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A graphical approach to select the nested structure of a CES function

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Abstract

The existing literature estimating nested CES production functions has so far neglected to provide a convincing method to discriminate among alternative nested structures. This study proposes a new approach to evaluate empirically which nested structure describes the production technology more closely. It is based on the estimation of translog point elasticities and on the graphical analysis of their distribution. A Monte Carlo simulation framework is used to demonstrate the applicability of the method given alternative parametrizations.

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1. Introduction

Computable general equilibrium (CGE) models are typically employed to support decision-making in evaluating the economic consequences of new public policies. These models require the definition of a set of functions that approximate, as closely as possible, agents' production, consumption, and accumulation behaviors. Traditionally, CGE modelers have employed nested CES functions to describe production technologies for their global regularity and constant elasticities. However, McKitrik (1998) noted that nested CES functions are based on restrictive properties (homogeneity and input separability) that may not be empirically observed, and Lecca *et al.* (2011) and Feng and Zhang (2018) showed that CGE results are sensible to the input hierarchical structure. Furthermore, Lagomarsino (2020) reported that a convincing approach for selecting among alternative nested CES structures is still lacking.

This study illustrates, in a Monte Carlo framework, a new empirical and graphical approach to evaluate whether a nested CES function is consistent with the data and to select between competing nested structures. The method is based on the concept of Allen and Hicks partial elasticities of substitution and on their underlying relationships in the nested CES context. Furthermore, this paper provides an empirical insight on the rules that characterize nested CES Morishima partial elasticities.

Section 2 describes the approach, Section 3 outlines the Monte Carlo setting, Section 4 shows results, Section 5 investigate Morishima partial elasticities of substitution, and Section 5 concludes.

2. A graphical approach

Let us consider a 3-input CES production function that relates inputs x_i, with $i = 1,2,3$, to output Y. This can be represented by any of the following 2-level nested structures:

$$Y_{\mathrm{t}}^{\mathrm{CES}} = \gamma\left(\alpha(\beta(x_{1t})^{-\rho_1} + (1-\beta)(x_{2t})^{-\rho_1})^{\frac{\rho}{\rho_1}} + (1-\alpha)(x_{3t})^{-\rho}\right)^{\frac{1}{\rho}} \tag{1}$$

$$Y_{\mathrm{t}}^{\mathrm{CES}} = \gamma\left(\alpha(\beta(x_{1t})^{-\rho_1} + (1-\beta)(x_{3t})^{-\rho_1})^{\frac{\rho}{\rho_1}} + (1-\alpha)(x_{2t})^{-\rho}\right)^{\frac{1}{\rho}} \tag{2}$$

$$Y_{\mathrm{t}}^{\mathrm{CES}} = \gamma\left(\alpha(\beta(x_{2t})^{-\rho_1} + (1-\beta)(x_{3t})^{-\rho_1})^{\frac{\rho}{\rho_1}} + (1-\alpha)(x_{1t})^{-\rho}\right)^{\frac{1}{\rho}}, \tag{3}$$

or by the 1-level non-nested CES:

$$Y_t^{CES} = \gamma(\alpha(x_{1t})^{-\rho} + \beta(x_{2t})^{-\rho} + (1-\alpha-\beta)(x_{3t})^{-\rho})^{\frac{1}{\rho}} \tag{4}$$

where $\gamma > 0$ is the productivity parameter, $\alpha, \beta \in (0,1)$ are the share parameters, ρ, $\rho_1 \in (-1, \infty)$ are the substitution parameters,[1] and $t = 1, \ldots, T$ indexes observations. The constant elasticities of substitution can be derived as $\sigma = 1/(1+\rho)$ and $\sigma_1 = 1/(1+\rho_1)$.

According to Sato's (1967) definition, nested CES functions have constant elasticities of substitution, σ and σ_1, and partial elasticities of substitution that follow specific rules determined by the nested structure of the function. Table I presents these rules for the three nested CES presented in (1)-(3). The partial elasticities considered in the definition are the Hicks elasticities of substitution (HES) and the Allen elasticities of substitution (AES).[2]

[1] The subscript *1* indicates that the parameter refers to the inner nest.

