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Identification of the Conditions for Increasing Dimensionality of the Income Expansion Path

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Abstract

This paper determines sufficient conditions for a utility function to be associated with an income expansion path whose dimensionality increases with income i.e. given a finite number of products, the relationship between income and dimensionality of the consumption vector is given by a step function such that successive intervals on the real number line (each real number gives an income level) map on to consumption vectors of increasing dimensionality. This constitutes an important investigation as the rich are observed to exhibit greater consumption variety than the poor and the same household exhibits increasing product variety as its income increases over time.

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1. Introduction

The objective of this paper is to specify the properties of a utility function which result in the dimensionality of the income expansion path increasing with income. In the case of a discrete number of goods, this phenomenon is reflected in a step function type relation between income and dimensionality of product consumption with the latter either remaining constant or registering a jump with increase in income till variety in product consumption equals that in product availability. Thus, for N products, an individual would consume n(M) products at any level of income M with $n(M) \leq N, and\ \forall M\ s.t. n(M) < N, \exists\ would\ exist\ a\ M^0 > M\ s.t\ N \geq n(M^0) > n(M)$.

The significance of this paper stems from the fact that none of the conventional utility functions result in increasing dimensionality of the demand vector with income increase. This is demonstrated in Section 2. This is at odds with reality: the rich exhibit much more product variety than the poor and a household increases its product variety as its income increases.

In Section 3 we first deduce the properties of an additively separable utility function which are sufficient for dimensionality of consumption to increase with income. In Section 4 we show that even an additively non-separable utility function can result in such varying dimensionality, and deduce the general properties of such utility functions which lead to this outcome. Section 5 concludes by discussing the implications of our research and ways and means of broadening the research agenda of this paper.

2. Dimensionality of Product Consumption and Income Increase: The Cases of Conventional Utility Functions

One of the most popular utility functions, the Cobb-Douglas utility function [Mas-Colell et al. (1995)] implies that every commodity available for consumption has to be consumed in a positive amount for utility to be positive. In the case of perfectly divisible goods, any positive level of income can be used for such consumption which therefore always characterizes utility maximization and demand functions. Thus, the dimensionality of product consumption equals the dimensionality of product availability, irrespective of the level of income, which is at odds with reality.

In the case of perfect substitutes [Samuelson (1951), Varian (1982)], it is always optimal to spend an additional unit of money on that good which results in the greatest addition to utility. In rare cases it might be true that two or more goods are tied in that regard .The goods chosen for consumption depends on the relative prices and the marginal utilities of various goods. For example, for two goods x and y in a two good world the idea is to compare $\frac{MU_x}{P_x}$ with $\frac{MU_y}{P_y}$. As both marginal utilities of consumption expenditure are constant, the equality or direction of inequality characterising these two magnitudes will, *ceteris paribus*, remain frozen with income increase, resulting in turn in a frozen dimensionality of consumption (1 or 2 if $\frac{MU_x}{P_x} = \frac{MU_y}{P_y}$). It is easy to see that this complete lack of sensitivity of dimensionality to income will hold in the general *n* product case.

In the case of the Leontief utility function [Mas-Colell et al. (1995), Varian (1992), and Samuelson (1951)] the form itself dictates that the ratio of consumption of any two goods will be fixed in equilibrium. Thus, such ratios will hold irrespective of the level of income and will therefore result in the same dimensionality of the income expansion path irrespective of income.

Next we come to the lexicographic utility function. The basic property of the lexicographic utility function is that for an appropriately ordered consumption vector, utility increases in the quantity of the good occupying the *i th* position in the ordered consumption vector if and only if the quantities of goods preceding this good in the consumption vector are unchanged, irrespective of whether the consumption level of goods following it decrease, increase or remaining constant. Thus, all income will necessarily be spent on the first listed good in equilibrium irrespective of the level of income. Therefore, the dimensionality of the income expansion path will be identically equal to 1.

