Economics Bulletin

Volume 39, Issue 2

Rational bubbles in altruistic economies: when Tirole meets Ramsey

Stefano Bosi
Université Paris-Saclay, EPEE

Thai Ha-Huy
Université Paris-Saclay, EPEE

Ngoc-Sang Pham
Montpellier Business School, MRM

Abstract

We consider an overlapping generations model à la Tirole (1985) augmented with altruism from parents to children as in Barro (1974). We compute the global dynamics and we show that, in the case of low altruism, bequests are zero and our model works exactly as the Tirole's model (1985) where rational bubbles can arise, while, in the case of high altruism, bequests are positive and bubbles are ruled out. This result holds whatever the share of altruistic agents in total population. Our contribution raises the question of the robustness of Tirole's conclusion about the existence of rational bubbles under a large degree of altruism.

Dear Editor, please find included the final version of our paper accepted by you for publication. Best regards.
Citation: Stefano Bosi and Thai Ha-Huy and Ngoc-Sang Pham, (2019) "Rational bubbles in altruistic economies: when Tirole meets Ramsey", *Economics Bulletin*, Volume 39, Issue 2, pages 816-829
Contact: Stefano Bosi - stefano.bosi@univ-evry.fr, Thai Ha-Huy - thai.hahuy@univ-evry.fr, Ngoc-Sang Pham - pns.pham@gmail.com.
Submitted: November 27, 2017. **Published:** April 03, 2019.

1 Introduction

We consider an economy with altruistic and selfish agents. Here, altruism means that parents endow their children with bequests. Bequests from parents increase the disposable income when young; bequests to children reduce the disposable income when old. Preferences of altruistic agents depends on children's utility (one-sided altruism). More explicitly, the parents' utility function depends on their two-period consumption demand and the utility function of progeny. The weight of children's utility in parents' preferences is an exogenous parameter. We assume, for simplicity, a kind of cultural transmission: altruists (egoists) give birth to altruists (egoists). The initial fraction of altruists in total population is given. Since we assume a constant demographic growth rate for both the classes, the fraction of altruists remains constant over time. Therefore, in our model, altruism is captured by two parameters: the weight of children in altruistic parents' utility and the fraction of altruists in total population.

It is known that the introduction of one-sided intergenerational altruism transforms an overlapping generations (OLG) model with capital accumulation à la Diamond (1965) in a dynastic model à la Ramsey (1928) (Barro, 1974; Barro and Becker, 1989). Dynamics in OLG models are different from those arising in Ramsey models either in the short or in the long run. The introduction of partial altruism bridges OLG models and Ramsey models. Thus, an important question to address concerns the critical degree of altruism beyond which an OLG economy behaves as a Ramsey economy.

In this article, we consider a variant of the seminal Diamond (1965) model in order to encompass the possibility of financial bubbles. In the spirit of Tirole (1985), we consider the occurrence of a rational bubble in a dynamic general equilibrium context focusing on an asset which does not bring any dividend but has a positive price. Since Tirole's seminal contribution, it is known that a persistent bubble arises as a second steady state in an OLG model with capital accumulation à la Diamond (1965), if and only if the first steady state, that is the regime without bubble of the basic Diamond's model, experiences, at equilibrium, overaccumulation. Overaccumulation corresponds to an excessive level of capital with respect to the golden rule à la Phelps (1961), that is to an inefficiently low interest rate. In this case, surprisingly, the financial bubble is virtuous, absorbing the oversaving and restoring the golden rule.

In infinite-horizon general equilibriums with standard conditions, bubbles are possible only if the present value of aggregate outputs is infinite (or, equivalently, the so-called "low implied interest rates" condition holds in the terminology used by Alvarez and Jermann, 2000) and/or borrowing constraints are frequently binding (Santos and Woodford, 1997; Bosi, Le Van and Pham, 2018). In this respect, an important point to raise concerns the critical degree of altruism beyond which rational bubbles disappear in an OLG model à la Tirole (1985). Our model is pertinent to study the passage from OLG models with bubbles to Ramsey models without bubbles because of a twofold altruism (intergenerational link and fraction of altruists in total population).

We find a critical degree of altruism such that, below this threshold, bequests are zero and persistent rational bubbles arise if and only if the steady state à la Diamond experiences capital overaccumulation, and that, above this critical value, bequests are positive and bubbles become impossible.

In addition, these results hold whatever the share of altruists in total population. It is known that in a Ramsey model (with stationary technology), bubbles are rules out (see, among others, Bosi, Le Van and Pham, 2018). In the case of positive bequests, the altruistic agents behave as dynastic agents à la Ramsey: since they save more and, driving the capital accumulation, they become dominant, the OLG economy with positive bequests turns out to work as a Ramsey economy and, therefore, there is no longer room for bubbles. Thus, through our contribution, the robustness of Tirole's conclusions is questioned, at least in the case of a sufficiently degree of altruism (positive bequests).

Finally, differently from Nourry and Venditti (2001), we provide also a global analysis and an explicit trajectory of capital accumulation.

2 Model

We consider a productive economy with capital accumulation and a constant returns to scale technology involving capital and labor services. Under constant returns to scale, price-taker small firms can be represented by a unique aggregate firm with production function $F(K_t, N_t)$, where K_t denotes the entire stock of capital of the economy and N_t the aggregate labor demand. Let $f(k_t) \equiv F(k_t, 1)$ be the average productivity and $k_t \equiv K_t/N_t$ be the capital intensity with $f'(0) = \infty$ and $f'(\infty) = 0$ (Inada condition).