[2] AES are defined as:

$$\sigma_{ij}^{AES} = \frac{\sum_{k=1}^{n} f_k\, x_k}{x_i x_j}\frac{|D_{ij}|}{|D|}$$

Table I – Partial elasticities of substitution in nested CES functions

Nested structures	$x_1$and x_2 elasticity		$x_1$and x_3 elasticity		$x_2$and x_3 elasticity	
	HES	AES	HES	AES	HES	AES
$((x_1,x_2),x_3)$	σ_1	$\sigma+\frac{(\sigma_1-\sigma)}{\theta^1}$	$\frac{\frac{1}{\theta^1_{x_1}}-\frac{1}{\theta^1}}{\sigma^1}+\frac{\frac{1}{\theta^2_{x_3}}+\frac{1}{\theta^1}}{\sigma}$	σ	$\frac{\frac{1}{\theta^1_{x_2}}-\frac{1}{\theta^1}}{\sigma^1}+\frac{\frac{1}{\theta^2_{x_3}}+\frac{1}{\theta^1}}{\sigma}$	σ
$((x_1,x_3),x_2)$	$\frac{\frac{1}{\theta^1_{x_1}}-\frac{1}{\theta^1}}{\sigma^1}+\frac{\frac{1}{\theta^2_{x_2}}+\frac{1}{\theta^1}}{\sigma}$	σ	σ_1	$\sigma+\frac{(\sigma_1-\sigma)}{\theta^1}$	$\frac{\frac{1}{\theta^1_{x_2}}-\frac{1}{\theta^1}}{\sigma^1}+\frac{\frac{1}{\theta^2_{x_3}}+\frac{1}{\theta^1}}{\sigma}$	σ
$((x_2,x_3),x_1)$	$\frac{\frac{1}{\theta^1_{x_1}}-\frac{1}{\theta^1}}{\sigma^1}+\frac{\frac{1}{\theta^2_{x_2}}+\frac{1}{\theta^1}}{\sigma}$	σ	$\frac{\frac{1}{\theta^1_{x_1}}-\frac{1}{\theta^1}}{\sigma^1}+\frac{\frac{1}{\theta^2_{x_3}}+\frac{1}{\theta^1}}{\sigma}$	σ	σ_1	$\sigma+\frac{(\sigma_1-\sigma)}{\theta^1}$

Notes: θ^s is the expenditure share of the *s*th class inputs, $\theta^s_{x_i}$ is the relative share of the *i*th element of *s*th class.

Summing up, HES between inputs belonging the same nest are constant, AES between each of the inputs inside the nest and the input outside are constant and identical.

In this paper, we argue that to understand if a given dataset supports a nested CES production function a researcher should check if these rules are met empirically by the non-constant point elasticities of substitution of a translog production function. Indeed, under certain conditions, the translog is a linear approximation to a nested CES and its elasticities' distribution follows closely the rules presented in Table I. To evaluate whether the elasticities are close to being constant, one can look at the dimension of the point elasticities' prediction interval that indicates in which range an estimated elasticity of substitution obtained from a new level of inputs and output quantities should fall 95% of the times.[3]

Together with a descriptive analysis, several insights can be derived by observing the graphical representation of the elasticities' distribution. First, the concentration of point elasticities around a limited range of values (i.e. a clear peak in the distribution) represents an evidence in support of a constant elasticity. Second, narrow prediction intervals suggest that point elasticities are well predicted for different levels of inputs and output and not expected to vary significantly. Third, the distribution of different partial elasticities peaking around the same value provides an indication on which nested structure best fits the data.

To summarize, the approach proposed in this study consists in the following three steps: i) estimation of a translog function[4] and the derivation of AES and HES partial elasticities, ii) derivation of prediction intervals, iii) descriptive and graphical analysis of the point elasticities' distribution.