Let us now consider the CES (constant elasticity of substitution) or Dixit-Stiglitz type of utility function [Mas-Colell et al. (1995), Varian (1992), Atkinson and Stiglitz (1980), Frisch (1965)] . The CES function is usually stated as:

$u(x_1, x_2,.....x_N) = (\sum_{i=1}^{N} \alpha_i x_i^{\rho})^{1/\rho}$ where $0 < \rho < 1$ and $\alpha_i > 0$ for all i or equivalently

$u(x_1, x_2,.....x_N) = \left(\sum_{i=1}^{N} \alpha_i x_i^{\left(\frac{\sigma-1}{\sigma}\right)}\right)^{\frac{\sigma}{\sigma-1}}$ where $\sigma > 1$ and is the elasticity of substitution.

where x_i denotes quantity of consumption good i and σ is the elasticity of substitution among varieties. It is assumed that $\sigma > 1$.By substituting $\rho = \frac{\sigma-1}{\sigma}$ in the first version of the function we get the second version of the function.

The equilibrium level of consumption of product i is given by

$$x_i = \frac{\left(\frac{\alpha_i}{p_i}\right)^{\frac{1}{1-\rho}}}{\sum_{j=1}^{N} \left(\frac{\alpha_j}{p_j{}^{\rho}}\right)^{\frac{1}{1-\rho}}} \mathrm{M} \quad (1)$$

Thus, if all prices are positive and $\alpha_i > 0$ for all i we get a positive level of x_i for all i at every level of income. The step function relationship between income and the number of goods consumed does not exist.

We end with the Klein-Rubin (Stone Geary) utility function which underlies the linear expenditure system that is often used by empiricists to estimate demand functions. This can be written as follows:

$$U = \prod_{i=1}^{N} (x_i - \beta_i)^{\alpha_i} \; where \; x_i > \beta_i \; and \sum_{i=1}^{N} \alpha_i = 1 \quad (2)$$

The Stone Geary utility function only considers those cases where the consumption of each good i exceeds a certain subsistence amount β_i. Thus, $x_i > \beta_i \geq 0 \Rightarrow x_i > 0$. Thus, by assumption, all goods are consumed in positive amounts by the individual.

We therefore see that all the mentioned conventional utility functions correspond to a dimensionality of the income expansion path that is invariant with income. The objective of this paper therefore becomes non trivial i.e. to discover the general properties of utility functions which make dimensionality of product consumption sensitive to the level of income. Finding out the conditions corresponding to such varying dimensionality makes sense: at any point in time we observe the much greater variety of consumption of the rich than of the poor; in 'rags to riches' sagas, variety surely increases over time. It could be the case that the form of the utility function itself is sensitive to income increase but this paper establishes that such sensitivity is not a necessary condition for the consumed product variety to be sensitive to income. Moreover, even if the concerned individual was to jump from one form of conventional utility function to another because of external influences such as those exercised by peers, the various outcomes that are possible are exemplified by the following: the dimensionality increasing from 1 to the dimensionality of product availability (say, because of a switch from a lexicographic to a Cobb Douglas utility function) or registering the same change in the opposite direction; and the dimensionality remaining fixed (say, a switch from a utility function characterizing perfect substitutes to a lexicographic utility function). Both cases are hardly observed in reality **(**for related references see [Hicks (1956), Marshall (1890), Samuelson (1948), Samuelson (1951), Varian (1982) and Varian (1985)**].**

3. **Deducing Conditions under which Dimensionality of Consumption Increases with Income: The Case of Additive Separability of Utility Functions**

In the beginning, instead of assuming specific forms of utility functions we consider a general specification. The theorem below assumes additive separability. The theorem is followed by an example of a functional form which meets the assumptions listed in the theorem and thus is consistent with the result stated therein. Subsequently in Section 4, we try to grasp how increasing dimensionality of the income expansion path can be consistent with a non-additive utility function and even give an example of such a function.

