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Multivariate Stochastic Dominance: A Parametric Approach

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Abstract

This paper proposes parameterized multivariate stochastic dominance (PMSD) rules under different distributional assumptions for a class of non-satiable risk-seeking investors. In particular, it determines the PMSD rules for both stable symmetric and Student's t distributions. Methodologically, the PMSD rules for ordering are based on comparison of i) location parameters, ii) dispersion parameters, and iii) either stability indices or degrees of freedom. In addition, it presents the main steps for evaluating such rules. This paper confirms that return tail behavior plays a crucial role in determining non-satiable investors' optimal choices.

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1. Introduction

Modern portfolio theory largely focuses on risk-averse investors' optimal choices (see, e.g., Markowitz (1952), Sharpe (1994), and related studies). In particular, it is well known that second-order stochastic dominance is consistent with the preferences of non-satiable risk-averse investors (see, e.g., Levy (1992), Müller & Stoyan (2002)). In general, investors are either risk-averse, risk-neutral, or risk-seeking and are, respectively, characterized by monotonically increasing concave, linear, or convex utility functions (Ingersoll (1987)). In recent years, several studies examining the implications of stochastic dominance in portfolio selection have been presented (see, e.g., Post & Kopa (2017), Arvanitis et al. (2019), Kouaissah & Ortobelli (2020)). However, few papers focus on risk-seeking investors and operationalizing dominance rules for parametric families of distributions (Ali (1975), Bawa (1976), and Crainish et al. (2013)). Unlike these studies, this work proposes and evaluates dominance rules for parametric multivariate families of distributions that are suitable for determining non-satiable risk-seeking investors' optimal choices.

Evidence provided by several authors suggests that investors do not always behave in a risk-averse manner. Instead, under certain circumstances, they exhibit risk-seeking characteristics and may hold non-diversified portfolios (Statman (2004), Ortobelli et al. (2018), and Arvanitis et al. (2019)). Accordingly, this paper considers a general class of non-satiable risk-seeking investors and examines the impact of stochastic dominance on their optimal choices. In particular, this research attempts to answer the following questions: i) If we assume elliptical distributions, which multivariate rules should be used with respect to their parameters? and ii) Which impact do distributional assumptions have on the choices of non-satiable risk-seeking investors? Since the decision problem concerns multivariate random variables, we propose parameterized multivariate stochastic dominance (PMSD) rules that are simple and tractable. Furthermore, empirical evidence shows that asset returns present heavy tails and slight skewness (see, e.g., Rachev & Mittnik (2000)). Several studies accordingly suggest using the symmetric-stable distribution or the Student's t distribution for the observed fat tails (see, e.g., Mandelbrot (1963), Fama (1965), Blattberg & Gonedes (1974), Fergusson & Platen (2006), Ortobelli et al. (2016)).

Motivated by these concerns, we develop PMSD rules that are suitable for ordering either Student's t or sub-Gaussian random vectors. Thereby, we provide criteria for deriving a convex type stochastic dominance between multi-parametric families. This enables us to determine the admissible set of portfolios for all non-satiable risk-seeking investors.[1] The optimal efficient sets are determined by verifying conditions on the parameters of heavy-tailed multivariate distributions. Specifically, we prove that in the case of stable Paretian and Student's t distributions, the stability index and the degree of freedom are respectively crucial for establishing the PMSD rule. This approach has the advantage of making it possible for decision makers to properly utilize the distributions' parameters to verify whether preference exists according to PMSD rule. The effectiveness of the proposed PMSD rule may

[1]For normal distribution of returns Bawa (1976) proved that the set of nondiversified portfolios contains at most two securities or one security if and only if it has the largest mean and the largest variance.

be evaluated by applying it to the equity market. In this regard, we present the main steps to examine whether there exists PMSD among financial benchmarks. This study confirms the relevance of returns' tail behavior in determining optimal portfolios for non-satiable risk-seeking investors.

