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A Note on the Likelihood of the Absolute Majority Paradoxes

Mostapha Diss
Univ Lyon, UJM Saint-Etienne, GATE UMR 5824, F-42023 Saint-Etienne, France.

Eric Kamwa
Université des Antilles, LC2S UMR CNRS 8053, F-97275 Schoelcher Cedex, France

Abdelmonaim Tlidi
University Cadi Ayyad of Marrakesh, GREER, National School of Applied Science, Safi, Morocco

Abstract

For three-candidate elections, we compute under the Impartial Anonymous Culture assumption, the conditional probabilities of the Absolute Majority Winner Paradox (AMWP) and the Absolute Majority Loser Paradox (AMLP) under the Plurality rule, the Borda rule, and the Negative Plurality rule for a given number of voters. We also provide a representation of the conditional probability of these paradoxes for the whole family of weighted scoring rules with large electorates. The AMWP occurs when a candidate who is ranked first by more than half of the voters is not selected by a given voting rule; the AMLP appears when a candidate who is ranked last by more than half of the voters is elected. As no research papers have tried to evaluate the likelihood of these paradoxes, this note is designed to fill this void. Our results allow us to claim that ignoring these two paradoxes in the literature, particularly AMWP, is not justified.

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Contact: Mostapha Diss - diss@gate.cnrs.fr, Eric Kamwa - eric.kamwa@univ-antilles.fr, Abdelmonaim Tlidi - mtlidi2010@gmail.com.
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1 Introduction

Let us assume n individuals (voters, decision-makers, judges, etc.) with strict rankings on a set of three alternatives or candidates denoted by $\mathcal{C} = \{a, b, c\}$. Individuals rank the candidates from the most desirable candidate to the least desirable one. Each ranking is assumed to be a linear order, i.e., a transitive, antisymmetric, and total relation. In addition, each voter is assumed to vote sincerely and act according to her true preferences. The six possible strict rankings on $\mathcal{C}$ are displayed in Table 1. In this table, it is indicated that n_1 voters have the ranking $a \succ b \succ c$, i.e., they rank candidate a at the top followed by candidate b and candidate c is the least preferred. In this framework, a *voting situation* in defined by $\tilde{n} = (n_1, \dots, n_6)$ and indicates the number of voters endowed with each linear order such that $\sum_{i=1}^{6} n_i = n$.

Table 1: The six possible strict rankings on $\mathcal{C} = \{a, b, c\}$

$a \succ b \succ c$	n_1	$a \succ c \succ b$	n_2	$b \succ a \succ c$	n_3
$b \succ c \succ a$	n_4	$c \succ a \succ b$	n_5	$c \succ b \succ a$	n_6

A *Condorcet Winner* (CW) is a candidate who is majority preferred to any other candidate, while a *Condorcet Loser* (CL) is majority dominated by each of the other candidates. An *Absolute Majority winner* (AMW) is a candidate who is top ranked by more than half of the voters. Similarly, an *Absolute Majority Loser* (AML) is a candidate who is bottom ranked by more than half of the voters. Using the labels of Table 1, we get what follows:

$$\begin{array}{lll}
\text{Candidate } a \text{ is the CW} & \Rightarrow & n_1+n_2+n_5 > n_3+n_4+n_6 \text{ and } n_1+n_2+n_3 > n_4+n_5+n_6 \\
\text{Candidate } a \text{ is the CL} & \Rightarrow & n_1+n_2+n_5 < n_3+n_4+n_6 \text{ and } n_1+n_2+n_3 < n_4+n_5+n_6 \\
\text{Candidate } a \text{ is the AMW} & \Rightarrow & n_1+n_2 > \frac{n}{2}, \text{ i.e., } n_1+n_2-n_3-n_4-n_5-n_6 > 0 \\
\text{Candidate } a \text{ is the AML} & \Rightarrow & n_4+n_6 > \frac{n}{2}, \text{ i.e., } -n_1-n_2-n_3+n_4-n_5+n_6 > 0
\end{array}$$

Notice that the CW (resp. the CL, the AMW and the AML) does not always exist as this can be the case when there is a *majority cycle*.[1] In addition, the AMW (resp. the AML), when she exists, is also the CW (resp. the CL). When a voting rule always picks the CW, when she exists, this rule is said to be *Condorcet consistent*. However, when a voting rule fails to select the CW, when she exists, this defines the *Condorcet Winner Paradox*. There are also situations in which a voting rule can select the CL and this defines the *Condorcet Loser Paradox*, also known as the *Strong Borda Paradox*.