Assumption 1 *The production function F is C^2, homogeneous of degree one, strictly increasing and concave. The intensive production function f satisfies the Inada conditions.*

The firm maximizes the net profit $F(K_t, N_t) - (R_{kt} - 1) K_t - \delta K_t - w_t N_t$ where R_{kt} and w_t denote the return on capital and the real wage. The price of final good is normalized to one. In our model, two generations overlap. Thus, the length of the period is equal to the half-life of a generation and the depreciation rate of capital is close to one. We set for simplicity $\delta = 1$. Profit maximization equalizes prices and productivities: $R_{kt} = f'(k_t) \equiv R(k_t)$ and $w_t = f(k_t) - k_t f'(k_t) \equiv w(k_t)$.

Agents live two periods, work and consume when young, dissave and consume when old. Heterogeneity is twofold: young coexist with old, altruists coexists with egoists. In the following, we will denote by $i = a, e$ the altruistic and egoist (selfish) households respectively. The number of individuals born at time t and belonging to the class i is given by N_t^i. Each household supplies one unit of labor when young: $N_t = N_t^a + N_t^e$. For simplicity, we assume that the classes of altruists and egoists grow at the same rate: $n = N_{t+1}^i/N_t^i$ (each household give birth to n children). Thus, $n = N_{t+1}/N_t$ and the fraction of altruists in total population $\pi = N_t^a/N_t$ remains a constant over time.

Savings are diversified at time t through two asset demands: physical capital K_{t+1}^i and a pure bubble B_{t+1}^i. Capital letters denote aggregate variables, while small letters the same variables normalized by the size of a class: $\left(b_t^i, k_t^i\right) = \left(B_t^i/N_t^i, K_t^i/N_t^i\right)$. The portfolio of an individual of generation t becomes $n\left(b_{t+1}^i, k_{t+1}^i\right)$, where the notational choice of for asset demands with the temporal subscript $t+1$ stresses the forward-looking nature of these variables. The whole stocks of assets are given by $B_t = N_t^a b_t^a + N_t^e b_t^e$ and $K_t = N_t^a k_t^a + N_t^e k_t^e$, while the per capita value by:

$$b_t = \pi b_t^a + (1-\pi) b_t^e \tag{1}$$

$$k_t = \pi k_t^a + (1-\pi) k_t^e \tag{2}$$

Agents consume c_t^i when young and d_{t+1}^i when old. They supply one unit of labor, paid w_t when young, and spend, when old, the return on portfolio $n\left(R_{bt+1} b_{t+1}^i + R_{kt+1} k_{t+1}^i\right)$, where R_{bt+1} and R_{kt+1} denote the gross interest rates on bubble and capital respectively. A selfish agent maximizes the utility function $u\left(c_t^e, d_{t+1}^e\right)$ under her budget constraints.

Assumption 2 *The utility function u is C^2, strictly increasing and strictly concave.*

An egoist born at time t maximizes the two-period utility function

$$U_t^e \equiv u\left(c_t^e, d_{t+1}^e\right) \tag{3}$$

under two budget constraints (one per period) and two additional non-negativity constraints:

$$\max_{c_t^e, d_{t+1}^e, b_{t+1}^e, k_{t+1}^e} u\left(c_t^e, d_{t+1}^e\right)$$

subject to

$$\begin{aligned} c_t^e + n\left(b_{t+1}^e + k_{t+1}^e\right) &\leq w_t \\ d_{t+1}^e &\leq n\left(R_{bt+1} b_{t+1}^e + R_{kt+1} k_{t+1}^e\right) \\ b_{t+1}^e, k_{t+1}^e &\geq 0 \end{aligned}$$

An altruistic agent born at time t maximizes the utility function U_t^a and takes in account her own children' welfare (nU_{t+1}^a). The simplest representation is a linear combination of a two-period selfish utility (as above) and a weighted altruistic component:

$$U_t^a \equiv u\left(c_t^a, d_{t+1}^a\right) + \alpha n U_{t+1}^a \tag{4}$$

with

$$\alpha n < 1 \tag{5}$$

Here, the weight α captures the degree of individual altruism, the strength of intertergenerational link. As shown by Barro and Becker (1989), an OLG model of capital accumulation where all agents share the same preferences (4) is equivalent to a Ramsey model with intertemporal utility:

$$\sum_{\tau=0}^{\infty} (\alpha n)^\tau u\left(c_{t+\tau}^a, d_{t+\tau+1}^a\right) \tag{6}$$

The Inada condition $f'(+\infty) = 0$ (Assumption 1) implies that any feasible sequence of capital (and consumption) is bounded. Hence, condition (5) ensures the convergence of this series.

