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Insurance Contracts under Beliefs Contamination

Rodrigo J Raad
Department of Economics - Universidade Federal de Minas Gerais

Gilvan Guedes
Demography Department - Universidade Federal de Minas Gerais

Lucélia Vaz
School of Management - Centro Federal de Educação Tecnológica

Abstract

We propose an insurance model introducing a global contamination on agents' beliefs over exogenous and endogenous variables. We establish conditions for the elasticity of contamination such that insurance demand declines with an increase in the level of such contamination. Our model assumes that agents are risk averse and distort their beliefs about exogenous events. Distortion also influences expectations over insurance transfers by strategic default, and leads to a price markup in relation to its actuarilly fair level as markets select larger insurers, increasing their market power. We impose boundaries on the relation between risk aversion and the elasticity of contamination, which is robust in the sense that insurance demand will decline even when the beliefs contamination leads to an increase in the probability of large loss. We further show that contracts are not efficient, resulting in a long run welfare loss.

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Contact: Rodrigo J Raad - rjraad@gmail.com, Gilvan Guedes - gilvan@gilvanguedes.com, Lucélia Vaz - luceliavv@gmail.com.

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1 Introduction

The effects of a change in wealth, price (see also Dionne et al., 2013) and risk distribution (see also Tibiletti, 1995; Dionne and Gollier, 1992) have long been studied in insurance economics. These effects are relevant as they may impact the demand for insurance, with negative consequences for economic welfare (see also Guedes et al., 2019; Einav et al., 2010). The magnitude and direction of these effects combined are ambiguous however, with little evidence of their consequences for insurance demand.

In cases where prices are actuarially fair, Mossin's Theorem (see also Mossin, 1968) assures that the demand does not vary because agents are always fully insured, regardless of their level of risk aversion and of their background risk. When insurance is priced above its actuarially fair level, Dionne et al. (2013) show that the demand for insurance can either increase, decrease or stay unaltered as a result of an increase in wealth. The direction in this case will be driven by the nature of risk aversion. The authors go further decomposing wealth and substitution effects explicitly, characterizing the insurance as a Giffen good. Finally, Dionne et al. (2013) and Tibiletti (1995) show how a change in the background risk can influence the demand. Tibiletti (1995) in particular studies changes in background risk represented by an increase in the correlation between a non-insured asset and a random loss. The author concludes that the demand for insurance decreases whenever a beneficial change in the distribution of final wealth occurs.[1]

All results discussed above hold only under *ceteris paribus* conditions. Certain aggregate shocks however could lead to a simultaneous change in price, wealth and beliefs. Contemporary climate change is a good example of how global contamination in insurance markets may occur. Insurers establishing premiums based on probability of catastrophic events from past experience are likely to default under mutual claims after event occurrence (simultaneous contamination on price and transfer). This was actually observed in the American insurance market after Hurricane Andrew devastated many houses along the American East coast (see also Browne and Hoyt, 2000). Prices quickly increased due to market power, but the declining demand put the market under risk of collapse. This situation required government inter-

[1] Despite the similarity in terms of results between ours' and Tibiletti's model, the main substantive difference can be summarized in where and how distortion is allocated in the model structure. In our model, we consider a global distortion and isolate a behavioral effect on the demand, such as the distortion elasticity of beliefs.

vention and reinforced the role of the National Flood Insurance Program in subsidizing the insurance premiums for houses in areas under risk of flooding (see also Kunreuther et al., 1993). Under this type background risk, contamination on contractual contingencies is difficult to be implemented as a typical contingency claim in an insurance contract because this contamination is governed by some latent exogenous variable. This restriction can be viewed as a market incompleteness.[2]

Despite all previous efforts trying to understand stylized facts in insurance markets, results characterizing the consequences of a global contamination are lacking. Carlier et al. (2003) is one of the few exceptions. The authors show that a contamination from ambiguity aversion results in full insurance for high values of the loss. In this paper, we establish conditions for the elasticity of contamination over agents' beliefs leading to a reduction in the demand for insurance. Our approach differs from Carlier et al. (2003) in two aspects: first, agents are risk averse and distort their beliefs about exogenous events; second, we study the result by extending this distortion to insurance payments and also to insurance prices that sit above their actuarially fair levels.[3] Markup arises when extreme events select larger insurers, increasing their market power. To preclude ambiguous variations on demand, as found in Dionne et al. (2013), we impose a boundary on the relation between risk aversion and the elasticity of contamination. The boundary is robust in the sense that insurance demand will decline even when the beliefs contamination leads to an increase in the probability of large loss. In cases where the background risk changes, the optimal insurance choice declines (see also Tibiletti, 1995). We further show that contracts are not efficient, resulting in a long run welfare loss.