A vast and rich literature has been devoted to the study of the *Condorcet Winner Paradox* as well as the *Condorcet Loser Paradox* both normatively and in terms of assessing their probabilities of occurrence. For a quick and non-exhaustive review, the reader may refer to the works of Cervone et al. (2005), Diss and Gehrlein (2015, 2012), Gehrlein and Lepelley (2011), Kamwa and Valognes (2017), among others. Up to our knowledge, just a little attention has been paid to the *Absolute Majority Winner Paradox* (AMWP) and the *Absolute Majority Loser Paradox* (AMLP) which respectively appear as special cases of the Condorcet Winner Paradox and the Condorcet Loser Paradox. The AMWP occurs when a candidate who is the AMW is not selected by a given voting rule and the AMLP appears when the AML is elected. Does the fact that no works have tried to evaluate the likelihood of these two paradoxes mean that they never appear or that they are very rare events? This note aims to fill this void by providing an answer to this question.

By considering the AMWP and the AMLP, we deal with the whole family of weighted scoring rules which are voting systems that give points to candidates according to the rank they have in

[1] If a is majority preferred to b, b is majority preferred to c, and c is majority preferred to a, this describes one of the two possible majority cycles with three candidates.

voters' preferences and the winner is the candidate with the highest total number of points. In three-candidate elections, the normalized vector $(1, \lambda, 0)$ with $0 \leq \lambda \leq 1$ can be used to represent the whole family of weighted scoring rules. With this vector, a candidate will receive 1 point each time she is ranked first in each individual preference, λ point when she is second and 0 point when she is ranked last. The *Plurality rule* (PR), *Borda rule* (BR), and the *Negative Plurality rule* (NPR) are respectively defined for $\lambda = 0$, $\lambda = \frac{1}{2}$ and $\lambda = 1$. We denote respectively by $S(a)$, $S(b)$, and $S(c)$ the accumulated scores obtained by candidates a, b, and c under the weighted scoring rule. In this paper, we compute for three-candidate elections, the conditional probabilities of the AMWP and the AMLP for a small number of voters under PR, BR and NPR. In addition, by assuming large electorates, we provide a representation of the conditional probability of the two paradoxes for the whole family of weighted scoring rules as a function of λ.

In order to compute the conditional probabilities of AMWP and AMLP we need to set an assumption on the voters' preferences. In this note we rely on the well-known assumption of Impartial and Anonymous Culture (IAC). Under IAC, each voting situation is assumed equally likely to occur. This hypothesis has been introduced by Kuga and Hiroaki (1974) and later developed by Gehrlein and Fishburn (1976). Under this assumption, the likelihood of a given event X is calculated in respect with the following ratio:

$$\frac{\text{Number of voting situations in which event } X \text{ occurs}}{\text{Total number of possible voting situations } \tilde{n}} \tag{1}$$

It is worthwhile to mention that we use the *parameterized Barvinok's algorithm* developed by Verdoolaege et al. (2004) in order to provide the probability of the two paradoxes under PR, BR and NPR as a function of the number of voters. This algorithm is encoded to compute the number of lattice points in a rational convex polytope and the output is a given in the form of *Ehrhart polynomials* (Ehrhart, 1962, 1967). For more details on this algorithm and the related subjects, the reader may refer to the works of Barvinok and Pommersheim (1999), Bruynooghe et al. (2005), Clauss and Loechner (1998), Diss et al. (2012), Lepelley et al. (2008), Verdoolaege et al. (2004). As noticed before, we also provide for large electorates the probability of the two paradoxes for the whole family of weighted scoring rules. For this case, we will directly follow a procedure that was developed in Cervone et al. (2005) and recently used in many research papers such as Diss and Gehrlein (2015, 2012), Gehrlein et al. (2015), Moyouwou and Tchantcho (2017), among others.[2]