Consider a utility function $u = u(x_1, x_2, \ldots\ldots x_N)$. Under usual assumptions, the first and second own partial derivatives for each argument are positive and negative respectively whereas the cross partial derivatives are all non-negative. Below we impose the restriction of additive separability i.e. zero cross partial derivatives

Theorem 1: *Consider a utility function $u = u(x_1, x_2, \ldots\ldots x_N)$ and assume that all prices are positive.*

Assume that the following conditions are satisfied

(i) $u_i(x_1, x_2, \ldots\ldots\ldots x_N) > 0 \; \forall \; i$ i.e. the first derivative with respect to the ith argument is always positive regardless of the value of i

(ii) $u_{ii}(x_1, x_2, \ldots\ldots\ldots x_N) < 0 \; \forall \; i$ i.e. the second derivative with respect to the ith argument is always negative regardless of the value of i

(iii) $u_{ij}(x_1, x_2, \ldots\ldots\ldots x_N) = 0$ for $j \neq i$ i.e. $u(x_1, x_2, \ldots\ldots\ldots x_N) = \sum_{i=1}^{N} u^i(x_i)$ where $u_{ij}(.)$ refers to the cross partial derivatives with respect to any two arguments i and j and $u^i(x_i)$ is the utility purely drawn from the amount of the i th good which does not influence overall utility from consumption in any other way.

(iv)(a) $u_i(0)$ *is defined and therefore finite for all i; and (b) prices are such that for each i,* $\frac{u_i(0)}{P_i} \neq \frac{u_j(0)}{P_j}$ *for some* $j \neq i$ *.*[1]

(v) $\lim_{x_i \to \infty} u_i(x_i) = 0 \; \forall i$.

Let $x^* = (x_1^*, x_2^* \ldots\ldots x_N^*)$ *solve the problem: Max* $u(x)$ *such that* $\sum_{i=1}^{N} p_i x_i = M$. *Then it is true that there exists at least one level of* $M = \overline{M_z}$ *such that the consumed product variety for* $M \leq \overline{M_z}$ *is lower than that for* $M > \overline{M_z}$*. Further at most N-1 such levels, corresponding to* $z = 1, 2, \ldots\ldots n-1$ *and satisfying the property that* $\overline{M_z}$ *is increasing in z, exist.*

Proof*:* Initially the commodities (commodity) with the highest level of $\frac{u_i(0)}{P_i}$ are (is) consumed as income increases from 0. The consumption of these consumed commodities in equilibrium at every level of income is such that $\frac{u_i(x_i)}{P_i}$ is the same in equilibrium at each level of income; given property (ii) and (iii) the equalized level or the marginal utility of income falls as M increases. Now because of (v) and (iv), at a certain level of income, denoted as $\overline{M_1}$ this marginal utility of income will ultimately reach the level of $\frac{u_j(0)}{P_j}$ where *j* is some hitherto unconsumed commodity. This commodity j will enter the set of consumed commodities and this consumed product variety will increase. If this variety is still not equal to the available product variety some positive level of hitherto unconsumed products will surely be consumed for all incomes higher than a certain $\overline{M_2} > \overline{M_1}$ (again from(v) and (iv)). The process will continue till consumption levels of all products turn positive at a level of income higher than $\overline{M_K}$ where $K \leq N-1$. (Q.E.D.)

Condition (iii) is very important in the sense that it is a sufficient condition, in the presence of diminishing marginal utility of consumption of all commodities, for the marginal utility of income to be diminishing in income. What condition (iii) implies is that utility function is additive in nature; thus the marginal utility of any one commodity is independent of the quantity consumed of other commodities. This implies that the marginal utility of commodity consumption always declines with an increase in the quantity of the commodity under question irrespective of whether this increase is marked by an increase in the consumption of other commodities. Thus, additivity ensures that when income increases and gives rise to an increase in consumption of one or more than one good under consideration the marginal utility of consumption of each of the consumed goods still decreases and the marginal utility of income which is nothing but the equated marginal utility of consumption expenditure across goods also decreases. The declining marginal utility of income with 0 as the highest lower bound on it, as highlighted and discussed above, makes sure that it falls below 'marginal utility divided by price' of hitherto unconsumed goods in a step wise manner. This in turn leads to a step function relationship between income and number of goods consumed.

