t) > 0$ we deduce that $k_0 - c_0 z_0^{\frac{\beta-\sigma}{\sigma}} R(t) \geq 0$. Transversality condition for k requires that

$$\lim_{t\to\infty} [k_0 - c_0 z_0^{-\frac{\sigma-\beta}{\sigma}} R(t)] = k_0 - c_0 z_0^{\frac{\beta-\sigma}{\sigma}} R_* = 0 \;\Rightarrow\; c_0 = k_0 R_*^{-1} z_0^{\frac{\sigma-\beta}{\sigma}}.$$

Substituting this result into the equations (17) and (19), we get the solutions (12) and (13). In order to determine the solutions for h and u we need first the ratio $\frac{c}{k}$, that is given by

$$\frac{c(t)}{k(t)} = \frac{z^{\frac{\sigma-\beta}{\sigma}} e^{-\xi t}}{R_* - R(t)} \tag{20}$$

Passing to the limit yields into the above relation we immediately obtain $\frac{c_*}{k_*} = \xi$. Then we combine the second and the fourth equations of the system (5), consider the above result and after some algebraic manipulations, we arrive to the following differential equation

$$\frac{\dot{u}}{u} + \frac{1-\beta+\delta}{1-\beta} \frac{\dot{h}}{h} = \phi - \frac{z^{\frac{\sigma-\beta}{\sigma}} e^{-\xi t}}{R_* - R(t)} \tag{21}$$

whose solution is

$$[h(t)]^{\frac{1-\beta+\delta}{1-\beta}} u(t) = \frac{h_0^{\frac{1-\beta+\delta}{1-\beta}} u_0}{R_*} \left[R_* - R(t)\right] e^{\phi t}. \tag{22}$$

In fact this solution can also be obtained directly from the solution for k. Now we are able to obtain the solution for the control variable u. We substitute the relation (20) into the fourth equation of the system (5) and arrive to the following differential equation

$$\frac{\dot{u}}{u} = \varphi - \frac{z^{\frac{\sigma-\beta}{\sigma}} e^{-\xi t}}{R_* - R(t)} + \delta\theta u,$$

whose solution is given by (15). What we need now is to prove that (15) is an admissible solution, that means $u \in (0,1)$ and $\lim_{t\to\infty} u(t) = u_*$. Passing to the limit into (15) and applying successively l'Hôpital rule we get

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} \frac{u_0 z_*^{\frac{\sigma-\beta}{\sigma}} e^{-(\xi-\varphi)t}}{(\varphi + \delta\theta u_0) R_* - \theta u_0 B(t)}.$$

Let us consider the function $f(t) = (\varphi + \delta\theta u_0) R_* - \delta\theta u_0 B(t)$. Since $f(0) = (\varphi + \delta\theta u_0) R_* > 0$ and $\dot{f}(t) = -\delta\theta u_0 \dot{B}(t) < 0$ we conclude that f is a decreasing function of time and therefore we need

$$\lim_{t\to\infty} \left[(\varphi + \delta\theta u_0) R_* - \delta\theta u_0 B(t)\right] = 0, \Rightarrow B_* = \left(1 + \frac{\varphi}{\delta\theta u_0}\right) R_* \tag{23}$$

and finally yields

$$u_* = \lim_{t\to\infty} u(t) = \frac{\xi - \varphi}{\delta\theta}.$$

In fact, B_* and R_* are functions of the unknown variable u_0 and the starting values k_0 and h_0. We can write the above equation as is given in proposition 3 where we proved that it has a unique admissible solution u_0. This equation is a nonlinear one and only numerical procedures could be considered to determine u_0.

Let us now consider the function $g(t) = f(t)e^{-\varphi t} - \delta\theta u_0[R_* - R(t)]$. Since $g(0) = \varphi R_*$ and $\dot{g}(t) = -\varphi \left[(\varphi + \delta\theta u_0) A_* - \delta u_0 B(t)\right] e^{-\varphi t} < 0$, we deduce that g is a decreasing function of time. Passing to the limits into the definition of the function g we get $\lim_{t\to\infty} g(t) = 0$ and thus g is a positive decreasing function of time. Since both functions f and g are positive, we conclude that u is a positive function of time. In order to prove that $u < 1$, using the solution for u, we can write

$$1 - u(t) = \frac{\left[(\varphi + \delta\theta u_0) R_* - \delta\theta u_0 B(t)\right] e^{-\varphi t} - (\varphi + \delta\theta) u_0 [R_* - R(t)]}{\left[(\varphi + \delta\theta u_0) R_* - \delta\theta u_0 B(t)\right] e^{-\varphi t} - \delta\theta u_0 [R_* - R(t)]}.$$

Let us now consider the function

$$g_1(t) = \left[\left(\varphi + \delta\theta u_0 \right) R_* - \delta\theta u_0 B(t) \right] e^{-\varphi t} - (\varphi + \delta\theta) u_0 [R_* - R(t)].$$