The rest of the paper is organized as follows: in Section 2, we propose the main results of PMSD rules under different distributional assumptions. In Section 3, we suggest some practical steps to evaluate the relevance, if any, of the derived PMSD on optimal investors' choices. Finally, Section 4 summarizes our conclusions.

2. Weak Convex Stochastic Dominance

This section develops PMSD rules that are relevant for ordering different financial benchmarks from a non-satiable risk-seeking investor's point of view. In general, stochastic dominance conditions among two random variables, X and Y, involve the comparison of cumulative distribution functions, say F_X and F_Y (see, e.g., Levy (1992), Bruni et al. (2012)). The following definition recalls the conditions of increasing and convex stochastic orders.

Definition 1. *Increasing and convex order (ICX): we say that X dominates Y with respect to the ICX (in symbols, X ICX Y or $X \geq_{icx} Y$) if and only if*

$$\int_t^{+\infty} (1 - F_X(z))\, dz \geq \int_t^{+\infty} (1 - F_Y(z))\, dz,\ \forall\, t \in \mathbb{R}$$

or, equivalently, $X \geq_{icx} Y$ if and only if $E(g(X)) \geq E(g(Y))$ for any increasing and convex function g, i.e., any non-satiable risk-seeker investor prefers X to Y.

Recall that investors are risk seekers if they have a convex utility function (Ingersoll (1987)). It is not trivial to extend any stochastic order to a multivariate setting and several attempts have been made to provide stochastic dominance conditions for multivariate distribution functions (Levhari et al. (1975), Scarsini (1988), Müller & Stoyan (2002), and references therein). However, it is computationally challenging to verify a multivariate ordering rule that is based on comparisons between functionals (Ortobelli et al. (2016)). In the following subsections, we propose tractable PMSD rules that are based either on stable Paretian (sub-Gaussian) or Student's t distributional assumptions.

2.1. Stable Paretian Sub-Gaussian Distribution

Let $r_t = [r_{1,t}, \ldots, r_{n,t}]'$ be a vector of gross returns[2] on date t that follows an α-stable distribution $S_\alpha(\sigma, \beta, \mu)$, where $\alpha \in (0, 2]$ is the stability index, $\sigma > 0$ is the dispersion parameter, $\beta \in [-1, 1]$ is the skewness parameter, and μ is the location parameter (see Rachev & Mittnik (2000) for an excellent treatment). Let us assume that r is an α-stable sub-Gaussian distributed vector. Recall that an α-stable sub-Gaussian distribution is elliptically

[2] We define the i-th gross return between time $t-1$ and time t as $r_{i,t} = \frac{P_{t,i}}{P_{t-1,i}}$ and the i-th log return as $\ln(r_{i,t})$, where $P_{t,i}$ is the price of the i-th asset at time t.

distributed (because $\beta = 0$), and when $\alpha = 2$ we reach the normal distribution. According to Ali (1975) and Ortobelli et al. (2016), it is possible to characterize stochastic dominance by verifying conditions on the distribution parameters. In particular, it has been proven that ICX between α-stable sub-Gaussian distributions can be easily verified by a comparison between location, scalar, and tail parameters. Hence, the following theorem can be recalled (Ortobelli et al. (2016)):

Theorem 1. *Let $X_1 \sim S_{\alpha_1}(\sigma_1, 0, \mu_1)$, and $X_2 \sim S_{\alpha_2}(\sigma_2, 0, \mu_2)$. Suppose that $\alpha_1 > \alpha_2 > 1$, and $\sigma_1 \leq \sigma_2$. If $\mu_2 \geq \mu_1$, then $X_2 \geq_{icx} X_1$.*

The results of Theorem 1 establish a stochastic dominance rule among α-stable sub-Gaussian distributions. This rule can be generalized to a multivariate framework that considers the asymptotic behavior of the tail distributions. In particular, we state and prove a PMSD rule that can be used for ordering financial benchmarks.