In the absence of additivity we might have cases when the diminishing marginal utility of income is not achieved. This is because as consumption of more than one commodity increases with increase in income the marginal utility of consumption of each of the consumed products might not diminish. Though ceteris paribus, the negativity of the second

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[1]Apart from a remarkable coincidence this is always going to happen; in fact in most cases $\frac{u_1^i(0)}{P_i}$ being different for each i is highly likely.

own partial derivative of utility with respect to consumption of any product implies that there is a tendency of marginal utility of product consumption to decrease there is at the same time the force of complementarity as a result of which an increased consumption of one product tends to drive the marginal utility of other products up. Thus, when consumption of more than one product increases with increase in income the two mentioned forces generated might be such that complementarity might have a stronger influence and drive the equated ‘marginal utility divided by price’ across commodities up.

An example of the mentioned utility function is given by

$$u(x_1, x_2, \ldots x_N) = \sum_{i=1}^{N} u^i(x_1, x_2, \ldots x_N) = \sum_{i=1}^{N} a_i(x_i + c_i)^{0.5} \qquad (3)$$

where a_i and c_i are positive constants which are decreasing and increasing in *i* respectively. For the sake of simplicity in computation let us assume that all prices are equal to unity.

Note that $u_i = 0.5a_i\,(x_i + c_i)^{-0.5} > 0$ and $u_{ii} = -0.25a_i\,(x_i + c_i)^{-1.5} < 0$.Thus, properties (i) and (ii) are satisfied. By the formulation of the utility function (additive separability) property (iii) is satisfied. Further, $u_i(0) = 0.5a_i\,c_i^{\;-1.5}$ is finite and well defined for all i as well as decreasing in i; thus, property (iv) (a) and (iv) (b) are satisfied. Further property (v) is also satisfied as $\lim_{x_i \to \infty} 0.5a_i\,(x_i + c_i)^{-0.5} = 0$ i.e. if you take any positive number the marginal utility can be made smaller than that number by choosing$x_i > x_i^0$, where x_i^0 is a suitably large level of the i th commodity. Thus, this utility function will display the results of the theorem. In this case, since $u_i(0)$is different for all i, there will be N successive ranges of income with consumed product variety increasing by 1 for a movement from one range to the subsequent range.

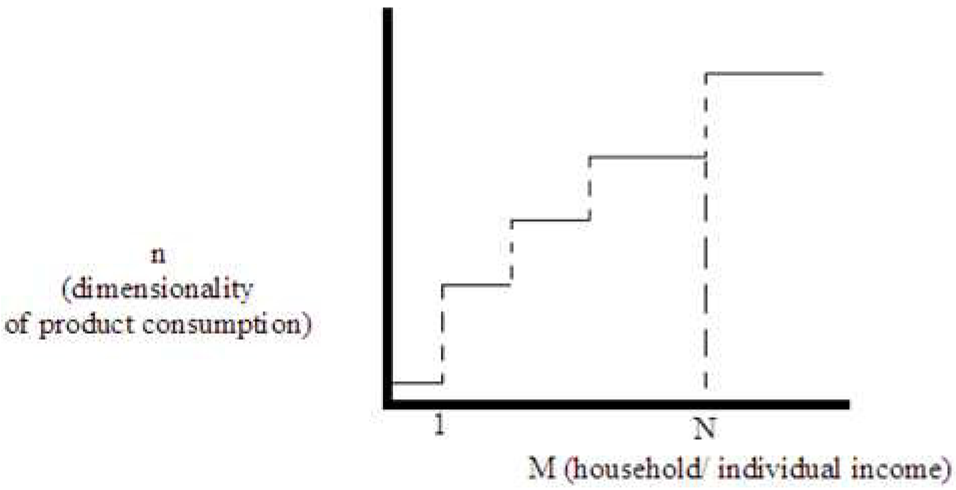

Figure 1: Relationship between M and n(M) in graphical form.

