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Sufficient conditions for the existence of stable sets of cooperative games

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Abstract

In 1944, von Neumann and Morgenstern introduced a stable set of n-person cooperative games in characteristic function form, with transferable utility (called TU-games for short), which is the first solution concept for cooperative games with at least three players. It is known that every n-person game has a stable set if $n in {3,4}$. On the other hand, Lucas constructed a 10-person TU-game which has no stable set. However, for $5 leq n leq 9$, it is not known whether every n-person TU-game has a stable set. In this paper, we show two sufficient conditions for an n-person TU-game to have a stable set for any $ngeq 5$.

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1 Introduction

In game theory, a *cooperative game* consists of two elements; a set of players and a characteristic function specifying the value created by different subsets of the players in the game. Formally, a cooperative game (N, v) with transferable utility, called a TU-game, consists of; $N = \{1, 2, \ldots, n\}$ is a (finite) set of players and v is the characteristic function with $v : 2^N \to \mathbb{R}$. As we know, the theory of cooperative games has a great deal of applications for economics, business administration, sociology, physical science, and so on. (For details of cooperative games, see books Driessen 1988, Peleg and P. Sudhölter 2007, and Peter 2008.)

In 1944, von Neumann and Morgenstern introduced a solution of n-person cooperative games in characteristic function form, called a *stable set*. The stable set is the first solution concept for cooperative games with at least three players. After that, many solution concepts were introduced so far. For example, the *Shapley value* is one of the most famous solution concepts which is introduced by Shapley 1953. The study of solution concepts is still active; for example, Ju et al. (2007) developed a new solution concept, called the *consensus value*. For other solution concepts, see also Spinetto 1971.

For solution concepts, a problem of interest is to show whether a given cooperative game has a solution. In particular, the existence of stable sets is deeply studied from long ago. It is proved in von Neumann and Morgenstern 1944 that every 3-person TU-game has a stable set. For TU-games with more persons, Bondareva et al. (1979) proved the existence of a stable set in every 4-person TU-game, but Lucus (1968a) constructed a 10-person TU-game which has no stable set. However, for $5 \leq n \leq 9$, it is not yet known whether or not a stable set exists in any n-person TU-game. For other topics for properties of solution concepts, see Lucchetti et al. 1987 for example.

In this paper, we show two sufficient conditions for n-person TU-games with $n \geq 5$ to have a stable set. In the next section, we introduce several terms and known results. In Section 3, we shall prove our main results.

2 Preliminaries

Throughout this section, let $G = (N, v)$ be an n-person TU-game, where $N = \{1, 2, \ldots, n\}$ is a finite set of players and v is the characteristic function with $v : 2^N \to \mathbb{R}$.

Definition 1 (Superadditivity). *The characteristic function v is* superadditive *if $v(S) + v(T) \leq v(S \cup T)$ for any two coalitions $S, T \subset N$ with $S \cap T = \emptyset$.*

Definition 2 (Imputation). *A payoff vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is an* imputation *of G if the following hold:*

1. *$x_i \geq v(i)$ for each $i \in \{1, 2, \ldots, n\}$ (Individual rationality)*
2. *$\sum_{i=1}^{n} x_i = v(N)$ (Efficiency)*

In what follows, $A(G)$ is the set of all imputations of G.

Definition 3 (Dominate). *For two imputations $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ and a coalition $S \subset N$, if*

1. $v(S) \geq \sum_{i \in S} x_i$ and
2. $x_i > y_i$ for any $i \in S$,

then $\mathbf{x}$ dominates $\mathbf{y}$ via S*, denoted by* $\mathbf{x} \succ^S_{\mathrm{dom}} \mathbf{y}$*. Furthermore,* $\mathbf{x}$ dominates $\mathbf{y}$*, denoted by* $\mathbf{x} \succ_{\mathrm{dom}} \mathbf{y}$*, if and only if there exists a coalition* $S \subset N$ *such that* $\mathbf{x} \succ^S_{\mathrm{dom}} \mathbf{y}$*.*

Definition 4 (Stable set)**.** *A subset* $V(v) \subseteq A(G)$ *is a* stable set *if the following hold:*