Assumption 1. *Suppose two sectors, A and B, composed of n and m assets respectively. Let $x = [x_1, x_2, \ldots, x_n]'$ and $y = [y_1, y_2, \ldots, y_m]'$ be the vectors of portfolio weights of sectors A and B, respectively, whose sum is equal to 1 (i.e., $\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j = 1$).*

1. *Assume that the returns r_A and r_B of sectors A and B are jointly α-stable sub-Gaussian distributed with vector means μ_A and μ_B and dispersion matrices Q_A and Q_B respectively.*
2. *Assume that $\alpha_A \geq \alpha_B > 1$.*

Corollary 1. *Under Assumption 1, and for any portfolio of returns $y'r_A$ exists a portfolio of returns $x'r_B$ such that $y'Q_A y \leq x'Q_B x$ (and at least one inequality holds strictly). Then, $x'\mu_B \geq y'\mu_A$ implies $x'r_B$ ICX $y'r_A$ (i.e., sector B weakly ICX dominates sector A).*

Proof. Since any linear combination $x \in \mathbb{R}^n$ of α-stable sub-Gaussian vector r ($\alpha \geq 1$) with mean μ and dispersion Q is still an α-stable sub-Gaussian distributed with mean $x'\mu$ and dispersion $\sigma = \sqrt{x'Qx}$, then the result follows as a logical consequence of Theorem 1. $\square$

Observe that any non-satiable risk-seeking investor prefers a portfolio with a single asset. As a matter of fact, if u is a convex utility function, it follows that $E(u(\sum x_i r_i)) \leq E(\sum_{i=1}^{n} x_i u(r_i))) \leq max_i E(u(r_i))$. However, according to the efficient frontier definition, the ICX dominance between sectors implies the dominance among all non-ICX-dominated assets (i.e., there could be also portfolios which are not ICX dominant and are not ICX dominated).

The stochastic dominance obtained by stable sub-Gaussian distribution is relatively stronger than that of mean-risk approaches as proved in the following Corollary in case of the mean absolute deviation $MAD_X = E(|X - E(X)|)$ (see Konno & Yamazaki (1991)).

Corollary 2. *Let $X_1 \sim S_{\alpha_1}(\sigma_1, 0, \mu_1)$, and $X_2 \sim S_{\alpha_2}(\sigma_2, 0, \mu_2)$. If $\alpha_2 \geq \alpha_1 \geq 1$, $\mu_1 \geq \mu_2$, and $MAD_{X_1} \geq \frac{\Gamma(1-\frac{1}{\alpha_1})}{\Gamma(1-\frac{1}{\alpha_2})} MAD_{X_2}$, then X_1 ICX X_2.*

Proof. Since $\forall\ \alpha > 1$ we have that $\sigma = \frac{\pi}{2\Gamma(1-\frac{1}{\alpha})} MAD_X$, where $MAD_X = E(|X - E(X)|)$, then the thesis holds. $\square$

Observe that when X and Y are two random variables having the same mean and different mean absolute deviations ($MAD_X \neq MAD_Y$), then X ICX Y implies that $MAD_X > MAD_Y$. However, the converse may not hold. Thus, Corollary 2 provides sufficient conditions to invert the relationship between stable sub-Gaussian distributions.

2.2. Student's *t* Distribution

Several studies have used Student's t distributions to model asset returns (see, e.g., Blattberg & Gonedes (1974) and Fergusson & Platen (2006)). The Student's t distribution $T_\nu(\sigma, \mu)$ has three parameters: μ is the location parameter, σ is the scale parameter, and ν is the degree of freedom. It possesses the following properties: i) all moments of order $m < \nu$ are finite; ii) it has fatter tails than the density function of a normal distribution; and iii) for values $\nu \to \infty$ it converges to the normal distribution.