To illustrate, consider a two good version of this utility function s.t. $a_i = \frac{1}{i}, c_i = i$ for $i = 1,2$. For $i = 1, j = 2$ and $x_1 \in (7,\infty)$, $u_2(0) > u_1(x_1)$. In other words, marginal utility from consumption of the first commodity is less than the marginal utility of the second commodity at consumption level equalling zero if and only if the quantity consumed of the former exceeds 7. Given prices equalling unity, therefore, it has to be true that M>7 will give rise to consumption of both commodities. For $7 \geq x_1$ and therefore $M \leq 7$ exactly the opposite is true i.e. only the first commodity will be consumed. Thus, consumed product variety is indeed a step function of income and registers a jump from 1 commodity to 2 commodities at $M = 7$.

Note that substitution possibilities exist in our framework: if price of the good originally chosen for consumption at M just exceeding zero rises sufficiently then this will cease to be the good first chosen for consumption as M rises. Instead some other good might display a higher marginal utility divided by price at zero level of consumption and displace the good mentioned as the good consumed at low levels of income. The good mentioned would be chosen for consumption after the goods chosen for consumption attain a marginal utility divided by price which is equal to the marginal utility divided by price of the mentioned good i.e. at an income level sufficiently greater than zero.

In regard to Theorem 1: u_i(0) could be negative for large M and some u_i(0)could be negative for small M and some i. This is a limitation of this paper: we do not allow for the possibility that u_i(0) is negative at any value of M i.e. the good is never inferior at any level of consumption. However, the problem can be alleviated, we believe, by defining goods suitably broadly so that they remain normal at all levels of income.

The consequence of inferiority at a finite level of income (u_i(0)varying with income and becoming negative) is that certain goods might never come into the consumption set. The tendency for a diminishing marginal utility of income to fall towards its largest lower bound, 0 will be counteracted by the tendency of u_i(0) to become negative at some level of income. As a result the diminishing marginal utility of income never falls to the level of u_i(0) and the concerned good never enters the consumption set. But this means that a certain subset of goods will fail to enter the consumption set. There is no reason, however, to expect that the step function relationship will not hold because of this tendency.

4. **Deducing Conditions under which Dimensionality of Consumption Increases with Income: The General Case**

Now consider the general case of non-negative cross partial derivatives: $u_{ij}(x_1, x_2, \ldots\ldots x_N) \geq 0$ (iii') where $j \neq i$ (Property iii')

Theorem 2: *Consider a utility function $u(x_1, x_2, \ldots\ldots x_N)$ which satisfies properties (i), (ii) (iii') and (iv) as well as the following property: (vi) There exists a positive monotonic transformation, $U(x_1, x_2, \ldots\ldots x_N)$, of the utility function such that the marginal utility of income[2]diminishes and in the limit approaches zero[3].Then it is true that there exists at least one level of $M = \overline{M}_z$ such that the consumed product variety for $M \leq \overline{M}_z$ is lower than for $M > \overline{M}_z$, and at most N-1 such levels, corresponding to $z = 1, 2, \ldots\ldots n-1$ and satisfying the property that $\overline{M}_z$ is increasing in z, exist.*

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[2] An indirect utility function characterised by positive and negative second derivatives with respect to income

[3] Examples of $u(.)$ *and* $U(.)$ respectively are (i) $x^{1/2}y^{1/2}$ and $x^{1/4}y^{1/4}$; and (ii) $x + y$ and $\sqrt{x + y}$.

Proof: $u(.)$ and $U(.)$ clearly give rise to the same demand function. Initially the commodities (commodity) with the highest level of $\frac{u_i(\mathbf{x=0})}{P_i}$ are (is) consumed as income increases from 0. This (equalized) level of marginal utility of consumption expenditure is the marginal utility of income for initial levels of income. The consumption of these commodities in equilibrium at every level of income is such that $\frac{u_i}{P_i}$ is the same in equilibrium at each level of income. Now (iii') indicates that $\frac{u_j}{P_j}$, where j is an unconsumed commodity as M initially goes above zero, will rise or remain constant. By (vi), the marginal utility of income will have to equal $\frac{u_j}{P_j}$ at some level of income. For all levels of incomes above this level, product j will be consumed. Now it is possible that the levels of income at which this equality is attained is not the same for all j and even different for each j. This gives us the result (Q.E.D.)