In our model, parents affect the utility level of progeny through bequests. Any altruistic agent born at time $t + \tau$ receives a bequest $h_{t+\tau} = H_{t+\tau}/N^a_{t+\tau}$ from parents when young and leaves a bequest $h_{t+\tau+1}$ to the n children when old.[1] An altruist maximizes the utility (6) facing a sequence of budget constraints and nonnegativity constraints:

$$\max_{\left(c^a_{t+\tau}, d^a_{t+\tau+1}, h_{t+\tau+1}, b^a_{t+\tau+1}, k^a_{t+\tau+1}\right)^{\infty}_{\tau=0}} \sum_{\tau=0}^{\infty} (\alpha n)^{\tau} u\left(c^a_{t+\tau}, d^a_{t+\tau+1}\right)$$

subject to

$$\begin{aligned} c^a_{t+\tau} + n\left(b^a_{t+\tau+1} + k^a_{t+\tau+1}\right) &\leq w_{t+\tau} + h_{t+\tau} \\ d^a_{t+\tau+1} + nh_{t+\tau+1} &\leq n\left(R_{bt+\tau+1} b^a_{t+\tau+1} + R_{kt+\tau+1} k^a_{t+\tau+1}\right) \\ h_{t+\tau+1}, b^a_{t+\tau+1}, k^a_{t+\tau+1} &\geq 0 \end{aligned}$$

for $\tau = 0, \ldots, \infty$. At equilibrium, we find $R_{bt} = R_{kt} \equiv R_t$ for $t = 1, 2, \ldots$

Selfish and altruistic agents maximize two different objectives: the utility function (3) and the series (6). Even if they smooth consumption similarly:

$$\frac{u_c(c^e_t, d^e_{t+1})}{u_d(c^e_t, d^e_{t+1})} = R_{t+1}, \quad \frac{u_c(c^a_{t+\tau}, d^a_{t+\tau+1})}{u_d(c^a_{t+\tau}, d^a_{t+\tau+1})} = R_{t+\tau+1}$$

they face different constraints. The selfish agent satisfies two standard budget constraints: $c^e_t + n\left(b^e_{t+1} + k^e_{t+1}\right) = w_t$ and $R_{t+1} n\left(b^e_{t+1} + k^e_{t+1}\right) = d^e_{t+1}$, now binding, while the altruistic agent faces an additional first-order condition: $\alpha u_c\left(c^a_{t+\tau+1}, d^a_{t+\tau+2}\right) = u_d\left(c^a_{t+\tau}, d^a_{t+\tau+1}\right) - \nu_{t+\tau}$, and specific budget constraints: $c^a_{t+\tau} + n\left(b^a_{t+\tau+1} + k^a_{t+\tau+1}\right) = w_{t+\tau} + h_{t+\tau}$ and $R_{t+\tau+1} n\left(b^a_{t+\tau+1} + k^a_{t+\tau+1}\right) = d^a_{t+\tau+1} + nh_{t+\tau+1}$, with $\nu_{t+\tau} h_{t+\tau+1} = 0$. Thus, if $h_{t+\tau+1} > 0$, then $\nu_{t+\tau} = 0$ and $\alpha u_c\left(c^a_{t+\tau+1}, d^a_{t+\tau+2}\right) = u_d\left(c^a_{t+\tau}, d^a_{t+\tau+1}\right)$, while, if $\nu_{t+\tau} > 0$ and, then, $h_{t+\tau+1} = 0$, $\alpha u_c\left(c^a_{t+\tau+1}, d^a_{t+\tau+2}\right) < u_d\left(c^a_{t+\tau}, d^a_{t+\tau+1}\right)$. This program is time-consistent.

Without loss of generality, we will consider equations with $\tau = 0$ from now on. Moreover, for simplicity, we will assume that the public authority faces an oversimplified budget constraint: $B_{t+1} = R_t B_t$.[2] Equivalently, we find $nb_{t+1} = R_t b_t$.

[1] In our model, we do not allow for a transition from altruistic parents to selfish offsprings or vice-versa. For simplicity, altruism is a matter of cultural transmission. The introduction of a Markovian transition matrix with exogenous probabilities could be a natural extension of the model.

[2] Because of its tractability, this case is often considered in the macroeconomic literature on the sovereign debt. A government facing a budget constraint without taxes and public spending and paying only the debt service is said to roll over the debt.

3 Equilibrium

In equilibrium, aggregate demand equals aggregate supply in every market (assets, labor and consumption good).

Definition 1. *An equilibrium is a positive sequence of prices and quantities*

$$\left(R_t, w_t, h_t, b_t, k_{t+1}, \left(b_t^i, k_{t+1}^i, c_t^i, d_{t+1}^i\right)_{i=a,e}\right)_{t=0}^{\infty}$$

such that the producers maximize the profit at any period; selfish households maximize the utility function (3) and altruist households the utility function (6) under their respective budget constraints; markets of financial asset, capital, labor and goods clear.

Proposition 1. *The equilibrium trajectory satisfies the following market clearing and first-order conditions that, under Assumptions 1 and 2, are not only necessary but also sufficient for profit and utility maximizations.*