2 Results

We first identify in Section 2.1 the range of weights for which the scoring rules (never) exhibit each of the two paradoxes in three-candidate elections. Then, we provide in Section 2.2 the likelihood of the two paradoxes under PR, BR and NPR for a given number of voters when the considered paradox can be observed. In Section 2.3 we focus on the whole family of weighted scoring rules and large electorates.

2.1 First results about these paradoxes

Proposition 1 tells us that in three-candidate elections, the AMLP never occurs for all the weighted scoring rules located between BR and NPR but it may occur out of this range. Proposition 2 focuses on AMWP and shows that this paradox may occur for all the weighted scoring rules except PR.

Proposition 1. *For three-candidate elections, the Absolute Majority Loser Paradox never occurs for* $\frac{1}{2} \leq \lambda \leq 1$ *and it may occur for* $0 \leq \lambda < \frac{1}{2}$.

[2] Our computation files are available upon request.

Proof. For $\dfrac{1}{2} \leq \lambda \leq 1$, suppose without loss of generality that candidate a is selected by the weighted scoring rule defined by the weight λ. This implies that $n_1 + n_2 + \lambda(n_3 + n_5) > n_3 + n_4 + \lambda(n_1 + n_6)$ and $n_1 + n_2 + \lambda(n_3 + n_5) > n_5 + n_6 + \lambda(n_2 + n_4)$. These two inequalities are equivalent to $n_4 < (1-\lambda)n_1 + n_2 + \lambda n_5 - (1-\lambda)n_3 - \lambda n_6$ and $n_6 < n_1 + (1-\lambda)n_2 + \lambda n_3 - (1-\lambda)n_5 - \lambda n_4$. Adding and collecting terms of the two last inequalities, we get $n_4 + n_6 < \dfrac{2-\lambda}{1+\lambda}n_1 + \dfrac{2-\lambda}{1+\lambda}n_2 + \dfrac{2\lambda-1}{1+\lambda}n_3 + \dfrac{2\lambda-1}{1+\lambda}n_5$ Hence, with $\dfrac{1}{2} \leq \lambda \leq 1$, we obtain $n_4 + n_6 < n_1 + n_2 + n_3 + n_5$ since $\dfrac{2-\lambda}{1+\lambda} \leq 1$ and $\dfrac{2\lambda-1}{1+\lambda} \leq 1$. Thus, $n_4 + n_6 < \dfrac{n}{2}$. This contradicts that candidate a is the AML; thus, the the AMLP never occurs for $\dfrac{1}{2} \leq \lambda \leq 1$. To show that the AMLP can happen for all $\lambda \in [0, \dfrac{1}{2}[$, just consider the following profile: $n_1 = n_2 = n_6 = x$, $n_3 = n_5 = 0$ and $n_4 = x + 1$ where x is an integer such that $x > \frac{1}{1-2\lambda}$. The reader can easily check that for all $0 \leq \lambda < \dfrac{1}{2}$, candidate a is the AML in this profile. The scores are the following: $S(a) = 2x$, $S(b) = x + 1 + 2\lambda x$ and $S(c) = x + (2x+1)\lambda$. Since $x > \frac{1}{1-2\lambda}$, it can easily be checked that $S(a) - S(b) = x - 1 - 2\lambda x > 0$ and $S(a) - S(c) = x - (2x+1)\lambda > 0$. Thus, with this profile, the AML may be elected for all $0 \leq \lambda < \dfrac{1}{2}$. Therefore, this profile can be used to show that the AMLP may occur for $0 \leq \lambda < \dfrac{1}{2}$. □