Note that the above theorem states conditions for product variety of consumption increasing from a level less than N to N, the variety of product availability, as income increases. However, consider the case where the positive monotonic transformation, $U(x_1,x_2,.........x_N)$ of the utility function is such that the marginal utility of income[4] diminishes and in the limit approaches a positive number greater than zero. In such a case, if there is a commodity whose price is high enough, the entire locus of marginal utilities of that commodity at zero level of consumption but at different levels of income will lie below that positive number and that good will never be consumed. But that does not imply that product variety will not increase with income. All that is needed for product variety to increase with income is a) diversity in marginal utility per unit expenditure on various commodities, $\frac{u_i}{P_i}$, at a zero level of consumption of all goods i.e. they are not all identical and b) low enough prices of some of the commodities which are not consumed at infinitesimally small positive levels of income in addition to c) the diminishing marginal utility of income. Property b) will ensure that the marginal utility of expenditure of some of the commodities not consumed at infinitesimally small levels of income will equal the marginal utility of income at some level of income as these would rise and would eventually equal a diminishing marginal utility of income for some high level of income. A certain weak version of the above theorem can therefore be stated.

Theorem 2a: *Consider a utility function $u(x_1,x_2,.........x_N)$ which satisfies properties (i), (ii) (iii') and (iv) as well as the following property: (vi') There exists a positive monotonic transformation, $U(x_1,x_2,.........x_N)$ of the utility function such that the marginal utility of income diminishes. Then it is true that for some vector of prices satisfying (iv) there exists at least one level of $M = \overline{M_z}$ such that the consumed product variety for $M \leq \overline{M_z}$ is lower than for $M > \overline{M_z}$ and at most N-1 such levels, corresponding to $z = 1, 2, ...n - 1$ and satisfying the property that $\overline{M_z}$ is increasing in z, exist.*

An example can be used to show that utility functions that exhibit the properties mentioned in Theorem 2a exists. For the sake of simplicity we consider the case where all prices equal unity.

$$u = (x_1 + c)^{\alpha}(x_2 + c)^{1-\alpha} \text{where } 0 < 1 - \alpha < \frac{1}{2} < \alpha < 1$$

[4] An indirect utility function characterised by positive and negative second derivatives with respect to income

$$u_1 = \alpha(x_1+c)^{\alpha-1}(x_2+c)^{1-\alpha} > 0; u_2 = (1-\alpha)(x_1+c)^{\alpha}(x_2+c)^{-\alpha} > 0; u_{11} = (\alpha-1)\alpha(x_1+c)^{\alpha-2}(x_2+c)^{1-\alpha} < 0; u_{22} = -\alpha(1-\alpha)(x_1+c)^{\alpha}(x_2+c)^{-(\alpha+1)} < 0;\ u_{12} = u_{21} = \alpha(1-\alpha)(x_1+c)^{\alpha-1}(x_2+c)^{-\alpha} > 0$$

Thus, properties (i), (ii) and (iii') are satisfied.

$$u_1(0,0) = \alpha > u_2(0,0) = 1-\alpha$$

Hence property (iv) will be satisfied.

Now only good 1 will be consumed at income level M if and only if the marginal utility of good 1 is greater than the marginal utility of good 2at the allocation (M,0):

$$(1-\alpha)(M+c)^{\alpha}c^{-\alpha} < \alpha(M+c)^{\alpha-1}c^{1-\alpha}$$

$$\Leftrightarrow M < c(\frac{\alpha}{1-\alpha}\text{-}1) > 0$$

Thus, for $0 < M \leq c(\frac{\alpha}{1-\alpha}\text{-}1)$ only good 1 will be consumed and for $M > c(\frac{\alpha}{1-\alpha}\text{-}1)$ both goods will be consumed.