$$\begin{aligned}
\frac{u_c\left(c_t^e, d_{t+1}^e\right)}{u_d\left(c_t^e, d_{t+1}^e\right)} &= \frac{u_c\left(c_t^a, d_{t+1}^a\right)}{u_d\left(c_t^a, d_{t+1}^a\right)} = R\left(k_{t+1}\right) && (7)\\
\nu_t + \alpha u_c\left(c_{t+1}^a, d_{t+2}^a\right) &= u_d\left(c_t^a, d_{t+1}^a\right) \text{ with } \nu_t h_{t+1} = 0 && (8)\\
c_t^e + n\left(b_{t+1}^e + k_{t+1}^e\right) &= w\left(k_t\right) && (9)\\
c_t^a + n\left(b_{t+1}^a + k_{t+1}^a\right) &= w\left(k_t\right) + h_t && (10)\\
nR\left(k_{t+1}\right)\left(b_{t+1}^e + k_{t+1}^e\right) &= d_{t+1}^e && (11)\\
nR\left(k_{t+1}\right)\left(b_{t+1}^a + k_{t+1}^a\right) &= d_{t+1}^a + nh_{t+1} && (12)\\
\pi k_t^a + (1-\pi) k_t^e &= k_t && (13)\\
\pi b_t^a + (1-\pi) b_t^e &= b_t && (14)\\
R\left(k_t\right) b_t &= nb_{t+1} && (15)
\end{aligned}$$

and the transversality condition: $\lim_{t\to\infty} (\alpha n)^t u_c\left(c_t^a, d_{t+1}^a\right)\left(b_{t+1}^a + k_{t+1}^a\right) = 0$, *for altruistic agents.*

Proof. See the Appendix. □

In the following, we will focus on equilibrium trajectory close to the steady state. Two steady state regimes may arise: without bequests ($h = 0$) or with bequests ($h > 0$). The bequests of an equilibrium transition sufficiently close to a steady state with null (positive) bequests will be null (positive) by continuity.

3.1 Zero bequests

Altruistic preferences do not necessarily imply positive bequests. Indeed, a positive intergenerational link ($\alpha > 0$) is compatible with null bequests as we will see below.

Let k^D be the solution of the following system:

$$\frac{u_c(c,d)}{u_d(c,d)} = R(k) \quad (16)$$
$$d = R(k)[w(k)-c] \quad (17)$$
$$d = nkR(k) \quad (18)$$

This solution is exactly the steady state of the seminal Diamond's model (1965).

Proposition 2 (steady state). *In the case without bequests, we recover the equilibrium à la Tirole (1985) (where a bubbly steady state coexists with a bubbleless one in the case of overaccumulation) with the additional restriction*

$$\alpha \leq \alpha^* \equiv \frac{1}{R(k^D)} \quad (19)$$

(low altruism degree) in the case of underaccumulation.

Proof. See the Appendix. □

Without bequests, the intergenerational link no longer works, altruism no longer matters[3] and there is room for bubbles. If the degree of altruism is sufficiently small ($\alpha \leq 1/R\left(k^D\right)$), the dominant consumer (the altruist) behaves as an agent à la Tirole (1985) instead of an agent à la Ramsey (1928) and, unsurprisingly, we recover the Tirole's conclusions.

As shown by Tirole, a persistent bubble exists if the steady state à la Tirole-Diamond is characterized by a low interest rate, that is capital overaccumulation. In this case, the bubble reabsorbs the oversaving and restores the golden rule à la Tirole-Phelps: $R(k) = n$.

3.2 Positive bequests

The main result of the paper is that bequests and bubbles are incompatible, whatever the proportion of altruistic agents.

Proposition 3 (steady state). *If bequests are positive, there are no bubbles, whatever the share π of altruists in total population.*

Proof. See the Appendix. □

Thus, even if there are very few altruists (π close to zero), if bequests are positive, bubbles are ruled out. This result holds not only at the steady state but also in a neighborhood of the steady state.[4] Nevertheless, we will show

[3] Altruistic and selfish agents turn out to have the same behavior in terms of consumption and portfolio choice.

[4] An equilibrium trajectory sufficiently close to a steady state with positive bequests exhibits also positive bequests at any date.

in a Cobb-Douglas example that positive bequests requires a sufficiently large intergenerational link α.

Let us explain the intuition of Proposition 1 and refer to the existing literature on rational bubbles. When bequests are positive, the interest factor becomes $R = 1/\alpha$. According to the restriction $\alpha n < 1$, the interest rate $R = 1/\alpha$ is larger than the population growth rate n. In this case, the bubbles are impossible. This interpretation is coherent with Proposition 1 (part a) in Tirole (1985).

Condition $\alpha n < 1$ also implies that, at the steady state, the present value $\sum_{t=0}^{\infty} (\alpha n)^t f(k)$ is finite. By consequence, our result is related to the nonexistence of bubbles in infinite-horizon general equilibrium models when the present value of aggregate outputs is finite (Santos and Woodford, 1997; Bosi, Le Van and Pham, 2018).[5]

4 A Cobb-Douglas example

To study the global dynamics, let us introduce the Cobb-Douglas fundamentals:

$$f(k_t) = Ak_t^s \text{ and } u\left(c_t^i, d_{t+1}^i\right) \equiv (1-\sigma)\ln c_t^i + \sigma \ln d_{t+1}^i \tag{20}$$

In this case, the trajectories of bubble, capital stock and consumption can be computed explicitly.

Profit maximization entails $R(k_t) = sAk_t^{s-1}$ and $w(k_t) = (1-s)Ak_t^s$.

Lemma 1. *In the case of Cobb-Douglas technology and logarithmic preferences, the dynamic system is given by*

$$\frac{1-\sigma}{\sigma}\frac{d_{t+1}^e}{c_t^e} = \frac{1-\sigma}{\sigma}\frac{d_{t+1}^a}{c_t^a} = R(k_{t+1}) \tag{21}$$

$$\nu_t + \alpha\frac{1-\sigma}{c_{t+1}^a} = \frac{\sigma}{d_{t+1}^a} \tag{22}$$

(equations (7)-(8), now explicit) jointly with equations (9) to (15).