1. *For any two imputations* $\mathbf{x}, \mathbf{y} \in V(v)$*,* $\mathbf{x} \not\succ_{\mathrm{dom}} \mathbf{y}$ *and* $\mathbf{y} \not\succ_{\mathrm{dom}} \mathbf{x}$*. (Internal stability)*
2. *For each imputation* $\mathbf{z} \notin V(v)$*, there exists* $\mathbf{x} \in V(v)$ *such that* $\mathbf{x} \succ_{\mathrm{dom}} \mathbf{z}$*. (External stability)*

Definition 5 ($(0,1)$-normal)**.** *The game G is* $(0,1)$-normal *if* $v(N) = 1$*,* $v(i) = 0$ *for each* $i \in N$ *and* $v(S) \in [0,1]$ *for each coalition* $S \subseteq N$ *with* $|S| \geq 2$*.*

Now we introduce another solution concept, called a *domination core*, which is introduced in von Neumann and Morgenstern 1944. This concept plays an important role to show the existence of a stable set in a 3-person TU-game.

Definition 6 (Domination core)**.** *A subset* $C(v) \subseteq A(v)$ *is a* domination core *if for any* $\mathbf{x} \in C(v)$*, there exists no imputation* $\mathbf{y} \in A(v)$ *such that* $\mathbf{y} \succ_{\mathrm{dom}} \mathbf{x}$*, that is, any imputation in a domination core is not dominated by any other.*

The following are well-known results for $(0,1)$-normal 3-person TU-games. Let $H = (\{1,2,3\}, \hat{v})$ be a $(0,1)$-normal 3-person TU-game.

Theorem 1 (von Neumann and Morgenstern 1944)**.** *The game H has a domination core if and only if* $\hat{v}(12) + \hat{v}(23) + \hat{v}(13) \leq 2\hat{v}(123) = 2$*.*

Theorem 2 (von Neumann and Morgenstern 1944)**.** *If H has a domination core, then it has a stable set* $V(\hat{v}) \subseteq A(\hat{H})$ *such that for any value* $0 \leq \alpha \leq \hat{v}(\{1,2,3\}) - \hat{v}(\{2,3\})$*, there exists an imputation* $\mathbf{x} = (x_1, x_2, x_3) \in V(\hat{v})$ *with* $x_1 = \alpha$*, where* x_i *is a payoff of the player* $i \in \{1,2,3\}$*.*

3 Main results

If there is a coalition S of i-person with $v(S) = v(N)$ for any $i \in \{3, 4, \ldots, n\}$ in an n-person TU-game (N, v), then the game has many imputations which are not dominated by any other imputation. As described in the previous section, any imputation in a domination core is not dominated by any other imputation and a domination core has a positive effect on the existence of a stable set. Thus, the above game (N, v) seems to have a stable set, and in fact, we can guarantee the existence of a stable set for any TU-game with such an assumption, as follows. In what follows, we suppose that any game is $(0,1)$-normal.

Theorem 3. *Let* $N = \{1, 2, \ldots, n\}$ *with* $n \geq 4$ *and let* (N, v) *be an n-person TU-game. If there exist coalitions* $S_3 \subset S_4 \subset \cdots \subset S_{n-1}$ *such that* $v(S_r) = 1$ *and* $|S_r| = r$ *for all* $r = 3, \ldots, n-1$*, then the game* (N, v) *has a stable set.*

Proof. The theorem holds when $n = 4$ (see Bondareva et al. 1979). Thus we may suppose that $n \geq 5$, and we prove the theorem by induction on n. Without loss of generality, let $S = \{2, 3, \ldots, n\}$ be the coalition with $v(S) = 1$ and let $(N \setminus \{1\}, \overline{v})$ be an $(n-1)$-person game with $\overline{v}(S) = v(S)$ for any coalition $S \subseteq N - \{1\}$. We show that a set of imputations

$$V = \left\{\mathbf{x} = (0, x_2, x_3, \ldots, x_n) \in A((N \setminus \{1\}, \overline{v})) : (x_2, x_3, \ldots, x_n) \in \overline{V}\right\}$$

is a stable set of the game (N, v), where $\overline{V}$ is a stable set $(n-1)$-person game which exists by inductive hypothesis (since there exists a coalition $S' \subset S$ such that $v(S') = \overline{v}(S') = 1$ and $|S'| = n - 2$).