[2]Latent contamination in financial markets is also likely to happen. Trade wars and political crises can increase subjective uncertainty on payoffs when derivatives are exercised before their expiration dates, weakening their function as an insurance instrument. The effect of the global contamination coupled with the decline in demand could induce a collapse in some derivative markets, as seen in global financial crises.

[3]Distortion on transfer is only due to the increase in default probability induced by some insurers bankrucpy or strategic default. Distortion on prices arises only because of market power induced by market selection of survival insurers in the long run.

2 The Model

We develop a theoretical framework based on Rothschild and Stiglitz (1976)'s model of private insurance demand. Different from the authors, we assume that agents' beliefs are exogenously contaminated by a latent uncertainty (see also Berger et al., 1986). Three types of contamination are considered. In the first, agents anticipate a markup above the actuarially fair price due to an increase in market power of insurers surviving bankruptcy after a change in the background risk.[4] In the second, agents consider the possibility of bankruptcy among surviving insurers because of the persistence in latent uncertainty over time. Finally, agents inaccurately anticipate the probability governing the states of nature, which embodies a contamination in their beliefs.

2.1 Basic Concepts

Exogenous uncertainty is characterized by a finite set S describing states of nature. Consider an underlying probability space $(Z, \mathscr{Z}, \zeta)$ where $\mathscr{Z}$ is the sigma-algebra containing agents' information and ζ is the objective probability. We assume that beliefs contamination is represented by a family of random variables[5] $\tilde{\epsilon} = \{\tilde{\epsilon}_s : Z \to E\}_{s \in S}$ where $E = [0, \bar{\epsilon}] \subset \mathbb{R}_+$ is the range of observations as a measurable space endowed with the Borel sigma-algebra. We assume that $\tilde{\epsilon}$ are privately observed and hence agents make plans contingent on their realizations. The value ϵ represents the beliefs' contamination in the accurate values of the insurance transfer, in the markup on insurance price and in the probabilities over S. Agents are subject to a loss l_s and trade an insurance contract in which each unit gives the right to receive a transfer t_s in each state $s \in S$. We further assume that all realizations of states are observed by insurers.[6] There is a single consumption good and the indemnity schedule $\{t_s\}_{s \in S}$ is given in units of this good. The observed price for each insurance unit, denoted by p, is taken as given.

In our model, the subjective uncertainty will be modeled based on the assumption that $\tilde{\epsilon}_s$ is continuous with differentiable probability density $\hat{f}_s : E \to \mathbb{R}_+$ defined over the realizations of $\tilde{\epsilon}$ for each $s \in S$. We denote

[4]We assume that this change is given *ex ante* in the model and agents anticipate it.

[5]That is, measurable functions defined on Z. In addition, we use the notation "˜" to represent random variables, "ˆ" for functions, and "∗" to distinguish any two functions.

[6]This model embodies the case of coinsurance.

by $\{\hat{\pi}_s : E \to [0,1]\}_{s\in S}$ the subjective probability distribution describing agents' contamination in the probability law governing the states of nature. We assume by convention that $\hat{\pi}(0)$ is the objective probability without contamination.

Firms choose the amount of insurance that maximizes their expected profit. We assume that a representative firm chooses the aggregate insurance supply, denoted by α. The firm's problem is then given by

$$\hat{v}_f(\epsilon) = \max\left\{\hat{p}(\epsilon)\alpha - \alpha\sum_{s\in S}\hat{\pi}_s(\epsilon)t_s : \alpha \in [0,1]\right\}. \tag{1}$$

Therefore, the representative firm offers a positive supply if and only if $\hat{p}(\epsilon) \geq \sum_{s\in S}\hat{\pi}_s(\epsilon)t_s$. The restriction $\alpha \in [0,1]$ means that the total amount of insurance units available is finite and normalized to one. This assumption is a consequence of some regulation stating that firms must be solvent on extreme events. The actuarially fair price is given by $p = \sum_{s\in S}\hat{\pi}_s(0)t_s$. For the sake of simplicity, we assume that $\hat{p}(\epsilon) = (1+\epsilon)p$ for all $\epsilon \in E$. Finally, suppose that management costs are negligible.[7]