Proof. See the Appendix. □

4.1 Zero bequests

The trajectories of bubble, capital stock and consumption can be computed explicitly in the Cobb-Douglas case (20). These solutions are known (see, among others, Bosi, Ha-Huy, Le Van, Pham and Pham, 2018) and we would like just to observe that, even if the system determines the total savings of agent $i = a, e$: $n\left(b^i + k^i\right) = \sigma w(k)$, it does not determine the sharing between the assets b^i and k^i. The reason is that the returns on capital and the bubble are the same

[5] The reader is referred to Pham (2018) for the relationship between OLG models and general equilibrium models with infinite-lived agents.

(R) and agents are indifferent to invest in capital or in the bubble. However, in equilibrium, the total saving of an individual $\left(n\left(b^i+k^i\right)\right)$ is determined.

In the case of fundamentals (20), the solution of the Diamond's system (16) to (18) gives $n/R\left(k^D\right)=\sigma\left(1-s\right)/s$, that is

$$k^D=\left[\left(1-s\right)A\frac{\sigma}{n}\right]^{\frac{1}{1-s}}$$

Overaccumulation in the Diamond's steady state corresponds to $R\left(k^D\right)<R\left(k^*\right)\equiv n$, that is to a larger propensity to save: $\sigma>s/\left(1-s\right)$. We have seen that the positivity of the bubble requires a low interest rate or, equivalently, a capital overaccumulation in the bubbleless steady state à la Diamond.

It is easy to check that, in the Cobb-Douglas case, the critical degree of altruism in (19) becomes

$$\alpha^*\equiv\frac{1}{R\left(k^D\right)}=\frac{\sigma}{n}\frac{1-s}{s} \tag{23}$$

As we will see in the next section (Proposition 4), in the Cobb-Douglas case, α^* is exactly the critical degree of altruism such that below ($\alpha\leq\alpha^*$) bequests are zero and above ($\alpha>\alpha^*$) bequests are positive.

We observe also that, in the Cobb-Douglas case with zero bequests, restriction (19) is always satisfied and becomes superfluous. Indeed, according to Proposition 4, zero bequests entail $\alpha\leq\alpha^*$.

4.2 Positive bequests

Positive bequests imply null multipliers ($\nu_t=0$) and no bubbles ($b_t=b_t^e=b_t^a=0$) not only at the steady state but also along the equilibrium transition. We compute first all the variables at the steady state and, then, the explicit trajectories for capital and consumption.

As we will see in the next two propositions, the properties of the steady state and the equilibrium path depend, in both the cases, on the same parametric expressions (the eigenvalues):

$$\lambda_\pm\equiv\frac{1}{2\alpha ns}\left[\gamma\pm\sqrt{\gamma^2-4s\sigma\left(1-\pi\right)\left(1-s\right)}\right]$$

with $\gamma\equiv1-\left(1-\pi\right)\left(1-s\right)\left(1-\sigma\right)$ and $\gamma^2-4s\sigma\left(1-\pi\right)\left(1-s\right)\geq0$.

Proposition 4 (steady state). *In the Cobb-Douglas case, the aggregate capital intensity is given by* $k=\left(\alpha sA\right)^{1/(1-s)}$. *The distribution of wealth and* *consumption between the selfish (e) and altruistic (a) agents is given by*

$$
\begin{aligned}
k^e &= \frac{1-s}{s}\frac{\sigma}{\alpha n}k && (24)\\
k^a &= \left(\frac{1}{\pi} - \frac{1-\pi}{\pi}\frac{1-s}{s}\frac{\sigma}{\alpha n}\right)k && (25)\\
(c,d)^e &= \left(1-\sigma, \frac{\sigma}{\alpha}\right)\frac{1-s}{s}\frac{k}{\alpha} && (26)\\
(c,d)^a &= \left(1-\sigma, \frac{\sigma}{\alpha}\right)\frac{1-s}{s}\frac{k}{\alpha}\left[\frac{1}{\pi}\frac{1-s\alpha n}{1-s}\frac{\alpha n}{\sigma+(1-\sigma)\alpha n} - \frac{1-\pi}{\pi}\right] && (27)
\end{aligned}
$$

Altruistic agents leave bequests to children:

$$h = \frac{1}{s\alpha\pi}\frac{s\alpha n-(1-s)\sigma}{\sigma+(1-\sigma)\alpha n}k \tag{28}$$

Thus, bequests are positive if and only if $\alpha > \alpha^*$ (the degree altruism is sufficiently large) where α^* is precisely given by (23).

Proof. See the Appendix. □

Nonnegativity of k^a requires $\alpha \geq (1-\pi)\alpha^*$ which is ensured by $\alpha \geq \alpha^*$, while the consumptions are nonnegative if and only if[6]

$$\lambda_- \leq 1 \leq \lambda_+ \tag{29}$$

We observe that the aggregate capital intensity $k = (\alpha s A)^{1/(1-s)}$ is increasing in the degree of altruism. Altruistic agents behave as patient agents à la Ramsey as shown by Barro (1974) and, *in principio*, they save more for future generations in terms of capital.

