Economics Bulletin

Volume 39, Issue 3

The Solow growth model with a CES production function and declining population

Hiroaki Sasaki
Graduate School of Economics, Kyoto University

Abstract

This study investigates the relationship between per capita output growth and population growth using the Solow growth model when population growth is negative. When the Cobb-Douglas production function is used, the per capita output growth rate can be positive even if the technological progress rate is zero. In contrast, when the CES production function is used, the per capita output growth rate is zero if the technological progress rate is zero and the elasticity of substitution between capital and labor is less than unity.

An earlier version of this paper was presented at seminars held at Ryukoku University and Saga University, and at the 2018 Japanese Economic Association Autumn Meeting in Gakushuin University. I would like to thank seminar participants and especially Ichiro Daito and Shin-ichi Fukuda for their useful comments and suggestions. Also, I thank an anonymous referee for constructive comments. I am grateful to JSPS KAKENHI Grant Number 16K03621 and The Asahi Glass Foundation for financial support. The usual disclaimer applies.
Citation: Hiroaki Sasaki, (2019) "The Solow growth model with a CES production function and declining population", *Economics Bulletin*, Volume 39, Issue 3, pages 1979-1988
Contact: Hiroaki Sasaki - sasaki@econ.kyoto-u.ac.jp.
Submitted: May 03, 2019. **Published:** September 03, 2019.

1 Introduction

This study investigates the long-run growth rate of per capita output by using the Solow growth model when population growth is negative and the production function takes the constant-elasticity-substitution (CES) form. Almost all growth models assume that population growth is non-negative, and do not consider negative population growth. However, as long as there are countries with negative population growth such as Japan, we need to investigate a case in which population growth is negative.

Japan's first postwar experience of a fall in population occurred in 2005, with negative population growth rates following in 2009 and 2011. Similarly, concern about population decline has been increasing in Italy and Germany (World Bank, 2013). Therefore, population decline is an urgent problem in developed economies.[1]

Few studies introduce negative population into economic growth models. One of the few examples is Ritschl (1985). He introduces negative population growth in the Solow's (1956) model and shows that if the population growth rate is negative, the positive steady-state value of capital stock per capita exists only when the saving rate is negative and that the steady state is unstable, and hence, capital stock per capita either approaches zero or goes to infinity.

Christiaans (2011) is also a preceding study that introduces negative population growth into a growth model. He builds a semi-endogenous growth model with an increasing returns to scale production function and shows that the long-run growth rate of per capita output can be positive even if population growth is negative. He also shows that when the constant returns to scale Cobb-Douglas production function is used and negative population growth is assumed, the long-run growth rate of per capita output can be positive even if the exogenously given technological progress rate is zero. This result suggests that technological progress may not be so important in economies with declining population. In contrast to Ritschl (1985), he assumes that the saving rate is positive.

Analytical method of Christiaans (2011) is important. As above-mentioned Ritschl (1985) shows, the steady-state does not exist when the saving rate is positive. Nevertheless, according to Christiaans (2011), we can obtain the long-run situation where the growth rate of per capita output stably approaches a constant value even if the saving rate is positive.

Christiaans (2011) only investigates the case in which the Cobb-Douglas production function is used. However, empirical studies with regard to the size of elasticity of substi-

[1] At first sight, few countries seem to have experienced negative population growth. However, we should consider the effect of immigrants. If we consider the rate of natural increase (i.e., the crude birth rate minus the crude death rate), several countries have experienced negative population growth. Indeed, according to United Nations (2013), the rates of natural population increase of 17 OECD countries were negative between 2005 and 2010.

tution suggest that the elasticity of substitution is less than unity (Rowthorn, 1999; Antràs, 2004; Klump, McAdam, and Willman, 2007; and Chirinko, 2008). Based on these empirical results, we use the CES production function. Our result shows that when the elasticity of substitution between capital and labor is less than unity, the long-run growth rate of per capita output converges to the exogenously given technological progress rate when population growth is negative. This result implies that the engine of long-run growth is technological progress. Therefore, the result obtained from the Cobb-Douglas case is largely different from the one obtained from the CES case.