Proposition 2. *For three-candidate elections, the Absolute Majority Winner Paradox never occurs under the Plurality rule ($\lambda = 0$) and it may occur for $0 < \lambda \leq 1$.*

Proof. Suppose without loss of generality that candidate a is an AMW. By definition, the AMW always gets the highest score for $\lambda = 0$; so, the AMWP never occurs under the Plurality rule. In order to show that the AMWP can occur for $0 < \lambda \leq 1$, just assume the following profile: $n_1 = n_2 = z + 1$, $n_3 = n_5 = 0$, $n_4 = 2z$ and $n_6 = 1$ where z is an integer such that $z > \frac{2-\lambda}{\lambda}$. The reader can easily check that a is the AMW with this profile. The scores are the following: $S(a) = 2z + 2$, $S(b) = 2z + (z+1)\lambda$ and $S(c) = 1 + (3z+1)\lambda$. Since $z > \frac{2-\lambda}{\lambda}$, $S(a) - S(b) = 2 - (z+1)\lambda < 0$. Thus, with this profile, the AMW is not elected for all $0 < \lambda \leq 1$. So, this profile can be used to show that the AMWP may occur for $0 < \lambda \leq 1$. □

2.2 Probabilities of the paradoxes for well-known scoring rules

Notice that, when computing the probability of the two paradoxes in concern in this note, we are more interested in the conditional probability. More precisely, we compute the probability that the AMWP (resp. the AMLP) occurs given that the AMW (resp. the AML) exists. We begin first by Proposition 3 which provides, for three-candidate elections, the existence probability of the AMW. By symmetry, this probability is the same as the one of the AML.

Proposition 3. *For three-candidate elections and a given number of voters, the existence probability of the Absolute Majority Winner (Loser) under IAC assumption is given by:*

$$P_{AMW}(3, n, IAC) = P_{AML}(3, n, IAC) = \begin{cases} \frac{9(n+6)n}{16(n+1)(n+5)} & \text{for even } n \\ \\ \frac{3\,(3\,n^2+28\,n+49)}{16(n+2)(n+4)} & \text{for odd } n \end{cases}$$

We are now able to compute the conditional probability of the two Absolute Majority paradoxes. The conditional probability of the AMWP (AMLP) is obtained according to the following formula:

$$\frac{\text{Number of voting situations in which the AMWP (AMLP) occurs}}{\text{Number of voting situations in which the AMW (AML) exists}} \tag{2}$$

Recall that the AMWP does not occur under PR while the AMLP can not be observed for BR and NPR. For the other possibilities, our results are summarized in Propositions 4 to 6.

Proposition 4. *For three-candidate elections with $n \geq 7$, the conditional probability of the Absolute Majority Loser Paradox under the Plurality rule is given by:*

$$P^{PR}_{AMLP}(3,n,IAC) = \begin{cases} \frac{2n^3+3n^2-78n-72}{81n^3+729n^2+2106n+1944} & \text{for } n \equiv 0 \text{ mod } 6 \\ \frac{2n^4+30n^3+70n^2-190n+88}{81n^4+1080n^3+4590n^2+7560n+3969} & \text{for } n \equiv 1 \text{ mod } 6 \\ \frac{2n^4+7n^3-168n^2-268n+1120}{(81n^3+891n^2+2916n+2916)n} & \text{for } n \equiv 2 \text{ mod } 6 \\ \frac{2n^4+34n^3+118n^2-354n-1080}{81n^4+1242n^3+6264n^2+11718n+6615} & \text{for } n \equiv 3 \text{ mod } 6 \\ \frac{2n^4+11n^3-82n^2-216n+960}{(81n^3+1053n^2+4374n+5832)n} & \text{for } n \equiv 4 \text{ mod } 6 \\ \frac{2n^3+24n^2-66n-520}{81n^3+837n^2+2727n+2835} & \text{for } n \equiv 5 \text{ mod } 6 \end{cases}$$