Assume a generic agent with utility function $u : C \to \mathbb{R}$ representing the consumption benefit where $C \subset \mathbb{R}_+$. Agents' endowment is denoted by w. We suppose that u is twice differentiable with u' and u'' continuous, bounded away from zero and that agents are risk averse, that is, $u'' < 0$. Each consumption choice is defined on a compact set C and contingent upon the states of nature and on realizations of the subjective variable representing a proxy for the contamination. We say a contingent consumption plan

$$\hat{c} = \{\hat{c}_s : E \to C\}_{s\in S}$$

is feasible when there is an insurance choice $\hat{\alpha} : E \to \mathbb{R}_+$ satisfying

$$\hat{c}_s(\epsilon) + (1+\epsilon)p\hat{\alpha}(\epsilon) \leq w - l_s + \hat{\alpha}(\epsilon)(1-\epsilon)t_s \text{ for all } \epsilon \in E \text{ and all } s \in S. \tag{2}$$

The definition below characterizes the states with high loss and the states with low loss using the ϵ-contaminated net transfers function.[8]

[7] We assume that insurers can invest the premium in a market with an interest rate enough to cover all operational costs.

[8] We can assume without loss of generality that contamination is the same for all variables. Results do not change because derivatives only depend on local behavior and optimal choices are interior.

Definition 2.1. Write $\hat{\nu}_s : E \to \mathbb{R}$ by $\hat{\nu}_s(\epsilon) = t_s - p - \epsilon(t_s + p)$ for all $(s, \epsilon) \in S \times E$ as the ϵ-contaminated net transfers function and assume that $\hat{\nu}_s(\epsilon) \neq 0$ for all $(s, \epsilon) \in S \times E$. Moreover, define the subsets S_+ and S_- of high and low loss respectively[9] by

$$S_+ = \{s \in S : \hat{\nu}_s > 0\} \text{ and } S_- = \{s \in S : \hat{\nu}_s < 0\}.$$

Since $\hat{\nu}_s(\epsilon) = (1 - \epsilon)t_s - (1 + \epsilon)p$ for all $(s, \epsilon) \in S \times E$, then a feasible consumption plan can be written as

$$\hat{c}_s(\epsilon) = w - l_s + \hat{\alpha}(\epsilon)\hat{\nu}_s(\epsilon) \text{ for all } \epsilon \in E \text{ and all } s \in S$$

Agents' indirect utility function[10] is then given by

$$\hat{v}_a(p) = \max\left\{\sum_{s\in S}\int_E \hat{\pi}_s(\epsilon)u(\hat{c}_s(\epsilon))\hat{f}_s(\epsilon)d\epsilon\right\} \tag{3}$$

over all feasible consumption plans $\{\hat{c}_s\}_{s\in S}$. The function $\hat{v}_a(p)$ represents the optimal expected value for the benefit evaluated over all feasible consumption plans. The following notation is used hereafter.

Notation 2.2. Consider a differentiable function $\hat{g} : X \to Y$ where $X \subset \mathbb{R}$ and $Y \subset \mathbb{R}$. Denote by $\hat{\xi}(\hat{g}, x)$ the elasticity of $\hat{g}$ evaluated at $x \in X$, that is, $\hat{\xi}(\hat{g}, x) = x\hat{g}'(x)/\hat{g}(x)$.

The definition below characterizes a measure of absolute risk aversion as the composition of two effects. The first is the well known Arrow-Pratt (see also Pratt, 1964; Pratt and Zeckhauser, 1987) measure of absolute risk-aversion. The second is a measure of risk-aversion relative to the net savings on a given state of nature. This measure can also be viewed as the savings elasticity of marginal utility.

Definition 2.3. Consider the Arrow-Pratt measure of absolute risk-aversion $\hat{a} : C \to \mathbb{R}_+$ given by $\hat{a}(c) = -u''(c)/u'(c)$ for all $c \in C$. Define $\hat{r}_s : C \to \mathbb{R}$ for each $s \in S$ by $\hat{r}_s(c) = \hat{a}(c)(w - c - l_s)$.

Remark 2.4. To see that $\hat{r}_s$ can be viewed as the savings elasticity of marginal utility, consider the function $\hat{g}_s(x) = u'(w - l_s - x)$ where $x = w - c - l_s$ is the total of savings in the absence of an insurance market for each $s \in S$. Then

[9] In general, $t_s = 0$ for all $s \in S_-$.

[10] That is, the utility evaluated at the optimal consumption level.

$\hat{r}_s(c) = x\hat{g}'(x)/\hat{g}(x)$ is the measure of risk-aversion relative to savings. Note also that $\hat{r}_s(c) = \hat{a}(c)(w - l_s) - c\hat{a}(c)$ where $c\hat{a}(c)$ is the Arrow-Pratt measure of relative risk-aversion, which can be viewed as the elasticity of marginal utility, that is, $\hat{\xi}(u', c)$ for all $c \in C$.