We notice also that, if all the agents are altruistic ($\pi = 1$), then $k^a = k$ and the consumption demands simplify:

$$(c,d)^a = (c,d) = \left(1-\sigma, \frac{\sigma}{\alpha}\right)\frac{1-s\alpha n}{\sigma s+(1-\sigma)s\alpha n}nk$$

Under a Cobb-Douglas specification, the explicit trajectories can be computed even in the case of altruism and bequests. Let

$$
\begin{aligned}
x &\equiv \frac{1}{1-\sigma}\frac{s\alpha n-(1-s)\sigma}{s\alpha n(1-\alpha n)+(1-s)[\pi\alpha n-\sigma(1-\pi)(1-\alpha n)]} && (30)\\
y &\equiv \frac{1}{1-\sigma}\frac{\pi[\sigma+(1-\sigma)\alpha n]}{s\alpha n(1-\alpha n)+(1-s)[\pi\alpha n-\sigma(1-\pi)(1-\alpha n)]} && (31)
\end{aligned}
$$

and $z_0 \equiv [(\lambda_- - a_{22})y - a_{21}x] / [(\lambda_- - a_{22})Ak_0^s - a_{21}h_0]$ with $a_{21} = \pi/(s\alpha n)$ and $a_{22} = [\pi + (1-\pi)\sigma](1-s)/(s\alpha n)$.

[6] If, for instance, $\alpha = \pi = s = \sigma = 1/2$ and $n = 2-\varepsilon$ with $\varepsilon > 0$ arbitrarily small in order to satisfy (5), (23) is verified and (29) becomes

$$0 < \alpha\lambda_- = 0.084\,86 \leq \alpha^* = 0.25 \leq \alpha\lambda_+ = 0.265\,14 < 1/2$$

Proposition 5 (global dynamics with bequests). *In the Cobb-Douglas case, the steady state is a saddle point and the aggregate capital sequence along the stable branch is explicitly given by*

$$k_t = k_0^{s^t} \prod_{\tau=1}^{t} \left[\alpha s A \frac{\lambda_- - a_{22} - a_{21} \frac{x + (h_0 z_0 - x)\lambda_-^{t-\tau}}{y + \left(A k_0^s z_0 - y\right)\lambda_-^{t-\tau}}}{\lambda_- - a_{22} - a_{21} \frac{x + (h_0 z_0 - x)\lambda_-^{t-\tau+1}}{y + \left(A k_0^s z_0 - y\right)\lambda_-^{t-\tau+1}}} \right]^{s^{\tau-1}} \tag{32}$$

while the bequest dynamics by the capital path (32):

$$h_t = h_0 \prod_{\theta=1}^{t} \frac{\sigma}{\sigma-1} \frac{R(k_\theta)}{n} + \sum_{\tau=0}^{t-2} g_\tau \prod_{\theta=\tau+2}^{t} \frac{\sigma}{\sigma-1} \frac{R(k_\theta)}{n} + g_{t-1} \tag{33}$$

with

$$g_t \equiv \frac{R(k_{t+1})}{1-\sigma} \frac{1}{\pi} \left[k_{t+1} - \frac{\sigma}{n} w(k_t) \right] \tag{34}$$

The trajectories of individual consumption and wealth are determined by the sequences of aggregate capital and bequest ((32) and (33)):

$$\left(c_t^e, d_{t+1}^e\right) = (1-\sigma, \sigma R(k_{t+1}))\, w(k_t) \tag{35}$$

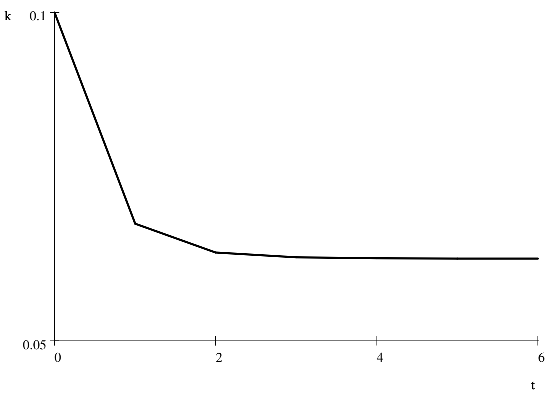
$$\left(c_t^a, d_{t+1}^a\right) = (1-\sigma, \sigma R(k_{t+1})) \left[\frac{n}{\sigma} \frac{1}{\pi} k_{t+1} - \frac{n}{\sigma} \frac{h_{t+1}}{R(k_{t+1})} - \frac{1-\pi}{\pi} w(k_t) \right] \tag{36}$$

$$k_{t+1}^e = \frac{\sigma}{n} w(k_t) \tag{37}$$

$$k_{t+1}^a = \frac{1}{\pi} k_{t+1} - \frac{1-\pi}{\pi} \frac{\sigma}{n} w(k_t) \tag{38}$$

Proof. See the Appendix. □

In the case of bequests, the aggregate capital path can be easily computed according to (32). If, for instance, $\alpha = \pi = s = \sigma = 1/2$, $A = 1$, $n = 2 - \varepsilon$ and $h_0 = k_0 = 0.1$, we obtain the following convergence to the steady state along the saddle path taking $\varepsilon > 0$ close to zero:



4.3 Summing up

The following concluding corollary sums up all the main results of the section.