By the definition, V clearly satisfies internal stability. Thus, it suffices to prove that V satisfies external stability. Let $\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n) \in A \setminus V$. If $x_1 = 0$, then there exists $\mathbf{z} \in V$ with $\mathbf{z} \succ^{S}_{\text{dom}} \mathbf{x}$ via some coalition $S \subseteq N \setminus \{1\}$ since $(x_2, x_3, \ldots, x_n) \notin \overline{V}$ by the definition of V. Thus, we may suppose that $x_1 > 0$. Let $\mathbf{y} = (y_1, y_2, y_3, \ldots, y_n)$ with $y_1 = 0$ and $y_i = x_i + \epsilon_i$ for each $i \in \{2, 3, \ldots, n\}$ and $\sum_{i=2}^{n} y_i \leq v(N \setminus \{1\})$, where $\epsilon_i = \frac{x_1}{n-1}$ (since $\mathbf{y}$ should be an imputation). Since $\mathbf{y} \succ^{S}_{\text{dom}} \mathbf{x}$ for some coalition $S \subseteq N \setminus \{1\}$, we are done if $\mathbf{y} \in V$. Suppose that $\mathbf{y} \notin V$. In this case, we can find an imputation $\mathbf{z} \in V$ with $\mathbf{z} \succ^{S'}_{\text{dom}} \mathbf{y}$ with $S' \subseteq N \setminus \{1\}$ similarly to the previous case. Therefore, since each dominating relation uses a coalition contained in $N \setminus \{1\}$, we have $\mathbf{z} \succ_{\text{dom}} \mathbf{x}$. □

Next we give another sufficient condition with different type from Theorem 3.

Theorem 4. *Let $N = \{1, 2, \ldots, n\}$ with $n \geq 5$ and let (N, v) be an n-person TU-game. For a subset $T = \{j, k, l\} \subseteq N$ and some $i \in N \setminus T$, if the following holds;*

1. *for the subgame (T, v_T) with $v_T(S) = v(S)$ for any $S \subseteq T$, the characteristic function v_T is superadditive,*
2. *$v(T) < v(N)$,*
3. *$v(ij) + v(kl) \leq v(N)$ and*
4. *$v(N) - v(T) < \frac{1}{2n-6}\big(v(ij) + v(ik) + v(il) - v(N)\big)$,*

then the game (N, v) has a stable set.

Proof. Without loss of generality, we may assume that the set $T = \{n-2, n-1, n\}$. Let c be a real number such that for some $i \in N \setminus T$ and the tuple $T = \{j, k, l\}$,

$$v(N) - v(T) \leq c < \frac{1}{2n-6}\big(v(ij) + v(ik) + v(il) - v(N)\big). \tag{1}$$

We show that the set of imputations

$$V(c) = \left\{\mathbf{x} = (c, c, \ldots, c, x_{n-2}, x_{n-1}, x_n) \in A(N, v) : (x_{n-2}, x_{n-1}, x_n) \in \overline{V}\right\}$$

is a stable set of the game (N, v), where $\overline{V}$ is a stable set in the 3-person game $(\{n-2, n-1, n\}, \overline{v})$ with $\overline{v}(\{n-2, n-1, n\}) = v(N) - (n-3)c$ and $\overline{v}(S) = v(S)$ for any other coalition. By the definition of $V(c)$, the internal stability of $V(c)$ clearly holds. Thus, we prove the external stability of $V(c)$.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_{n-1}, x_n)$ be an imputation in $A(N, v) \setminus V(c)$, and let $R = \sum_{i=1}^{n-3} x_i$. We consider the following cases.

Case 1. $R > (n-3)c$.