2.2 Main Results

In the case of an interior solution, the following lemma holds.

Lemma 2.5. *Consider agents' problem* (3). *Then in an interior solution* $\hat{\alpha} : E \to \mathbb{R}_+$

$$\sum_{s \in S} \hat{f}_s(\epsilon)\hat{\pi}_s(\epsilon)\hat{\nu}_s(\epsilon)u'(\hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon) + w - l_s) = 0 \text{ for all } \epsilon \in E. \tag{4}$$

Proof: See appendix. □

Assumption 2.6 below establishes a threshold for the relative ϵ total elasticity of beliefs compared to the elasticity of marginal utility of savings. This threshold is relative to the elasticity of net transfers $\hat{\xi}(\hat{\nu}_s, \cdot)$ for each $s \in S$ and hence it is robust over different insurance markets. Under this assumption, Theorem 2.7 states that the insurance demand declines with increasing beliefs contamination.

Assumption 2.6. Suppose that $\hat{\nu}_s(\epsilon) \neq 0$ for all $(\epsilon, s) \in E \times S$. Assume that

$$1 + \frac{\hat{\xi}(\hat{\pi}_s, \epsilon) + \hat{\xi}(\hat{f}_s, \epsilon)}{\hat{\xi}(\hat{\nu}_s, \epsilon)} + \hat{r}_s(c) > 0 \text{ for all } (s, \epsilon, c) \in S \times E \times C.$$

Theorem 2.7. *Suppose Assumption 2.6. Then* $\hat{\alpha}'(\epsilon) < 0$ *for all* $\epsilon \in E$.
Proof: See appendix.

□

We provide a numerical example to make Assumption 2.6 more intuitive.

Example 2.8. Consider a normalized contamination in the set $E = [0, 1]$. Assume that the density function $f : E \to E$ that governs contamination is given by[11] $f(\epsilon) = (5 + 2\epsilon)/6$. Suppose that preferences are represented by the utility function $u(c) = \log(c)$ with wealth $w = 2$. Beliefs over the states of nature are characterized by an ϵ-contamination with respect to the true uniform probability $\pi_{unif} = (0.5, 0.5)$ in relation to $\pi_{cont} = (0.6, 0.4)$, that is,

$$(\hat{\pi}_1(\epsilon), \hat{\pi}_2(\epsilon)) = (1 - \epsilon)\pi_{unif} + \epsilon\pi_{cont}.$$

The insurance price is given by[12] $p = 1/2$. Consider $S = \{1, 2\}$ where state one represents the loss and state two the absence of loss. Suppose that insurance transfers are given by $(t_1, t_2) = (0.5, 0)$ and losses are summarized by the vector $(l_1, l_2) = (0.5, 0)$. In this scenario, the agent could be fully insured but Assumption 2.6 is satisfied with the demand for insurance decreasing over the entire domain of contamination. Indeed, in this example

$$\hat{r}_s(c) \geq -0.2 \text{ and } \frac{\hat{\xi}(\hat{\pi}_s, \epsilon) + \hat{\xi}(\hat{f}_s, \epsilon)}{\hat{\xi}(\hat{\nu}_s, \epsilon)} > 0.8 \text{ for all } (c, \epsilon, s) \in C \times E \times S.$$

Figure 1 shows the graphic representation of optimal insurance choices $\hat{\alpha}$ over varying ϵ.

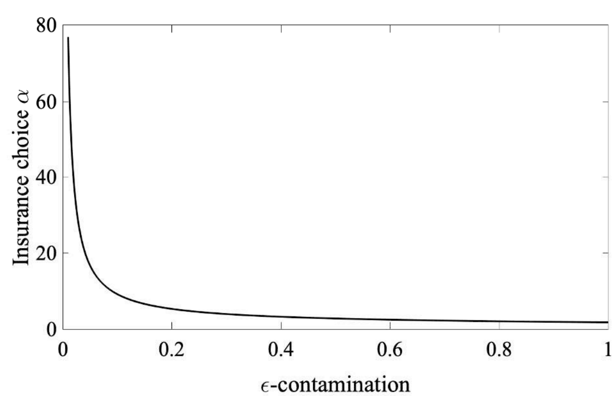

Figure 1: Insurance optimal choices under varying levels of ϵ-contamination

[11]This function concentrates mass on high values of contamination, that is, the higher the contamination the higher the probability of making a fixed prediction error.

[12]It is above the actuarially fair price $\bar{p} = 1/4$.

The following two definitions compare two kinds of contamination. They are used to ensure sufficient conditions for Theorem 2.12.