3. Monte Carlo setting

In the following, we use a Monte Carlo setting to illustrate the approach proposed and evaluate its power in identifying the assumed input-output relationship. We define a data generating processes (DGP) based on the 3-input 2-level CES presented in Equation (1). For simplicity, we call the three inputs energy (E = x_1), capital (K = x_2), and labour (L = x_3), and ((E;K);L)

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where x_i and x_j are two inputs, f_i,f_j, f_{ii}, and f_{ij} are the first and second partial derivatives of the production function with respect to input $x_i$and x_j respectively, $|D|$is the determinant of the bordered Hessian matrix D formed by the estimated coefficients and $|D_{ij}|$ represents the cofactor of the *ik*th term in the Hessian matrix. HES are defined by:

$$\sigma_{ij}^{HES}=\frac{(f_i x_i+f_j x_j)}{x_i x_j}\frac{f_i f_j}{(2f_{ij}f_i f_j-f_{jj}f_i^j-f_{ii}f_j^2)}$$

[3] For example, suppose to have data on different industrial sectors for the same year: the prediction interval informs on the range in which a new observation for a particular sector will fall 95% of the times. It should be noted that it is not possible to derive confidence intervals for the translog elasticities because these vary with input and output quantities.

[4] In the three-input case, the translog function can be written as: $y_t^{TL}=\ln a_0+a_1\ln x_1+a_2\ln x_2+a_3\ln x_3+0.5a_{11}\ln^2 x_1+0.5a_{22}\ln^2 x_2+0.5a_{33}\ln^2 x_3+a_{12}\ln x_1\ln x_2+a_{13}\ln x_1\ln x_3+a_{23}\ln x_2\ln x_3$

the nesting structure associated with Equation (2). Output is generated according to the following specification:

$$q_t = y_t^{CES} + \epsilon_t$$

where y_t is the logarithm of output Y_t, and $\epsilon_t \sim \ln N\,(0, \sigma_\epsilon)$. Parameters and inputs' distributions are listed in Table II.

Table II – DGP

Parameters	DGP
E, K, L	$\sim \ln N\,(0{,}0.5)$
γ	1.5
α, β	0.5
ρ, ρ_1	-0.4; 0.1; 0.9; 9
T	1.000

The value of the substitution parameters is determinant in the estimation of the CES because it affects its overall curvature, thus we let it vary across a range of values associated with high and low levels of substitutability (see Table III). We expect estimation results to become less precise the further ρ deviates from 0, the point in which the linear approximation is made.

Table III – Selected values for the substitution parameters and elasticities

ρ, ρ_1	-0.4	0.1	0.9	9.0
σ, σ_1	1.667	0.909	0.526	0.100

We repeated the simulations altering the remaining CES parameters to evaluate how results are affected. As econometric theory suggests, we found an increase in the number of observations to improve the estimates as well as a decrease in the error variance.

4. Simulation results

Table IV presents the translog estimated partial elasticities of substitution. Let us consider first the three cases highlighted in grey: i) $\rho = \rho_1 = 0.1$, ii) $\rho = 0.1$ and $\rho_1 = 0.9$, iii) $\rho = 0.9$ and $\rho_1 = 0.1$. We observe that, in all the cases considered, AES_{EL}, AES_{KL} and HES_{EK} approximate their assumed DGP values, and that prediction intervals are narrow overall, indicating that elasticities are well predicted for different levels of inputs and output and that they are not expected to vary much. Furthermore, results show that in case i) all partial elasticities range across the same values: this, indeed, is the case in which the nested CES reduces to the 1-level non-nested CES described in Equation (4). In cases ii) and iii), instead, AES_{EL} and AES_{KL} are very close to each other and their distributions are overlapping.

These first evidences indicate that, in the three cases considered, the estimated point elasticities are indeed following the rules presented in Table 1 and provide good predictions of the true value of the CES elasticities.