Proposition 5. *For three-candidate elections with $n \geq 2$, the conditional probability of the Absolute Majority Winner Paradox under the Borda rule is given by:*

$$P^{BR}_{AMWP}(3,n,IAC) = \begin{cases} \frac{3n^3+42n^2+8n-768}{81n^3+729n^2+2106n+1944} & \text{for } n \equiv 0 \text{ mod } 6 \\ \frac{3n^4+90n^3+800n^2+1670n-2563}{27(3n^4+40n^3+170n^2+280n+147)} & \text{for } n \equiv 1 \text{ mod } 6 \\ \frac{3n^4+48n^3+68n^2-992n+1280}{(81n^3+891n^2+2916n+2916)n} & \text{for } n \equiv 2 \text{ mod } 6 \\ \frac{3n^4+96n^3+962n^2+2784n-2565}{27(3n^4+46n^3+232n^2+434n+245)} & \text{for } n \equiv 3 \text{ mod } 6 \\ \frac{3n^4+54n^3+152n^2-1024n-2560}{(81n^3+1053n^2+4374n+5832)n} & \text{for } n \equiv 4 \text{ mod } 6 \\ \frac{3n^3+81n^2+581n+455}{27(3n^3+31n^2+101n+105)} & \text{for } n \equiv 5 \text{ mod } 6 \end{cases}$$

Proposition 6. *For three-candidate elections with $n \geq 3$, the conditional probability of the Absolute*

Majority Winner Paradox under the Negative Plurality rule is given by:

$$P_{AMWP}^{NPR}(3,n,IAC)=\begin{cases}\frac{127\,n^4+2250\,n^3+13920\,n^2+36720\,n+36288}{324\,n^4+4860\,n^3+25920\,n^2+58320\,n+46656} & for\ n\equiv 0\ mod\ 12\\ \frac{127\,n^4+2444\,n^3+15738\,n^2+41036\,n+44335}{324\,n^4+4968\,n^3+25056\,n^2+46872\,n+26460} & for\ n\equiv 1\ mod\ 12\\ \frac{127\,n^5+2250\,n^4+13920\,n^3+37360\,n^2+46608\,n+20640}{324\,n(n^4+15\,n^3+80\,n^2+180\,n+144)} & for\ n\equiv 2\ mod\ 12\\ \frac{127\,n^3+2317\,n^2+13101\,n+22815}{324\,n^3+4644\,n^2+20412\,n+26460} & for\ n\equiv 3\ mod\ 12\\ \frac{127\,n^4+1996\,n^3+10248\,n^2+20384\,n+8960}{(324\,n^3+4212\,n^2+17496\,n+23328)n} & for\ n\equiv 4\ mod\ 12\\ \frac{127\,n^4+1936\,n^3+9198\,n^2+18424\,n+13195}{324\,n^4+3672\,n^3+14256\,n^2+22248\,n+11340} & for\ n\equiv 5\ mod\ 12\\ \frac{127\,n^4+1488\,n^3+4992\,n^2+6768\,n+2160}{(324\,n^3+2916\,n^2+8424\,n+7776)n} & for\ n\equiv 6\ mod\ 12\\ \frac{127\,n^3+1809\,n^2+6693\,n+7571}{324\,n^3+3348\,n^2+8316\,n+5292} & for\ n\equiv 7\ mod\ 12\\ \frac{127\,n^4+1742\,n^3+6952\,n^2+9552\,n+1920}{(324\,n^3+3564\,n^2+11664\,n+11664)n} & for\ n\equiv 8\ mod\ 12\\ \frac{127\,n^3+1555\,n^2+4533\,n+4185}{324\,n^3+2700\,n^2+6156\,n+3780} & for\ n\equiv 9\ mod\ 12\\ \frac{127\,n^4+1996\,n^3+10248\,n^2+20384\,n+15440}{(324\,n^3+4212\,n^2+17496\,n+23328)n} & for\ n\equiv 10\ mod\ 12\\ \frac{127\,n^4+2698\,n^3+20052\,n^2+62758\,n+72925}{324\,n^4+5616\,n^3+34344\,n^2+87696\,n+79380} & for\ n\equiv 11\ mod\ 12\end{cases}$$

Table 2 provides some computed values of $P_{AMW}(3,n,IAC)$, $P_{AML}(3,n,IAC)$, $P_{AMWP}^{BR}(3,n,IAC)$, $P_{AMWP}^{NPR}(3,n,IAC)$, and $P_{AMLP}^{PR}(3,n,IAC)$ for various values of n.