Definition 2.9. Given a continuous $g : E \to \mathbb{R}$ and a random family of contamination $\tilde{\epsilon} := \{\tilde{\epsilon}_s : Z \to E\}_{s\in S}$, write

$$\hat{E}[g \circ \tilde{\epsilon}_s] = \int_E g(\tilde{\epsilon}_s(z))\zeta(dz) \text{ for all } s \in S$$

as the expected value of g over the realizations of $\tilde{\epsilon}_s$ for all $s \in S$.

Remark 2.10. The change variable Theorem (see also Billingsley, 2008) states that if $\tilde{\epsilon}$ has continuous probability densities[13] $\{\hat{f}_s : E \to \mathbb{R}\}_{s\in S}$ then

$$\hat{E}[g \circ \tilde{\epsilon}_s] = \int_E g(\epsilon)\hat{f}_s(\epsilon)d\epsilon \text{ for all } s \in S.$$

Given another random family of contamination $\tilde{\epsilon}^* := \{\tilde{\epsilon}^*_s : Z \to E\}_{s\in S}$ with continuous probability densities $\{\hat{f}^*_s : E \to \mathbb{R}\}_{s\in S}$ and a continuous $g : E \to \mathbb{R}$ then $\hat{E}[g \circ \tilde{\epsilon}^*_s] = \int_E g(\epsilon)\hat{f}^*_s(\epsilon)d\epsilon$ for all $s \in S$.

The following definition specifies a formal characterization of two types of contamination and establishes a necessary condition to prove Theorem 2.12.

Definition 2.11. We say that the random contamination $\tilde{\epsilon}$ is weaker than $\tilde{\epsilon}^*$ if

1. $\tilde{\epsilon}_s(z) \leq \tilde{\epsilon}^*_s(z)$ for all $(z, s) \in Z \times S$
2. $\hat{f}_s(\epsilon) > \hat{f}^*_s(\epsilon)$ for all $(\epsilon, s) \in E \times S_+$
3. $\hat{f}_s(\epsilon) < \hat{f}^*_s(\epsilon)$ for all $(\epsilon, s) \in E \times S_-$

where $\{\hat{f}_s : E \to \mathbb{R}\}_{s\in S}$ and $\{\hat{f}^*_s : E \to \mathbb{R}\}_{s\in S}$ are the families of differentiable probability densities of $\tilde{\epsilon}$ and $\tilde{\epsilon}^*$ respectively.

Condition 1 assures that $\tilde{\epsilon}^*_s$ contamination first order stochastic dominates $\tilde{\epsilon}_s$ for each $s \in S$. Given a fixed level of contamination, Condition 2 assures that the density induced by $\tilde{\epsilon}$ gives more probability of a certain variation in ϵ for events s with high loss. Condition 3 has a reverse analogous interpretation. The following theorem states that larger levels of contamination lead to an average lower amount of optimal insurance units.

[13] That is, satisfying $\sum_{s\in S} \int_E \hat{\pi}_s(\epsilon)\hat{f}_s(\epsilon)d\epsilon = 1$.

Theorem 2.12. *Suppose Assumption 2.6 and that the random contamination $\tilde{\epsilon}$ is weaker than $\tilde{\epsilon}^*$. If $\hat{\alpha}$ and $\hat{\alpha}^*$ are the optimal choices for $\tilde{\epsilon}$ and $\tilde{\epsilon}^*$ then*

$$\sum_{s\in S} \hat{\pi}_s(0)\hat{E}[\hat{\alpha}^* \circ \tilde{\epsilon}^*_s] < \sum_{s\in S} \hat{\pi}_s(0)\hat{E}[\hat{\alpha} \circ \tilde{\epsilon}_s].$$

Proof: See appendix. □

The following result shows that a persistent error in how agents assign probabilities in the occurrence of future natural events leads to a long run loss in welfare.

Theorem 2.13. *Let $\tilde{s} := \{\tilde{s}_n : Z \to S\}_{n\in\mathbb{N}}$ be an independent stochastic process with uniformly bounded variance and such that $(\pi_s)_{s\in S}$ is its induced probability distribution. Define the accurate optimal value*

$$\hat{v}_{ac}(p) = \max\left\{ \sum_{s\in S} \pi_s u(c_s) : (c_s, \alpha) \text{ satisfies } c_s \leq w - l_s + (t_s - p)\alpha \text{ for all } s \in S \right\}.$$

Then for each realization $s = \tilde{s}(z)$ and $\epsilon_n = \tilde{\epsilon}_{s_n}(z)$ for $n \in \mathbb{N}$ we have

$$\hat{v}_{ac}(p) > \lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} u(\hat{c}_{s_n}(\epsilon_n)).$$

Where $\hat{c}$ is the optimal choice of agent's problem (3).