Given the above conclusion, note that for $0 < M \leq c(\frac{\alpha}{1-\alpha})$ the marginal utility of income is given by $\alpha(M+c)^{\alpha-1}c^{1-\alpha}$, with its first derivative with respect to income given by $(\alpha-1)(M+c)^{\alpha-1}c^{1-\alpha} < 0$. Thus, at least for this range of M the entire group of functions will depict the property of diminishing marginal utility of income.

For income levels beyond this range, the marginal utility of income is found by solving the following equation:

$$(1-\alpha)(x_1+c)^{\alpha}(M-x_1+c)^{-\alpha} = \alpha(x_1+c)^{\alpha-1}(M-x_1+c)^{1-\alpha} \quad (4)$$

That is we equate the marginal utilities of consumption of both goods for allocations of income in the range, $M > c\left(\frac{\alpha}{1-\alpha}\right)$.This implies

$$\frac{1-\alpha}{\alpha} = \frac{M-x_1+c}{x_1+c} \Rightarrow x_1 = \alpha M + (2\alpha-1)c \ \ and\ x_2 = (1-\alpha)M - (2\alpha-1)c$$

This can then be substituted into the LHS or RHS of "(4)" to get the marginal utility of income in the range $M > c(\frac{\alpha}{1-\alpha}\text{-}1)$. We choose to substitute into the LHS. This yields

$$MU_M = (1\text{-}\alpha)(\alpha M + 2\alpha c)^{\alpha}((1\text{-}\alpha)M + 2(1\text{-}\alpha)c)^{\text{-}\alpha} \quad (5)$$

The derivative of this with respect to M is given by

$$(1-\alpha)\alpha(\alpha M+2\alpha c)^{\alpha-1}((1-\alpha)M+2(1-\alpha)c)^{-\alpha} - \alpha(1-\alpha)(\alpha M+2\alpha c)^{\alpha}((1-\alpha)M + 2(1-\alpha)c)^{-\alpha-1}$$
$$= (1-\alpha)\alpha(\alpha M+2\alpha c)^{\alpha-1}((1-\alpha)M+2(1-\alpha)c)^{-\alpha}[1-\frac{\alpha}{(1-\alpha)}]$$

Note that this is always negative for $\alpha > 1-\alpha$ as assumed. Therefore, we always have diminishing marginal utility.

5. Conclusion

This paper determines sufficient conditions for a utility function to be associated with an income expansion path whose dimensionality increases with income i.e. given a finite number of products, the relationship between income and dimensionality of the consumption vector is given by a step function such that successive intervals on the real number line (each real number gives an income level) map on to consumption vectors of increasing dimensionality. This constitutes an important investigation as the rich are observed to exhibit greater consumption variety than the poor and the same household exhibits increasing product variety as its income increases over time.

We consider the cases of additive and non-additive utility functions separately. We get a very clean result for the additive case: if marginal utilities are finite at zero levels of commodity consumption and tend towards zero as consumption of commodities tend towards infinity; and prices are such that marginal utilities of expenditure on various commodities are not all equal (ruling out of a freak case) at consumption expenditure equalling zero then it must be the case that there are at least two ranges of income associated with differing consumed product variety, with a range corresponding to a higher variety always following that with lower variety on the real number line.

In the case of non-additive utility functions the result is again clear: the property of variety of consumption increasing with income is definitely observed for those utility functions which can undergo a positive monotonic transformation to yield functions characterized by diminishing marginal utility of income as well as a pecking order of commodities such that the marginal utility of consumption of any good at zero level of consumption is higher for a good higher up on the pecking order but is always finite. This corresponds to a well arranged preference tree of commodities.

All our results hold for the case in which goods are perfectly divisible. If all goods are lumpy or discrete apart from savings, then the conclusion of product variety increasing with income is true. Consider a world in which there are 3 goods plus saving. Further assume that the price of each good is the same and given by p. Then any income less than *p* will be characterised by only savings and zero consumption of the other goods, it is only when incomes exceed p, 2p and 3p that consumed product variety can exceed 2, 3 and 4 respectively. But so far as the effective monthly price of the services of any good is small compared to existing income levels we can say that the our analysis for perfectly divisible goods continues to retain its bite.

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