First observe that $g_1(0) = \varphi(1 - u_0)R_* > 0$ and $\lim_{t \to \infty} g_1(t) = 0$.

$$\dot{g}_1(t) = \varphi \left\{ u_0 \dot{R}(t) - \left[\left(\varphi + \delta\theta u_0 \right) R_* - \delta\theta u_0 B(t) \right] e^{-\varphi t} \right\}.$$

$(\varphi + \delta u_0) R_* - \delta\theta u_0 B(t) = 0$ for all $t \geq t_*$, and therefore if there exists $t_1 > 0$ such that $\dot{g}_1(t_1) = 0$, then $t_1 < t_*$. Consequently $g_1(t)$ is an increasing function for all $t \in (0, t_1)$ and a decreasing function for all $t > t_1$. Since $\lim_{t \to \infty} g_1(t) = 0$, we conclude that $g_1(t) > 0$ for all $t > 0$ and consequently we can claim that $1 - u(t) > 0$. We can now substitute u from the equation (15) into the equation (22) to obtain the solution for h given by (14).

In order to obtain the solution for the dual variable μ we do not need to solve the last differential equation from the system (5). This solution can be determined directly from the equation (4), to obtain

$$\mu(t) = \frac{A(1 - \beta)}{\delta} \left[z(t) \right]^{-\beta} \left[h(t) \right]^{\frac{\gamma}{1-\beta}} \lambda(t). \tag{24}$$

The transversality condition for h can thus be written

$$\lim_{t \to \infty} h(t)\mu(t)e^{-\rho t} = \frac{A(1 - \beta)}{\delta} \lim_{t \to \infty} \frac{\left[z(t) \right]^{1-\beta}}{u(t)} k(t)\lambda(t)e^{-\rho t} = 0.$$

This is obviously true and thus the proof is completed. □

The same result concerning the uniqueness of the solution trajectories was also obtained by Ruiz-Tamarit. However, its result is partially dependent on the assumption that the inverse of the intertemporal elasticity of substitution equals the elasticity of output with respect to physical capital.

4 Numerical simulations and conclusions

The main aim of this paper was to provide a closed-form solution to the model of Lucas with externalities and to prove that this solution is unique. Moreover, the method we proposed here enables us to determine the unknown starting values of the control variables and thus to obtain the optimal trajectories of all variables of the system, first along the transitional path and then along the balanced growth path. Obviously this result is very important itself, but the main consequence of the solutions which we found, consists in the fact that we can supply some answers to Lucas's question. In order to do this, we consider the case of two economies, with exactly the same endowment levels, as was the case of the two countries (South Korea and Philippines) considered by Lucas. The benchmark values for the two economies are the following:

a. The case of country $C1$: $\beta = 0.25$, $\gamma = 0$, $A = 1.05$, $\delta = 0.05$, $\pi = 0.01$, $\rho = 0.04$, $\sigma = 1.5$, $h_0 = 10$, and $k_0 = 60$.

b. The case of country $C2$: $\beta = 0.25$, $\gamma = 0.417$, $A = 1.05$, $\delta = 0.05$, $\pi = 0.01$, $\rho = 0.04$, $\sigma = 1.5$, $h_0 = 10$, and $k_0 = 60$.

The trajectories of all the variables are presented in ten graphs consecutive. The starting values of control variables c and u are determined from the relations (16), and are given as follows.

a. For the country $C1$: $u_0 = 0.8839$ and $c_0 = 13.6662$.

b. For the country $C2$: $u_0 = 0.9294$ and $c_0 = 26.9542$.

Examining the graphs with the trajectories of variables, we observe that

a. The country $C1$ reaches the BGP after 25 years, and the country $C2$ after only 20 years.

b. The optimal level of the fraction of labor allocated to the production of physical capital at BGB is 86.67% for the country $C1$ and 67.61% for the country $C2$.

c. Along the transitional dynamics, the presence of the external effects of human capital, generates a higher economic growth rate. For this economy, the living standards double almost twenty years.

d. Along the BGP, GDP per capita grows at about 0.7 percent per year in the country with no external effect and grows at about 2.5 percent per year in the country with external effect.

To conclude, we may claim that physical capital accumulation plays an important role in the development process of an economy, but certainly not the essential one. The main engine of growth is the accumulation of human capital.

In spite of all these extremely relevant conclusions generated by the external effect of the human resources, the model developed by Lucas is appreciably conditioned by one of its hypotheses. We refer here to the fact that the external effect has the same and the constant influence throughout the transitional dynamic path. This hypothesis is difficult to be accepted. We think we need a more acceptable hypothesis. The external effect must have a small impact at the beginning of the transitional dynamic path and this impact should grow as one goes along this transition path. We think that in a future paper we will try to study this alternative.