Our model and Christiaans (2011) can be positioned as semi-endogenous growth (or non-scale growth) models when the population growth rate is positive. Traditional semi-endogenous growth models such as Jones (1995, 1999) explicitly assume increasing returns to scale in the production function while implicitly assume positive population growth. Christiaans (2011) shows that when the population growth rate takes a large negative value, endogenous growth occurs even if the traditional condition for endogenous growth is not satisfied, that is, the production function does not have the property of increasing returns to scale.

We use a growth model with exogenous technical change because we want to stress the difference between the case with the elasticity of substitution being less than unity and the case with the elasticity of substitution being equal to unity. Almost all existing growth models do not consider negative population growth, and accordingly, we decide to introduce negative population growth into the most basic growth model, that is, the Solow growth model with exogenous technical change. Sasaki and Hoshida (2017) introduce negative population growth into a growth model with endogenous technical change and the Cobb-Douglas production function. As long as the Cobb-Douglas production function is used, results obtained in the endogenous technical change case are similar to results obtained in the exogenous technical change case.[2]

The rest of the paper is organized as follows. Section 2 investigates the long-run growth rate of per capita output when the production function takes the Cobb-Douglas form and population growth is negative. Section 3 investigates the long-run growth rate of per capita output when the production function takes the CES form according to whether population growth is positive or negative. Section 4 concludes the paper.

[2]Introducing negative population growth into a growth model with endogenous technical change and the CES production function will be left for future research.

2 Cobb-Douglas production function with negative population growth

In this section, by using the Solow (1956) growth model, we investigate the long-run growth rate of per capita output when population growth is negative.

Suppose that the production function takes the following Cobb-Douglas form:

$$Y = K^{\alpha}(AL)^{1-\alpha}, \quad 0 < \alpha < 1, \tag{1}$$

where Y denotes output; K, capital stock; L, labor input; and A, an index of technology. The parameter α denotes the output elasticity with respect to capital.

Let capital stock per effective labor and output per effective labor be $k = K/(AL)$ and $y = Y/(AL)$, respectively. Then, from equation (1), we obtain $y = k^{\alpha}$. Suppose that a constant fraction s of total output Y is saved. We assume that the growth rate of population is constant $\dot{L}/L = n$, the growth rate of technological progress is constant $\dot{A}/A = \gamma > 0$, and the rate of capital depreciation is constant $\delta \in [0, 1]$. Then, the dynamics of k is given by

$$\dot{k} = sk^{\alpha} - (n + \gamma + \delta)k. \tag{2}$$

Following the procedure of Christiaans (2011), we investigate the case in which population growth is negative.

Suppose that $n < 0$ but $n + \gamma + \delta > 0$. Then, along the balanced growth path, we obtain $g^{*}_{Y/L} = \gamma$ as in the case of $n > 0$, where g_x denotes the growth rate of a variable x and "$*$" denotes the long-run value.

Suppose, on the other hand, that $n < -(\gamma + \delta) < 0$, that is, the population growth rate is negative and the absolute value of it is large. Then, from equation (2), we have $\dot{k} > 0$ as long as $k > 0$. Hence, k continues to increase. The growth rate of k is given by

$$g_k = sk^{\alpha-1} - (n + \gamma + \delta) > 0. \tag{3}$$

With $\alpha - 1 < 0$, as k increases, we obtain

$$\lim_{k\to+\infty} g_k = -(n + \gamma + \delta) > 0. \tag{4}$$

This suggests that the growth rate of k could be positive even if population growth is negative.

The growth rate of per capita output Y/L leads to

$$\begin{aligned} g^*_{Y/L} &= \gamma + g^*_y = \gamma + \alpha g^*_k \\ &= \gamma - \alpha(n + \gamma + \delta) > 0. \end{aligned} \tag{5}$$

When the technological progress rate is zero, that is, $\gamma = 0$, from equation (5), we obtain

$$g^*_{Y/L} = -\alpha(n + \delta) > 0. \tag{6}$$

Therefore, the long-run growth rate of per capita output could be positive even if the technological progress rate is zero as long as population growth is negative. In contrast, when population growth is positive, $g^*_{Y/L}$ is given by $g^*_{Y/L} = \gamma$. Accordingly, when the technological progress rate is zero, we have $g^*_{Y/L} = 0$. However, when population growth is negative, per capita output can continue to increase even without technological progress, which is a strong result.