Proof: See appendix. □

3 Conclusion

In this paper we established sufficient conditions on the total elasticity of contamination over agents' beliefs that lead to a reduction in the demand for insurance. The total elasticity of contamination can be split into two components. The first component represents the effect of the contamination on the probabilities over exogenous events. In the second component, this effect relies on the contamination probability distribution. Although the partial effect on the probabilities over exogenous events trivially reduces insurance demand, on total elasticity its magnitude and direction are undetermined.

The boundary conditions established in Assumption 2.6 however precludes this indeterminacy and is a sufficient condition to reduce demand, even in cases where the background risk changes. In addition to showing a decline in demand under these conditions, we find that contracts are not efficient. As a consequence agents aggregately incur in welfare loss in the long run. Future research should incorporate solutions to eliminate inefficiency caused by contamination. This is particularly important for insurance contracts against climate related extreme events as evidence of uncertainty brought with climate change quickly increases.

4 Appendix

Proof: **of Lemma 2.5.** Equation (2) becomes

$$\hat{c}_s(\epsilon) = \hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon) + w - l_s \text{ for all } (s, \epsilon) \in S \times E \tag{5}$$

where we recall that $\hat{\nu}_s(\epsilon) = (t_s - p) - \epsilon(t_s + p)$ for all $(s, \epsilon) \in S \times E$. Therefore,

$$\hat{v}_a(p) = \max \left\{ \sum_{s \in S} \int_E \hat{\pi}_s(\epsilon) u(\hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon) + w - l_s) \hat{f}_s(\epsilon) d\epsilon \right\} \tag{6}$$

over all measurable $\hat{\alpha} : E \to \mathbb{R}_+$ such that $\hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon) + w - l_s \geq 0$ for all $(s, \epsilon) \in S \times E$. The concavity of u and the interior solution assure that the F.O.C. is a sufficient condition for the optimality. Write

$$\hat{v}(\hat{\alpha}) = \sum_{s \in S} \hat{\pi}_s(\epsilon) \int_E u(\hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon) + w - l_s) \hat{f}_s(\epsilon) d\epsilon.$$

If $\hat{\alpha}$ is an interior solution of (6) then[14] $\lim_{\tau \to 0^+} (\hat{v}(\hat{\alpha} + \tau h) - \hat{v}(\hat{\alpha}))/\tau \leq 0$ for each $h : E \to \mathbb{R}$. Define $g(\tau) = \hat{v}(\hat{\alpha} + \tau h)$. Then this is the same as stating that $g'(0) \leq 0$. Therefore the F.O.C. evaluated at the optimal insurance choice $\hat{\alpha}$ satisfies

$$\int_E h(\epsilon) \left(\sum_{s \in S} \hat{\pi}_s(\epsilon) \hat{\nu}_s(\epsilon) u'(\hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon) + w - l_s) \hat{f}_s(\epsilon) \right) d\epsilon \leq 0$$

and hence, choosing

$$h(\epsilon) = \sum_{s \in S} \hat{f}_s(\epsilon) \hat{\pi}_s(\epsilon) \hat{\nu}_s(\epsilon) u'(\hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon) + w - l_s) \text{ for all } \epsilon \in E$$

[14] This is the Gateaux concept of derivative.

then we conclude that[15]

$$\sum_{s\in S}\hat{f}_s(\epsilon)\hat{\pi}_s(\epsilon)\hat{\nu}_s(\epsilon)u'(\hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon)+w-l_s)=0 \text{ for all } \epsilon\in E. \tag{7}$$

□

Proof: **of Theorem 2.7**. Recall by (5) that $\hat{c}_s(\epsilon)=\hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon)+w-l_s$ where $\hat{\nu}_s(\epsilon)=t_s-p-\epsilon(t_s+p)$ for all $s\in S$. Therefore $\hat{\nu}_s\hat{\alpha}(\epsilon)=\hat{c}_s(\epsilon)+l_s-w$ for all $\epsilon\in E$. Write

$$\hat{\xi}^*_s(\epsilon)=(\hat{\xi}(\hat{\pi}_s,\epsilon)+\hat{\xi}(\hat{f}_s,\epsilon)+\hat{\xi}(\hat{\nu}_s,\epsilon))/\hat{\xi}(\hat{\nu}_s,\epsilon) \text{ for all } (\epsilon,s)\in E\times S.$$