Table IV – Estimated median translog HES and AES and prediction intervals

		ρ=-0.4, σ =1.667			ρ=0.1, σ = 0.909			ρ=0.9, σ = 0.526			ρ=9, σ = 0.100		
		σ	low	up	**σ**	low	up	**σ**	low	up	**σ**	low	up
ρ_1=-0.4 σ_1 =1.667	EK_{HES}	**1.643**	1.602	1.686	**1.623**	1.583	1.662	**1.592**	1.548	1.636	**1.272**	1.169	1.379
	EL_{AES}	**1.633**	1.587	1.681	**0.913**	0.894	0.932	**0.547**	0.534	0.561	**0.244**	0.176	0.307
	KL_{AES}	**1.683**	1.634	1.780	**0.907**	0.887	0.926	**0.510**	0.497	0.524	**0.158**	0.984	0.219
ρ_1=0.1 σ_1 =0.909	EK_{HES}	**0.902**	0.891	0.914	**0.901**	0.889	0.913	**0.897**	0.885	0.908	**0.883**	0.828	0.934
	EL_{AES}	**1.672**	1.637	1.709	**0.909**	0.896	0.922	**0.528**	0.520	0.536	**0.251**	0.212	0.288
	KL_{AES}	**1.657**	1.621	1.694	**0.909**	0.896	0.923	**0.532**	0.525	0.540	**0.206**	0.167	0.244
ρ_1=0.9 σ_1 =0.526	EK_{HES}	**0.543**	0.537	0.549	**0.535**	0.530	0.540	**0.510**	0.502	0.518	**0.492**	0.462	0.522
	EL_{AES}	**1.717**	1.676	1.758	**0.909**	0.898	0.919	**0.512**	0.514	0.520	**0.185**	0.158	0.210
	KL_{AES}	**1.663**	1.623	1.704	**0.908**	0.897	0.919	**0.532**	0.524	0.540	**0.198**	0.174	0.224
ρ_1=9 σ_1 =0.100	EK_{HES}	**0.278**	0.263	0.293	**0.258**	0.244	0.271	**0.217**	0.204	0.231	**0.188**	0.165	0.215
	EL_{AES}	**1.841**	1.658	2.023	**0.922**	0.886	0.959	**0.504**	0.482	0.523	**0.144**	0.095	0.189
	KL_{AES}	**1.757**	1.589	1.920	**0.902**	0.867	0.936	**0.510**	0.488	0.530	**0.162**	0.116	0.209

Let us now consider three Figures (1-3) illustrating the distribution of the estimated translog point elasticities for each of the three cases presented above. In the figures, the three rows correspond to the distributions of the HES_{EK}, AES_{EL} and AES_{KL}, respectively. For each of them, the first graph shows the point elasticities with the upper and lower bounds of their prediction intervals; the second graph represents the distribution of the point elasticities; the third graph is a surface plot that combines the previous two graphs.

Figure 1 refers to the $\rho = \rho_1$= 0.1 case. The graphs show that the estimated point elasticities range across very narrow figures and that prediction intervals are not wider than one decimal place. Moreover, distributions clearly peak around the median value, providing a further evidence that the estimated partial elasticities are characterized by a very low variability.

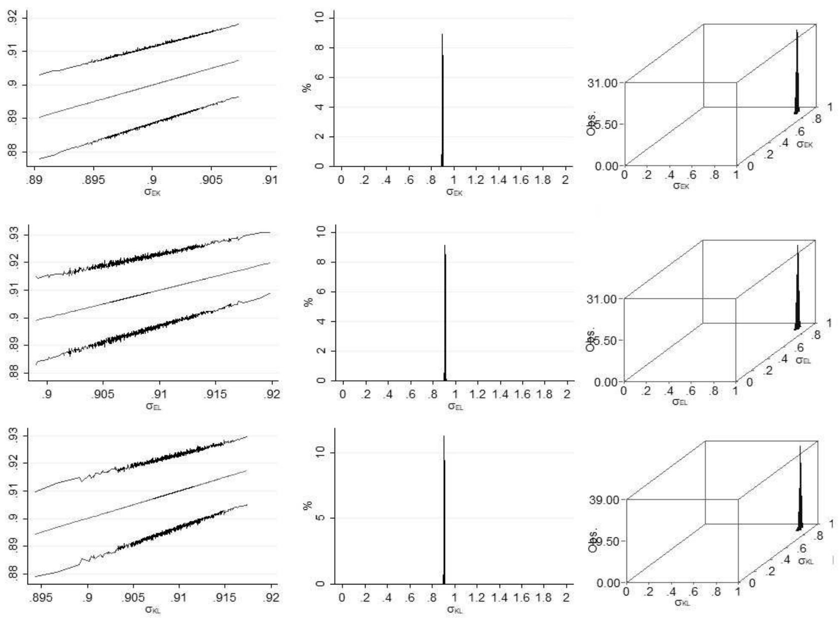

Figure 1 – Prediction intervals and elasticities distribution in case i)