Summarizing the above results, we obtain Figure 1 for the Cobb-Douglas production function case. In Figure 1, the horizontal axis corresponds to the population growth rate while the vertical axis corresponds to the long-run growth rate of per capita output.

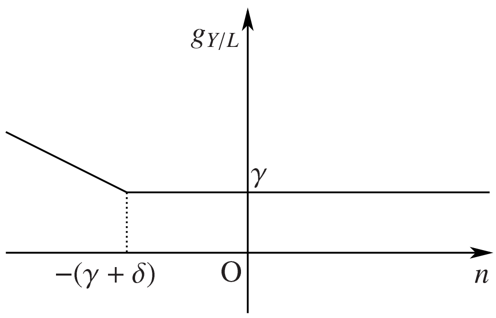

Figure 1: Long-run relationship between n and $g_{Y/L}$: Cobb-Douglas production function

The intuition behind the above result is as follows. When $n + \gamma + \delta < 0$, that is, the effective depreciation rate is sufficiently lower than zero, the effective depreciation rate will never catch up with investment, and accordingly, capital stock and hence output will grow forever in per effective labor units.

3 Solow model with a CES production function

3.1 Positive population growth

Suppose that the production function takes the following CES form.

$$Y = [\alpha K^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(AL)^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}}, \quad 0 < \alpha < 1,\ \sigma > 0, \tag{7}$$

where α denotes a constant parameter and σ, the elasticity of substitution between capital and labor.

From equation (7), output per effective labor is given by

$$y = f(k) = [\alpha k^{\frac{\sigma-1}{\sigma}} + (1-\alpha)]^{\frac{\sigma}{\sigma-1}}. \tag{8}$$

Using equation (8), we obtain the growth rate of per capita output as follows:

$$g_{Y/L} = \gamma + g_y, \tag{9}$$

$$g_y = \pi(k) g_k = \frac{1}{1 + \frac{1-\alpha}{\alpha} k^{\frac{1-\sigma}{\sigma}}} g_k, \tag{10}$$

where $\pi(k)$ denotes the capital share of income.

The equation of motion of k is given by

$$\dot{k} = s[\alpha k^{\frac{\sigma-1}{\sigma}} + (1-\alpha)]^{\frac{\sigma}{\sigma-1}} - (n+\gamma+\delta)k. \tag{11}$$

When the production function takes the CES form, Inada conditions may not hold in particular situations (Barro and Sala-i-Martin, 2003). Then, we may not have $k > 0$ such that $\dot{k} = 0$.

The marginal productivity of capital is given by

$$f'(k) = \alpha[\alpha + (1-\alpha)k^{\frac{1-\sigma}{\sigma}}]^{\frac{1}{\sigma-1}}. \tag{12}$$

When $0 < \sigma < 1$, with regard to the marginal productivity of capital, we obtain the following relationships:

$$\lim_{k\to 0} f'(k) = \alpha^{\frac{\sigma}{\sigma-1}}, \tag{13}$$

$$\lim_{k\to\infty} f'(k) = 0. \tag{14}$$

From these, k converges to $k = 0$ if the following condition holds:

$$n + \gamma + \delta > s\alpha^{\frac{\sigma}{\sigma-1}}. \tag{15}$$

We obtain the long-run growth rate of per capita output as follows:

$$g^*_{Y/L} = s\alpha^{\frac{\sigma}{\sigma-1}} - (n + \delta), \tag{16}$$

which can be positive or negative depending on the size of n.