Table 2: Computed values of $P_{AMW}(3,n,IAC)$, $P_{AML}(3,n,IAC)$, $P_{AMWP}^{BR}(3,n,IAC)$, $P_{AMWP}^{NPR}(3,n,IAC)$, and $P_{AMLP}^{PR}(3,n,IAC)$

n	$P_{AMW}(3,n,IAC)$ $= P_{AML}(3,n,IAC)$	$P_{AMWP}^{NPR}(3,n,IAC)$	$P_{AMWP}^{BR}(3,n,IAC)$	$P_{AMLP}^{PR}(3,n,IAC)$
3	0.8571	0.6327	0.1250	0
4	0.5000	0.5714	0	0
5	0.7857	0.5757	0.1212	0
6	0.5260	0.5062	0.0247	0
7	0.7424	0.5511	0.1021	0.0204
8	0.5385	0.4848	0.0346	0
9	0.7133	0.5126	0.0924	0.0210
10	0.5455	0.4781	0.0366	0.0092
11	0.6923	0.4961	0.0853	0.0179
50	0.5615	0.4117	0.0398	0.0207
51	0.5969	0.4198	0.0489	0.0251
100	0.5622	0.4023	0.0386	0.0229
101	0.5644	0.3935	0.0377	0.0247
.	.	.	.	.
∞	0.5625	0.3920	0.0370	0.0247

From Table 2, many comments can be drawn. The first one is that the probability of the existence of an AMW (AML) is substantially high. Despite the fact that this probability decreases as the number of voters is increasing, it remains significant and reaches 56.25% for large electorates. Obviously, this probability is lower than that of the existence of a CW (CL) but our results allow us to claim that the difference between the two probabilities for a given number of voters remains negligible, especially for small number of voters.[3] Second, we can notice that the AMLP appears to be a very rare event. Indeed, we have already shown that this paradox never occurs for $\frac{1}{2} \leq \lambda \leq 1$. For $0 \leq \lambda < \frac{1}{2}$, our results show that the occurrence of this paradox is pretty insignificant and never exceed 2.52%. Finally, we notice that the occurrence of the AMWP is surprisingly very high, especially for NPR. These first theoretical results indicate that the AMWP is not as rare as its absence from the literature suggests.

2.3 Probabilities of the paradoxes for all weighted scoring rules and infinite electorates

For large electorates, Propositions 7 and 8 respectively give the limiting conditional probability of the AMWP and the AMLP as a function of λ for three-candidate elections.

Proposition 7. *For three-candidate elections with large electorates, the limiting conditional probability of the Absolute Majority Winner Paradox under weighted scoring rules is given as follows:*

$$P^{\lambda}_{AMWP}(3, \infty, IAC) = \begin{cases} \dfrac{\lambda^3}{3\,(\lambda-2)\,(\lambda^2-1)} & \textit{for } 0 \leq \lambda \leq \frac{1}{2} \\[2ex] \dfrac{32\,\lambda^8 + 1008\,\lambda^7 - 1838\,\lambda^6 - 614\,\lambda^5 + 1596\,\lambda^4 - 619\,\lambda^3 + 56\,\lambda^2 - 6\,\lambda + 4}{486\,\lambda^4\,(-2+\lambda)\,(\lambda+1)} & \textit{for } \frac{1}{2} \leq \lambda \leq 1 \end{cases}$$