Corollary 1. *In the Cobb-Douglas case (20), under low altruism ($\alpha \leq \alpha^*$), bequests are zero and our model becomes exactly the Tirole (1985) model, while, under high altruism ($\alpha > \alpha^*$), bequests are positive, Proposition 4 applies and bubbles are ruled out.*

5 Conclusion

We have bridged the OLG literature on rational bubbles à la Tirole (1985) with the literature à la Barro (1974) with altruism from parents to children. We have considered a population of heterogenous agents (selfish and altruistic).

Positive bequests require a sufficiently large degree of descendent altruism. Under positive bequests, any rational bubble is ruled out not only at the steady state but also in a neighborhood. This result is robust and holds whatever the share of altruistic agents in total population. In this respect, our contribution addresses the question of robustness of rational bubbles in Tirole (1985).

Eventually, we have also computed the global dynamics in the case of Cobb-Douglas production and utility functions.

6 Appendix

Proof of Proposition 1.

Focus on necessity. By the no-arbitrage conditions, at equilibrium, we get $R_{bt} = R_{kt} = R(k_t)$ for any $t \geq 1$ and equation (15) as in Tirole (1985). Equation (7) captures the agents' consumption smoothing. Denoting by ν_t the Lagrangian multiplier associated with the non-negativity constraint $h_{t+1} \geq 0$, we obtain equation (8) as in Barro and Becker (1989). Equations (9), (10), (11), (12) represent the budget constraints, now binding. Equations (13) and (14) capture the capital and bubble markets clearing. Since the altruistic agent behaves like a Ramsey agent, the transversality condition is satisfied. These conditions are also sufficient. A standard proof applies: because of the convexity of maximization programs, a sequence $\left(R_t, w_t, h_t, b_t, k_{t+1}, \left(b_t^i, k_{t+1}^i, c_t^i, d_{t+1}^i\right)_{i=a,e}\right)_{t=0}^{\infty}$ satisfying equations (7) to (15) is an equilibrium.

Proof of Proposition 2.

In the case of null bequests, $h_t = 0$, the dynamical system simplifies:

$$\begin{aligned}
\frac{u_c\left(c_t^i, d_{t+1}^i\right)}{u_d\left(c_t^i, d_{t+1}^i\right)} &= R\left(k_{t+1}\right) && (39)\\
d_{t+1}^i &= R\left(k_{t+1}\right)\left[w\left(k_t\right) - c_t^i\right] && (40)\\
b_{t+1}^i + k_{t+1}^i &= \frac{d_{t+1}^i}{nR\left(k_{t+1}\right)} && (41)\\
R\left(k_t\right) b_t &= n b_{t+1}
\end{aligned}$$

where $i = a, e$, with $\pi k_t^a + (1-\pi) k_t^e = k_t$, $\pi b_t^a + (1-\pi) b_t^e = b_t$ and

$$\alpha \leq \frac{u_d\left(c_t^a, d_{t+1}^a\right)}{u_c\left(c_{t+1}^a, d_{t+2}^a\right)} = \frac{1}{R\left(k_{t+1}\right)} \tag{42}$$

Since both the types of agents share the same preferences u, under Assumption 2 (namely strict concavity), equations (39) to (41) imply: $c_t^a = c_t^e = c_t$, $d_{t+1}^a = d_{t+1}^e = d_{t+1}$ and $b_{t+1}^a + k_{t+1}^a = b_{t+1}^e + k_{t+1}^e = b_{t+1} + k_{t+1}$.

Therefore, the dynamical system becomes

$$\begin{aligned}
\frac{u_c\left(c_t, d_{t+1}\right)}{u_d\left(c_t, d_{t+1}\right)} &= R\left(k_{t+1}\right)\\
d_{t+1} &= R\left(k_{t+1}\right)\left[w\left(k_t\right) - c_t\right]\\
b_{t+1} + k_{t+1} &= \frac{d_{t+1}}{nR\left(k_{t+1}\right)}\\
R\left(k_t\right) b_t &= n b_{t+1}
\end{aligned}$$

as in Tirole (1985) with the additional restriction

$$\alpha \leq \frac{1}{R\left(k_{t+1}\right)} \tag{43}$$

inherited from (42).

Let k^D be the solution of the previous system with $b = 0$ computed at the steady state, that is of system (16) to (18) (Diamond, 1965).

As in Tirole (1985), there are three regimes.

(1) Capital underaccumulation (at the Diamond's steady state) when $k^D < k^* \equiv R^{-1}(n)$: there is no bubble.

(2) Golden rule (at the Diamond's steady state) when $k^D = k^*$: there is no bubble.

(3) Capital overaccumulation (at the Diamond's steady state) when $k^D > k^*$: a bubbly steady state (k^*, b) with $b > 0$ (golden rule) coexists with a bubbleless one $\left(k^D, 0\right)$.

The zero bequests assumption requires restriction (43) to be satisfied. Is this the case?

(1) In the case of underaccumulation, we observe that

$$\frac{1}{R\left(k^D\right)} < \frac{1}{n}$$

and, we require (43) as a stronger condition than inequality (5):

$$\alpha \leq \frac{1}{R\left(k^D\right)}$$

(2) In the case of golden rule at the Diamond's steady state, (5) implies (43).