Take an imputation $\mathbf{y} = (c, c, \ldots, c, y_{n-2}, y_{n-1}, y_n)$ with $y_i = x_i + \epsilon_i$ for each $i \in \{n-2, n-1, n\}$ and $\sum_{i=n-2}^{n} y_i = v(N) - c(n-3) \leq v(N \setminus \{1, 2, \ldots, n-3\})$. Since $\mathbf{y} \succ_{\rm dom}^{S} \mathbf{x}$ with some coalition $S \subseteq \{n-2, n-1, n\}$, we are done if $\mathbf{y} \in V(c)$. Suppose that $\mathbf{y} \notin V(c)$. In this case, we can find an imputation $\mathbf{z} \in V(c)$ with $\mathbf{z} \succ_{\rm dom}^{S'} \mathbf{y}$ for some coalition $S' \subseteq \{n-2, n-1, n\}$ by the definition of $V(c)$. Therefore, since each dominating relation uses a coalition contained in $\{n-2, n-1, n\}$, we have $\mathbf{z} \succ_{\rm dom} \mathbf{x}$.

Case 2. $R < (n-3)c$.

By the assumption, at least one of x_i's with $i \in N \setminus \{n-2, n-1, n\}$, say x_1, is less than c. Suppose that $x_k \geq v(1, k) - c$ for any $k \in \{n-2, n-1, n\}$. Since $v(N) \geq x_{n-2} + x_{n-1} + x_n \geq v(1, n-2) + v(1, n-1) + v(1, n) - 3c$, we have

$$c \geq \frac{1}{3}\left(v(1, n-2) + v(1, n-1) + v(1, n) - v(N)\right)$$

which contradicts to (1) when $n \geq 5$. Thus, there exists $t \in \{n-2, n-1, n\}$ such that $x_t < v(1, t) - c$, say $t = n-2$.

By the left relation of (1), we have $v(1, n-2) - c \leq v(1, n-2) + v(n-2, n-1, n) - v(N)$. Thus, by the condition 3, we have

$$v(1, n-2) - c \leq v(n-2, n-1, n) - v(n-1, n).$$

By the right relation of (1) and the condition 3, we have

$$v(n-2, n-1) + v(n-1, n) + v(n, n-2) < 2v(N) - (2n-6)c = 2\overline{v}(n-2, n-1, n).$$

Thus, the 3-person game has a domination core by Theorem 1, and hence, for any α with $0 \leq \alpha \leq v(n-2, n-1, n) - v(n-1, n)$, the game has a stable set $\overline{V}$ containing imputations $\mathbf{y} = (\alpha, y_{n-1}, y_n)$ by Theorem 2. Let $\mathbf{y} \in \overline{V}$ with $y_1 = \alpha = v(1, n-2) - c$. By the assumption, $y_{n-2} > x_{n-2}$, $c > x_1$ and $y_{n-2} + c = v(1, n-2)$. Therefore, $\mathbf{y}' = (c, c, \ldots, c, \alpha, y_{n-1}, y_n) \in V(c)$ dominates $\mathbf{x}$ via the coalition $\{1, n-2\}$.

Case 3. $R = (n-3)c$.

If $x_i = c$ for each $i \in \{1, 2, \ldots, n-3\}$, then we can find $\mathbf{z} \in V(c)$ with $\mathbf{z} \succ_{\rm dom} \mathbf{x}$ similarly to Case 1. Otherwise, there exists $j \in \{1, 2, \ldots, n-3\}$ such that $x_j < c$, and hence, we can prove the case similarly to Case 2. □

4 Concluding remarks

In the literature, von Neumann and Morgenstern (1944) (resp. Bondareva et al. (1979)) proved that any 3-(resp. 4-)person TU-game has a stable set, and Lucus (1968a) constructed a 10-person TU-game which has no stable set. In this paper, we could provide two sufficient conditions for an n-person TU-game to have a stable set for any $n \geq 5$. However, it is not yet known whether or not every n-person TU-game has a stable set for $n \in \{5, 6, 7, 8, 9\}$. For games with many persons, Rabie (1985) and Lucus (1968b) construct 9-person and 8-person TU-games with no *symmetric* stable set, respectively. On the other hand, it is known that there are several 5-person TU-games which have a stable set (cf. Lucus (1968b)). Thus a remaining problem of interest is to decide whether every 5-person TU-game has a stable set. Moreover, we deal only with TU-games in this paper but our sufficient conditions may possibly be extended to cooperative games with other types (e.g., a NTU-game). Therefore, one of future directions of the study is to prove similar sufficient conditions for other cooperative games.

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