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5 Appendix

Proof of Proposition 1. Applying *Lemma*1 to the functions $r_k = \frac{\dot{k}}{k}$, $r_c = \frac{\dot{c}}{c}$, $r_h = \frac{\dot{h}}{h}$ and $\frac{\dot{u}}{u}$, we obtain by direct computation from the corresponding equations of the system (5) the results given by the equations (6)-(8). From *eqns*. (6) and (7), in order to ensure the positivity of the growth rate of human capital and the fact that u_* is an admissible solution, $0 < u_* < 1$, we also get the set Ω. □

Proof of Lemma 1. The first two properties follow from the derivative of z with respect to time

$$(1-\beta)\frac{\dot{z}}{z} = \frac{\varphi\left(z_*^{1-\beta} - z_0^{1-\beta}\right)e^{-\varphi t}}{\left(z_*^{1-\beta} - z_0^{1-\beta}\right)e^{-\varphi t} + z_0^{1-\beta}}.$$

To prove the third property we need the derivative of z with respect to u_0

$$\frac{dz}{du_0} = \frac{\partial z}{\partial z_0}\frac{dz_0}{du_0} = \left(\frac{z}{u_0}\right)^{2-\beta}\left(\frac{k_0}{h_0}\right)^{1-\beta} e^{-\varphi t} > 0,$$

and thus the proof is completed. □

Proof of Proposition 3. Without loss of generality we consider only the case $z_0 < z_*$. The proof is similar for the case $z_0 > z_*$. Let us consider the function

$$f(u) = (\varphi + \delta\theta u)\, R_*(u; k_0, h_0) - \delta\theta u B_*(u; k_0, h_0)$$

defined on $[0, u_*]$. The functions $(\varphi + \delta\theta u)\, R_*(u; k_0, h_0)$ and $\delta\theta u B_*(u; k_0, h_0)$ are both positive and strictly increasing functions for all $u \in (0, u_*]$ and $f(0) = 0$. If $f(u_*) < 0$, then there exists a unique $u_0 \in (0, u_*)$ such that $f(u_0) = 0$.

$$f(u_*) = (\varphi + \delta\theta u_*)\, R_*(u_*; k_0, h_0) - \delta\theta u_* B_*(u_*; k_0, h_0)$$

$$= \xi R_*(u_*; k_0, h_0) - (\xi - \varphi) B_*(u_*; k_0, h_0)$$

$$= \int_0^\infty z(t, u_*)^{\frac{\sigma-\beta}{\sigma}} \left[\xi e^{-\xi t} - (\xi - \varphi) e^{-(\xi-\varphi)t}\right] dt.$$

$$f(u_*) \le |f(u_*)| = \left| \int_0^\infty z(t, u_*)^{\frac{\sigma-\beta}{\sigma}} \left[\xi e^{-\xi t} - (\xi - \varphi) e^{-(\xi-\varphi)t}\right] dt \right|$$

$$< z_*^{\frac{\sigma-\beta}{\sigma}} \left| \int_0^\infty \left[\xi e^{-\xi t} - (\xi - \varphi) e^{-(\xi-\varphi)t}\right] dt \right| = 0.$$

Consequently our equation has a unique solution in the interval $(0, u_*)$ and thus the proof is completed. $\square$