Define for each $\epsilon\in E$

$$g_s(\epsilon)=\hat{f}_s(\epsilon)\hat{\pi}_s(\epsilon)\hat{\nu}_s(\epsilon) \text{ and } h_s(\epsilon)=u'(\hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon)+w-l_s) \text{ for all } s\in S.$$

Thus

$$\begin{aligned}g_s'(\epsilon)&=\hat{f}_s'(\epsilon)\hat{\pi}_s(\epsilon)\hat{\nu}_s(\epsilon)+\hat{f}_s(\epsilon)\hat{\pi}_s'(\epsilon)\hat{\nu}_s(\epsilon)+\hat{f}_s(\epsilon)\hat{\pi}_s(\epsilon)\hat{\nu}_s'(\epsilon)\\&=\hat{f}_s(\epsilon)\hat{\pi}_s(\epsilon)\hat{\nu}_s(\epsilon)(\hat{f}_s'(\epsilon)/\hat{f}_s(\epsilon)+\hat{\pi}_s'(\epsilon)/\hat{\pi}_s(\epsilon)+\hat{\nu}_s'(\epsilon)/\hat{\nu}_s(\epsilon))\\&=\epsilon^{-1}\hat{f}_s(\epsilon)\hat{\pi}_s(\epsilon)\hat{\nu}_s(\epsilon)\hat{\xi}(\hat{\nu}_s,\epsilon)\hat{\xi}^*_s(\epsilon)\end{aligned}$$

and

$$\begin{aligned}h_s'(\epsilon)&=(\hat{\nu}_s'(\epsilon)\hat{\alpha}(\epsilon)+\hat{\nu}_s(\epsilon)\hat{\alpha}'(\epsilon))u''(\hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon)+w-l_s)\\&=-\hat{a}(\hat{c}_s(\epsilon))u'(\hat{c}_s(\epsilon))(\epsilon^{-1}\hat{\xi}(\hat{\nu}_s,\epsilon)\hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon)+\hat{\nu}_s(\epsilon)\hat{\alpha}'(\epsilon)).\end{aligned}$$

Since $\hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon)=\hat{c}_s(\epsilon)+l_s-w$ then $\hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon)\hat{a}(\hat{c}_s(\epsilon))=-\hat{r}_s(\hat{c}_s(\epsilon))$ and hence

$$h_s'(\epsilon)=\epsilon^{-1}u'(\hat{c}_s(\epsilon))\hat{\xi}(\hat{\nu}_s,\epsilon)\hat{r}_s(\hat{c}_s(\epsilon))-u'(\hat{c}_s(\epsilon))\hat{\nu}_s(\epsilon)\hat{\alpha}'(\epsilon)\hat{a}(\hat{c}_s(\epsilon))$$

Therefore, differentiating (7) with respect to ϵ we get

$$\begin{aligned}0&=\sum_{s\in S}g_s'(\epsilon)h_s(\epsilon)+g_s(\epsilon)h_s'(\epsilon)\\&=\epsilon^{-1}\sum_{s\in S}u'(\hat{c}_s(\epsilon))\hat{f}_s(\epsilon)\hat{\pi}_s(\epsilon)\hat{\nu}_s(\epsilon)\hat{\xi}(\hat{\nu}_s,\epsilon)\big(\hat{\xi}^*_s(\epsilon)+\hat{r}_s(\hat{c}_s(\epsilon))\big)\\&-\hat{\alpha}'(\epsilon)\sum_{s\in S}u'(\hat{c}_s(\epsilon))\hat{f}_s(\epsilon)\hat{\pi}_s(\epsilon)\hat{\nu}_s^2(\epsilon)\hat{a}(\hat{c}_s(\epsilon)).\end{aligned}$$

[15] This inequality holds because f is continuous and we are considering the Lebesgue integral.

Moreover, $\hat{\nu}_s(\epsilon)\hat{\xi}(\hat{\nu}_s, \epsilon) < -\epsilon(t_s + p) < 0$ for all $\epsilon \in E$ and all $s \in S$. By Assumption 2.6

$$\hat{r}_s(\hat{c}_s(\epsilon)) + \hat{\xi}^*_s(\epsilon) > 0 \text{ for all } \epsilon \in E \text{ and all } s \in S$$

and hence $\hat{\alpha}'(\epsilon) < 0$ since $\hat{a}(\hat{c}_s(\epsilon)) > 0$ for all $\epsilon \in E$. □