(3) In the case of overaccumulation, at the bubbly steady state (golden rule) (5) implies (43) and at the bubbleless steady state, (43) is also satisfied because

$$\alpha < \frac{1}{n} < \frac{1}{R\left(k^D\right)}$$

Proof of Proposition 3.

The dynamic system is given by (7)-(15). At the steady state, in the case of positive bequests, $\nu = 0$. Equations (7) and (8) imply $R(k) = 1/\alpha$. Equation (15) becomes $b = \alpha n b$. Since $\alpha n < 1$, we obtain $b = 0$. We observe that $R(k) = 1/\alpha > n$, that is we have underaccumulation of capital with respect to the golden rule ($R(k) = n$). Thus, an economy with bequests implies underaccumulation and, so, it turns out to be bubbleless irrespective of the proportion π of altruists.

Proof of Lemma 1.

Simply replace the fundamentals (20) in system (7)-(15).

Proof of Proposition 4.

$k = R^{-1}(1/\alpha) = (\alpha s A)^{1/(1-s)}$ is the capital intensity. Replacing consumption demands $c^e = w(k) - nk^e$, $c^a = w(k) - nk^a + h$, $d^e = nR(k)k^e$ and $d^a = nR(k)k^a - nh$ in the intertemporal arbitrages

$$\frac{1-\sigma}{\sigma}\frac{d^e}{c^e} = \frac{1-\sigma}{\sigma}\frac{d^a}{c^a} = \frac{1}{\alpha}$$

we find $nk^e = \sigma w(k)$ and $nk^a = \sigma w(k) + [\sigma + (1-\sigma)\alpha n]h$. Substituting them in $k = \pi k^a + (1-\pi)k^e$ and noticing that $w(k) = k(1-s)/(s\alpha)$, we obtain the bequest (28).

Proof of Proposition 5.

Combining (21) and (22) with $\nu_t = 0$ (because of the positive bequests), we find

$$\begin{bmatrix} x_t - x \\ y_t - y \end{bmatrix} = M^t \begin{bmatrix} x_0 - x \\ y_0 - y \end{bmatrix} \tag{44}$$

where $x_t \equiv h_t/c_t^a$, $y_t \equiv Ak_t^s/c_t^a$ and $(x, y)^T = (I - M)^{-1} N$ with

$$M \equiv \frac{1}{\alpha n}\begin{bmatrix} 1 & 1-s \\ \frac{\pi}{s} & [\pi + (1-\pi)\sigma]\frac{1-s}{s} \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } N \equiv -\frac{1}{\alpha n}\begin{bmatrix} \frac{1}{1-\sigma} \\ \frac{\pi}{s} \end{bmatrix}$$

We know that $c^a, d^a \geq 0$ if and only if $(0 <)\, \lambda_- \leq 1 \leq \lambda_+$, that is if and only if the steady state is a saddle point. We diagonalize (44) to obtain, along the saddle path:

$$y_0 - y = \frac{a_{21}}{\lambda_- - a_{22}}(x_0 - x) \tag{45}$$

a reduced dynamics:

$$\begin{aligned} x_t - x &= (x_0 - x)\,\lambda_-^t \\ y_t - y &= (y_0 - y)\,\lambda_-^t \end{aligned}$$

Solving this system, after tedious computations, we obtain (32) and, solving recursively forward

$$h_{t+1} = g_t + \frac{\sigma}{\sigma - 1}\frac{R\,(k_{t+1})}{n}h_t$$

from h_0, we find (33).

7 References

Alvarez, F. and U.J. Jermann (2000) "Efficiency, equilibrium, and asset pricing with risk of default" *Econometrica* **68**, 775-797.

Barro, R.J. (1974) "Are government bonds net wealth?" *Journal of Political Economy* **82**, 1095-1117.

Barro, R.J. and G.S. Becker (1989) "Fertility choice in a model of economic growth" *Econometrica* **57**, 481-501.

Bosi, S., C. Le Van and N.-S. Pham (2018) "Intertemporal equilibrium with heterogeneous agents, endogenous dividends and borrowing constraints" *Journal of Mathematical Economics* **76**, 1-20.

Bosi, S., T. Ha-Huy, C. Le Van, C.-T. Pham and N.-S. Pham (2018) "Financial bubbles and capital accumulation in altruistic economies" *Journal of Mathematical Economics* **75**, 125-139.

Diamond, P.A. (1965) "National debt in a neoclassical growth model" *American Economic Review* **55**, 1126-1150.

Nourry, C. and A. Venditti (2001) "Determinacy of equilibrium in an overlapping generations model with heterogeneous agents" *Journal of Economic Theory* **96**, 230-255.

Pham, N.-S. (2018) "On the relationship between general equilibrium models with infinite-lived agents and overlapping generations models" Mimeo.

Phelps, E.S. (1961) "Golden rule of accumulation: a fable for growthmen" *American Economic Review* **51**, 638-643.

Ramsey, F.P. (1928) "Mathematical theory of saving" *Economic Journal* **38**, 543-559.

Santos, M.S. and M. Woodford (1997) "Rational asset pricing bubbles" *Econometrica* **65**, 19-57.

Tirole, J. (1985) "Asset bubbles and overlapping generations" *Econometrica* **53**, 1499-1528.

Woodford, M. (1986) "Stationary sunspot equilibria in a finance constrained economy" *Journal of Economic Theory* **40**, 128-137.