Proposition 8. *For three-candidate elections with large electorates, the limiting conditional probability of the Absolute Majority Loser Paradox under weighted scoring rules is given as follows:*

$$P^{\lambda}_{AMLP}(3, \infty, IAC) = \begin{cases} \dfrac{(-\lambda^2 - 4\,\lambda + 6)\,(2\,\lambda - 1)^4}{243\,(\lambda-1)^4} & \textit{for } 0 \leq \lambda < \frac{1}{2} \\[2ex] 0 & \textit{for } \frac{1}{2} \leq \lambda \leq 1 \end{cases}$$

Table 3 provides some computed values of $P^{\lambda}_{AMWP}(3, \infty, IAC)$ and $P^{\lambda}_{AMLP}(3, \infty, IAC)$. We notice from this table that the likelihood of AMWP is quite low for $0 \leq \lambda < \frac{1}{2}$ but it grows considerably for $\frac{1}{2} \leq \lambda \leq 1$. This paradox is maximized under NPR ($\lambda = 1$) with 39.2% of chance. Since the figures we obtain concerning AMLP are very low for $\lambda < \frac{1}{2}$ and that the paradox vanishes for $\lambda \geq \frac{1}{2}$, we can say that the AMLP should actually be a rare event. This paradox is maximized by PR ($\lambda = 0$) with 2.47% of chance.

[3] The probability of the existence of a CW (CL) under IAC assumption can be found for instance in Gehrlein and Lepelley (2011, page 21).

Table 3: Computed values of $P^{\lambda}_{AMWP}(3,\infty,IAC)$ and $P^{\lambda}_{AMLP}(3,\infty,IAC)$

λ	$P^{\lambda}_{AMWP}(3,\infty,IAC)$	$P^{\lambda}_{AMLP}(3,\infty,IAC)$
0	0	0.0247
0.1	0.0002	0.0144
0.2	0.0015	0.0067
0.3	0.0058	0.0021
0.4	0.0159	0.0002
0.5	0.0370	0
0.6	0.0783	0
0.7	0.1414	0
0.8	0.2196	0
0.9	0.3055	0
1	0.3920	0

3 Concluding remarks

By exploring the literature related to the theoretical probability of various paradoxes afflicting voting procedures, we noticed that the AMWP and the AMLP did not receive enough interest. This work attempted to fill this gap by providing the conditional probabilities of the AMWP and the AMLP under PR, BR, and NPR for a given number of voters. We also provided a representation of the conditional probability of each of these paradoxes for the whole family of weighted scoring rules with large electorates. All our calculations are given under the well-known IAC assumption.

The central finding of this note is that the probabilities we obtained are not negligible. This then implies that ignoring these two paradoxes, and more particularly the AMWP, as it is the case in the literature is not justified. By way of comparison, the probability of the Strong Borda paradox[4] and the one of the Strict Borda paradox,[5] which have been the subject of several research papers in the recent literature, never exceed 3.15% and 1.11%, respectively, with the whole family of weighted scoring rules and large electorates.[6] Recall that the probability we obtain for the AMWP is equal to 3.70% under BR and even reaches 39.20% under NPR with large electorates.

Notice finally that we could also look at a combination of the two paradoxes by calculating the probability that the AMWP and the AMLP would appear at the same time. Given that the probabilities of the AMLP are already low and taking into account Proposition 1, we conclude that the probability of the combination of the two paradoxes will be equal to zero for $\frac{1}{2} \leq \lambda \leq 1$ and should be extremely rare for $0 \leq \lambda < \frac{1}{2}$. Thus, the study of the simultaneous occurrence of the AMLP and the AMWP is neglected in this note.

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[4]Recall that the Strong Borda Paradox occurs when a CL exists and is selected by a given voting rule.

[5]The Strict Borda Paradox occurs when the voting rule can completely reverse the ranking of the pairwise majority elections. In three-candidate elections, this means that the CL is elected and, at the same time, the CW is ranked last.

[6]The probability of Borda paradoxes under IAC can be found in Diss and Gehrlein (2012).

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