Fig. No. 1 Transitional dynamics for y(t) - Country C1

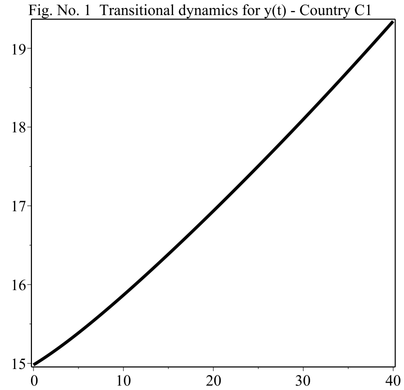


Fig. No. 6 Transitional dynamics for y(t) - Country C2

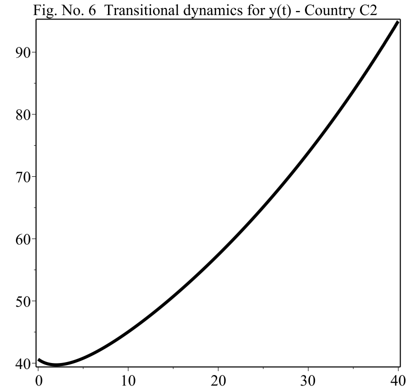


Fig. No. 2 Transitional dynamics for k(t) - Country C1

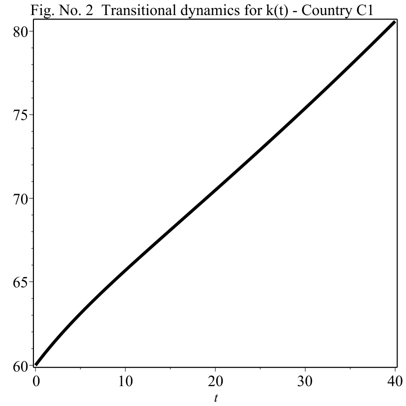


Fig. No. 3 Transitional dynamics for h(t) - Country C1

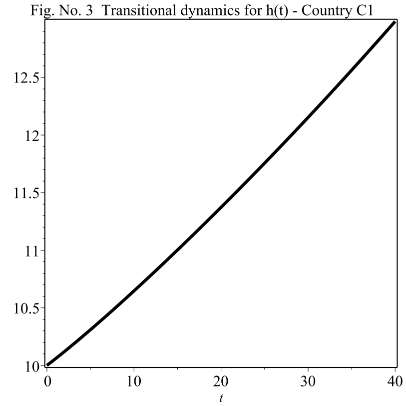


Fig. No. 4 Transitional dynamics for c(t) - Country C1

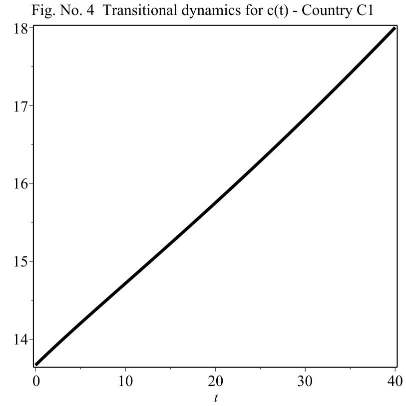


Fig. No. 7 Transitional dynamics for k(t)- Country C2

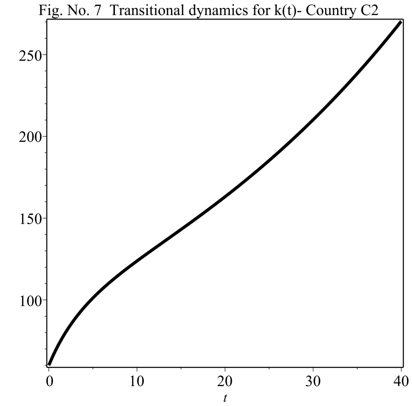


Fig. No. 8 Transitional dynamics for h(t) - Country C2

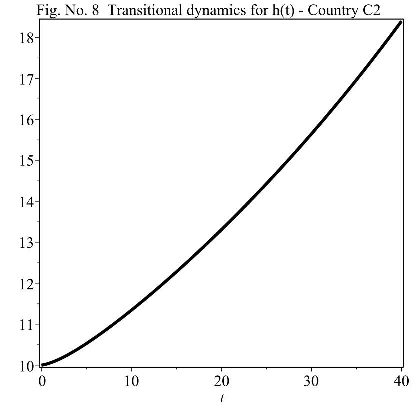


Fig. No. 9 Transitional dynamics for c(t) - Country C2

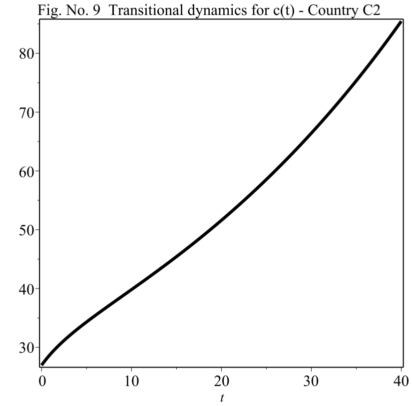


Fig. No. 5 Transitional dynamics for u(t) - Country C1

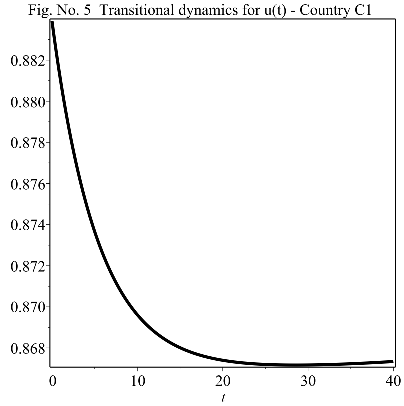
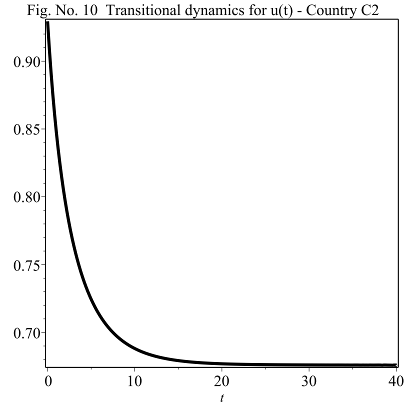


Fig. No. 10 Transitional dynamics for u(t) - Country C2



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