Proof: **of Theorem 2.12.** Equation (7) implies that

$$\sum_{s\in S} \hat{f}_s(\epsilon)\hat{\pi}_s(\epsilon)\hat{\nu}_s(\epsilon)u'(\hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon) + w - l_s) = 0 \text{ for all } \epsilon \in E.$$

and

$$\sum_{s\in S} \hat{f}^*_s(\epsilon)\hat{\pi}_s(\epsilon)\hat{\nu}_s(\epsilon)u'(\hat{\nu}_s(\epsilon)\hat{\alpha}^*(\epsilon) + w - l_s) = 0 \text{ for all } \epsilon \in E.$$

Thus given an arbitrary $\epsilon \in E$ there exists $s \in S$ such that

$$\hat{f}_s(\epsilon)\hat{\pi}_s(\epsilon)\hat{\nu}_s(\epsilon)u'(\hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon) + w - l_s) \leq \hat{f}^*_s(\epsilon)\hat{\pi}_s(\epsilon)\hat{\nu}_s(\epsilon)u'(\hat{\nu}_s(\epsilon)\hat{\alpha}^*(\epsilon) + w - l_s).$$

Suppose that $\hat{\nu}_s(\epsilon) > 0$. Then

$$\hat{f}_s(\epsilon)u'(\hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon) + w - l_s) \leq \hat{f}^*_s(\epsilon)u'(\hat{\nu}_s(\epsilon)\hat{\alpha}^*(\epsilon) + w - l_s).$$

By assumption $\hat{f}^*_s(\epsilon) < \hat{f}_s(\epsilon)$ and hence

$$u'(\hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon) + w - l_s) < u'(\hat{\nu}_s(\epsilon)\hat{\alpha}^*(\epsilon) + w - l_s)$$

that is, $\hat{\alpha}(\epsilon) > \hat{\alpha}^*(\epsilon)$. If $\hat{\nu}_s(\epsilon) < 0$ then

$$\hat{f}_s(\epsilon)u'(\hat{\nu}_s(\epsilon)\hat{\alpha}(\epsilon) + w - l_s) \geq \hat{f}^*_s(\epsilon)u'(\hat{\nu}_s(\epsilon)\hat{\alpha}^*(\epsilon) + w - l_s).$$

and the conclusions follow analogously by reversing the arguments. Therefore $\hat{\alpha}(\epsilon) > \hat{\alpha}^*(\epsilon)$ for all $\epsilon \in E$. Moreover, Theorem 2.7 states that $\epsilon \leq \epsilon'$ implies $\hat{\alpha}(\epsilon) \geq \hat{\alpha}(\epsilon')$. Therefore, using that $\tilde{\epsilon}$ is weaker than $\tilde{\epsilon}^*$

$$\hat{\alpha}(\tilde{\epsilon}_s(z)) \geq \hat{\alpha}(\tilde{\epsilon}^*_s(z)) > \hat{\alpha}^*(\tilde{\epsilon}^*_s(z)) \text{ for all } (z, s) \in Z \times S.$$

Therefore,

$$\int_Z \hat{\alpha}(\tilde{\epsilon}_s(z))\zeta(dz) > \int_Z \hat{\alpha}^*(\tilde{\epsilon}^*_s(z))\zeta(dz) \text{ for all } s \in S$$

and hence

$$\sum_{s\in S}\hat{\pi}_s(0)E[\hat{\alpha}\circ\tilde{\epsilon}_s] > \sum_{s\in S}\hat{\pi}_s(0)E[\hat{\alpha}^*\circ\tilde{\epsilon}_s^*]$$

□

Proof: **of Theorem 2.13.** The Kolmogorov strong law of large numbers[16] states that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leq N}u(\hat{c}_{s_n}(\epsilon_n)) = \sum_{s\in S}\pi_s\int_E u(\hat{c}_s(\epsilon))\hat{f}_s(\epsilon)d\epsilon.$$

Since $\hat{c}_s(\epsilon) < w - l_s + (t_s - p)\hat{\alpha}(\epsilon)$ for all $(\epsilon, s)\in E\times S$ with $\epsilon > 0$ then

$$\sum_{s\in S}\pi_s\int_E u(\hat{c}_s(\epsilon))\hat{f}_s(\epsilon)d\epsilon = \int_E\sum_{s\in S}\pi_s u(\hat{c}_s(\epsilon))\hat{f}_s(\epsilon)d\epsilon < \hat{v}_{ac}(p).$$

Thus

$$\hat{v}_{ac}(p) > \lim_{N\to\infty}\frac{1}{N}\sum_{n\leq N}u(\hat{c}_{s_n}(\epsilon_n)).$$

□

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[16]Note that for each Borel measurable $g : S\times E\to\mathbb{R}$ the stochastic process defined by $g(\tilde{s}_n(z), \tilde{\epsilon}_{\tilde{s}_n(z)}(z))$ for all $z\in Z$ is independent.

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