On the other hand, k converges to $k^* > 0$ if the following condition holds:

$$n + \gamma + \delta < s\alpha^{\frac{\sigma}{\sigma-1}}. \tag{17}$$

In this case, the economy converges to the balanced growth path. Then, the long-run growth rate of per capita output leads to

$$g^*_{Y/L} = \gamma > 0. \tag{18}$$

When $\sigma > 1$, with regard to the marginal product of capital, we obtain the following relationships.

$$\lim_{k \to 0} f'(k) = \infty, \tag{19}$$

$$\lim_{k \to \infty} f'(k) = \alpha^{\frac{\sigma}{\sigma-1}}. \tag{20}$$

From this, k converges to $k^* > 0$ if the following condition holds:

$$n + \gamma + \delta > s\alpha^{\frac{\sigma}{\sigma-1}}. \tag{21}$$

In this case, the economy converges to the balanced growth path. Then, the long-run growth rate of per capita output is given by

$$g^*_{Y/L} = \gamma > 0. \tag{22}$$

On the other hand, $\dot{k} > 0$ as long as $k > 0$ if the following condition holds.

$$n + \gamma + \delta < s\alpha^{\frac{\sigma}{\sigma-1}}. \tag{23}$$

Then, we obtain

$$\lim_{k\to\infty} g_k = s\alpha^{\frac{\sigma}{\sigma-1}} - (n+\gamma+\delta) > 0, \tag{24}$$

which implies endogenous growth. From this, the long-run growth rate of per capita output is given by

$$\begin{aligned} g_{Y/L}^* &= \gamma + g_y^* \\ &= s\alpha^{\frac{\sigma}{\sigma-1}} - (n+\delta) > 0. \end{aligned} \tag{25}$$

3.2 Negative population growth

We consider a case in which $n < -(\gamma+\delta) < 0$.

$$g_k = s[\alpha + (1-\alpha)k^{\frac{1-\sigma}{\sigma}}]^{\frac{\sigma}{\sigma-1}} - \underbrace{(n+\gamma+\delta)}_{-} > 0. \tag{26}$$

Accordingly, k continues to increase.

When $0 < \sigma < 1$, from equation (26), we obtain

$$\lim_{k\to\infty} g_k = -(n+\gamma+\delta) > 0. \tag{27}$$

Then, from equation (10), we obtain $\lim_{k\to\infty} g_y = 0$, and accordingly, from equation (9), the long-run growth rate of per capita output is given by

$$g_{Y/L}^* = \gamma > 0. \tag{28}$$

Therefore, $g_{Y/L}^*$ is equal to the technological progress rate.

When the elasticity of substitution is less than unity, Y/L cannot grow indefinitely without technological progress even if k grows indefinitely. This result is easy to understand if we consider an extreme case in which the production function takes the Leontief form $y = \min\{k, 1\}$, that is, the elasticity of substitution is zero. In this case, y has an upper limit $y = 1$ when $k \geq 1$. Therefore, even if k continues to increase, y remains constant, and Y/L also remains constant without technological progress. In traditional semi-endogenous growth model such Jones (1995), y increases as long as k increases. When population growth is negative, y may not increase even if k continues to increase.

When $\sigma > 1$, from equation (26), we obtain

$$\lim_{k\to\infty} g_k = s\alpha^{\frac{\sigma}{\sigma-1}} - (n+\gamma+\delta) > 0. \tag{29}$$

Then, from equation (10), we have $\lim_{k\to\infty} g_y = g_k$, and from equation (9), the long-run growth rate of per capita output is given by

$$g^*_{Y/L} = s\alpha^{\frac{\sigma}{\sigma-1}} - (n+\delta) > 0. \tag{30}$$

Summarizing the above results, we obtain Figures 2 and 3. In these figures, $\bar{n} = s\alpha^{\frac{\sigma}{\sigma-1}} - \gamma - \delta > 0$.