In practical applications, ν is the most important parameter (especially when $\nu > 2$). Student's t distribution belongs to the elliptically distributed family (see, e.g., Schoutens (2003)). It is therefore important to define a suitable order of preferences to deal with distributions for different values of ν. The following determines a ranking criterion that aims to compare univariate Student's t distributions according to ICX. The fact that ICX can be verified by comparing the degree of freedom, dispersion, and location parameters is shown below.

Theorem 2. *Let $X_1 \sim T_{\nu_1}(\sigma_1, \mu_1)$, and $X_2 \sim T_{\nu_2}(\sigma_2, \mu_2)$. Suppose $\nu_1 \geq \nu_2$, and $\sigma_1 \leq \sigma_2$ with at least one strict inequality. If $\mu_2 \geq \mu_1$, then $X_2 \geq_{icx} X_1$.*

Proof. Let $Z_i = \frac{X_i - \mu_i}{\sigma_i}$ (for $i = 1, 2$) be such that $Z_1 \sim T_{\nu_1}(1, 0)$ and $Z_2 \sim T_{\nu_2}(1, 0)$. Empirical and theoretical studies assert that the distribution functions of Z_1 and Z_2 cross only once. The mathematical formulations of the distribution function can be expressed as follows:

$$F_\nu(z) = \frac{1}{2}\left(1 + \operatorname{sgn}(z)\left(1 - I_{\left(\frac{\nu}{z^2+\nu}\right)}\left(\frac{\nu}{2}, \frac{1}{2}\right)\right)\right),$$

where $I_{(x)}(a, b)$ is the regularized Beta function (see, e.g., Johnson et al. (1995) and literature therein). From the properties of the regularized Beta function and given that $0 < \frac{\nu}{z^2+\nu} \leq 1$ for any $z \in \mathbb{R}$ and $\nu > 0$, it follows that:

1. $F_{\nu_1}(z) < F_{\nu_2}(z)$, $\forall\ z < 0$, and $\nu_1 > \nu_2$;
2. $F_{\nu_1}(z) > F_{\nu_2}(z)$, $\forall\ z > 0$, and $\nu_1 > \nu_2$;
3. $F_{\nu_1}(0) = F_{\nu_2}(0) = \frac{1}{2}$, $\forall\ \nu > 0$.

Thus, the distribution functions cross only once and $F_{\nu_1}(z)$ is below $F_{\nu_2}(z)$ to the left crossing point. This directly implies that Z_2 ICX Z_1 (which is equivalent to X_2 ICX X_1 as shown by Ortobelli et al. (2016)). Then the proof is complete. $\square$

As in the α-stable sub-Gaussian distribution case, the results of Theorem 2 establish a stochastic dominance rule that can be generalized to a multivariate framework. This extension yields the PMSD rule that is suitable for ordering financial benchmarks.

Assumption 2. *Consider previous two sectors, A and B.*

1. *Assume that the returns r_A and r_B of sectors A and B are jointly Student's t distributed with vector means μ_A and μ_B and dispersion matrices Q_A and Q_B respectively.*
2. *Assume that $\nu_A \geq \nu_B \geq 2$.*

Corollary 3. *Under Assumption 2, and for any portfolio of returns $y'r_A$ exists a portfolio of returns $x'r_B s$ such that $y'Q_A y \leq x'Q_B x$ (and at least one inequality holds strictly). Then, $x'\mu_B \geq y'\mu_A$ implies $x'r_B$ ICX $y'r_A$ (i.e., sector B weakly ICX dominates sector A).*

Proof. Since any linear combination $x \in \mathbb{R}^n$ of Student's t vector r ($\nu > 2$) with mean μ and dispersion Q is still a Student's t distribution with mean $x'\mu$ and dispersion $\sigma = \sqrt{x'Qx}$, then the thesis holds as a logical consequence of Theorem 2. □

This generalization allows for the establishment of the PMSD rule and thereby the application of this rule to complicated problems. The scale parameter σ of the Student's t distribution is a scaled standard deviation (SD) of the normal distribution. Similarly to the stable distribution, the stochastic dominance obtained from the Student's t distribution is relatively stronger than that of mean-variance ordering, as proved in the following Corollary.