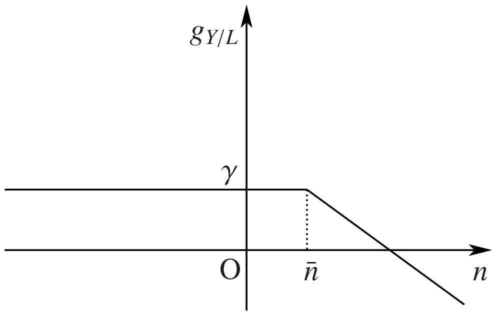

Figure 2: Long-run relationship between n and $g_{Y/L}$: CES production function and $\sigma < 1$

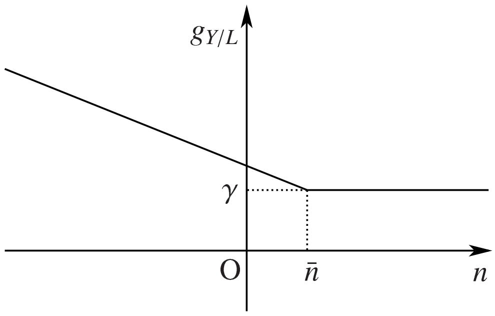

Figure 3: Long-run relationship between n and $g_{Y/L}$: CES production function and $\sigma > 1$

4 Conclusion

Our analysis shows that when the elasticity of substitution is less than unity and population growth is negative, the long-run growth rate of per capita output is given by the exogenous rate of technological progress. Therefore, as long as the technological progress rate is zero, the growth rate of per capita output is zero.

On the contrary, if the elasticity of substitution is unity, that is, the production function takes the Cobb-Douglas form, then the long-run growth rate of per capita output can be positive even without technological progress when population growth is negative.

We conclude that the result that the economy can attain sustainable growth even if population growth is negative and technological progress is zero depends on the specificity of the Cobb-Douglas production function. In a more realistic case where the elasticity of substitution between capital and labor is less than unity, technological progress plays an important role in economic growth irrespective whether population growth is positive or negative.

Finally, it should be noted that the difference between the case with the elasticity of substitution being less than unity and the case with the elasticity of substitution being equal to unity does not occur unless the population growth rate is extremely negative.

References

Antràs, P. (2004) "Is the US Aggregate Production Function Cobb-Douglas? New Estimates of the Elasticity of Substitution," *Contributions to Macroeconomics* 4 (1), Article 4.

Barro, R. J. and Sala-i-Martin, X. I. (2003) *Economic Growth*, 2nd edition, Cambridge MA: MIT Press.

Chirinko, R. S. (2008) "σ: The Long and Short of it," *Journal of Macroeconomics* 30, pp. 671–686.

Christiaans, T. (2011) "Semi-Endogenous Growth When Population is Decreasing," *Economics Bulletin* 31 (3), pp. 2667–2673.

Diamond, P. (1965) "National Debt in a Neoclassical Growth Model," *American Economic Review* 55, pp. 1126–1150.

Jones, C. I. (1995) "R&D-based Models of Economic Growth," *Journal of Political Economy* 103 (4), pp. 759–784.

Jones, C. I. (1999) "Growth: With or Without Scale Effects?" *American Economic Review* 89 (2), pp. 139–144.

Klump, R., McAdam, P., and Willman, A. (2007) "Factor Substitution and Factor-Augmenting Technical Progress in the United States: A Normalized Supply-Side System Approach," *The Review of Economics and Statistics* 89 (1), pp. 183–192.

Ritschl, A. (1985) "On the Stability of the Steady State when Population is Decreasing," *Journal of Economics* 45 (2), pp. 161–170.

Rowthorn, R. E. (1999) "Unemployment, Wage Bargaining and Capital-Labour Substitution," *Cambridge Journal of Economics* 23 (4), pp. 423–425.

Sasaki, H. and Hoshida, K. (2017) "The Effects of Negative Population Growth: An Analysis Using a Semi-Endogenous R&D Growth Model," *Macroeconomic Dynamics* 21 (7), pp. 1545–1560.

Solow, R. M. (1956) “A Contribution to the Theory of Economic Growth,” *Quarterly Journal of Economics* 70 (1), pp. 65–94.

United Nations (2013) *World Population Prospects: The 2012 Revision*, New York: United Nations.

World Bank (2013) *World Development Indicators*, Washington, DC: World Bank.