Corollary 4. *Let $X_1 \sim T_{\nu_1}(\sigma_1, \mu_1)$ and $X_2 \sim T_{\nu_2}(\sigma_2, \mu_2)$. If $2 \leq \nu_1 \leq \nu_2$, $\mu_1 \geq \mu_2$, and $SD_1 \geq \sqrt{\frac{\nu_1(\nu_2-2)}{\nu_2(\nu_1-2)}} SD_2$, then X_1 ICX X_2.*

Proof. Since $\forall\ \nu \geq 2$ we have $SD = \sigma\sqrt{\frac{\nu}{\nu-2}}$, then the thesis holds . □

Furthermore, if X and Y are two random variables with the same mean and different finite variances, then X ICX Y implies $Var(X) > Var(Y)$. However, this does not imply that the converse holds except under the conditions of Corollary 4.

3. Practical Applications

This section suggests the main steps to determine whether the conditions for the PMSD hold among market sectors empirically. In practice, we distinguish three steps to verify ICX dominance rules among sectors under the different distributional assumptions.

Step 1. Fit the ICX mean dispersion efficient frontier considering a unique index of stability for the stable distributional assumption and unique value for the degrees of freedom (df) of the multivariate Student't distribution. Observe that the optimal choices of non-satiable risk-seeking investors are not diversified and contain a single asset. However, when we fit the ICX mean-dispersion efficient frontier we have to consider all the portfolios that are not ICX dominated. These portfolios are the maximum dispersion portfolios for a fixed mean and under elliptical distributional assumptions are composed either of a single asset (optimal ICX choices) or of two assets, as discussed and proved by Bawa (1976, 1977).

Step 2. Identify whether ICX dominance exists among the sectors whose components have been chosen and those whose assets were not chosen among the assets of the fitted ICX efficient frontiers. Stochastic dominance always occurs under the Gaussian assumption, while for the stable and Student't distributional assumptions we need to consider different indexes of stability and df for the chosen and non-chosen sectors, to verify the existence of ICX dominance. Thus, we first value a common index of stability (df) of the non-chosen sectors from the first step and we fit the mean dispersion ICX efficient frontier. Second, we value a common index of stability (df) of the sectors which were chosen in the first step. If we observe that the common index of stability (df) of the chosen sectors is not smaller than the index of stability (df) of the non-chosen sectors, we compute a common index of stability (df) of the chosen sectors using only those sectors whose index of stability (df) was estimated to be smaller than the tail index of the non-chosen sectors. Then we fit the mean dispersion ICX efficient frontier of those sectors which were chosen in the first step. Then, according to the dominance rules of Corollaries 1 and 3, we verify weak ICX sector dominance by checking whether, for any portfolio belonging to the fitted efficient frontier of non-chosen sectors, there exists a portfolio of the fitted efficient frontier of the chosen sectors which ICX dominates.

Step 3. Test whether, for any ICX efficient portfolio of the non-chosen sector, there exists a portfolio of the chosen sectors that ICX dominates according to a non–parametric approach (see Davidson & Jean-Yves (2000)). For ICX dominance, this third step serves to validate the distributional assumption.

4. Conclusions

This paper introduces a methodological approach to compare various financial benchmarks based on PMSD orders. The proposed dominance rules can be used by non-satiable risk-seeking investors to identify the best market sectors to invest in. The main contribution of the work is to examine the distributional assumptions' impact on asset allocation decisions. Here, the crucial differences between the Gaussian and the stable Paretian approaches suggest that portfolio selection models must consider the asymptotic (tail) behavior of the returns. For this reason, this paper proposes another family of heavy-tailed distribution (Student's t) and proves that ICX could be verified by comparing the values of the df, dispersion, and location parameters. Finally, it also discusses the crucial steps for applying the proposed methods and their implications to optimal portfolio choices.

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