Figure 2 refers to the $\rho = 0.1$ and $\rho_1 = 0.9$ case. The range of estimated AES_{EL} and AES_{KL} values is still small and prediction intervals are narrow. Moreover, they both peak close to the assumed value of the CES elasticities. As anticipated, the three partial elasticities cover a wider range of values than in the previous case. Indeed, as we move away from the expansion point (ρ, $\rho_x = 0$), the translog ability to approximate the nested CES decreases. However, Figure 2 still shows a clear peak in the HES_{EK} distribution around its assumed value and narrow prediction intervals for each point elasticity, which suggests that the expected variation in the point elasticities is limited.

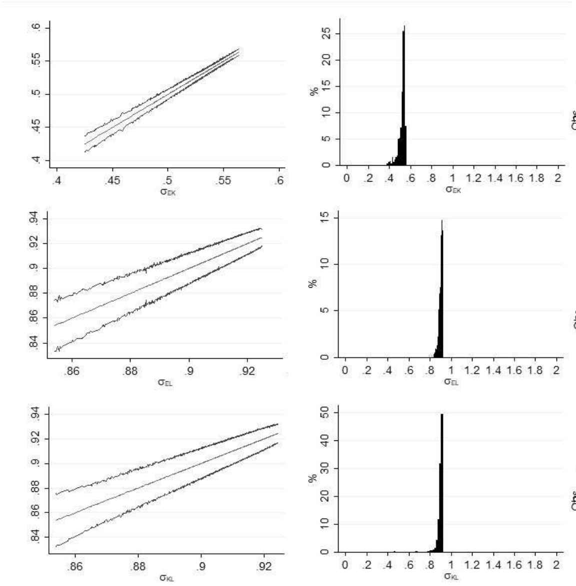

Figure 2 – Prediction intervals and elasticities distribution in case ii)

Figure 3 refers to the $\rho = 0.9$ and $\rho_1 = 0.1$ case. The graphs show that the estimates are less precise than in case i), but the distributions are still peaking around the assumed CES elasticities and prediction intervals are narrow.

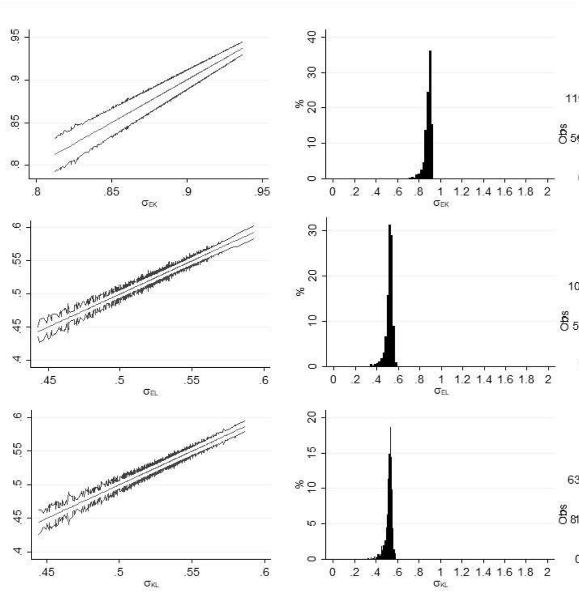

Figure 3 – Prediction intervals and elasticities distribution in case iii)

Thus, the graphical analysis of these three cases confirms that the approach is correctly identifying the assumed nested structure.

The remaining cases presented in Table IV show that, as the substitution parameters deviate from zero, HES and AES estimates become less accurate and the width of the estimated predicion interval increases, especially for values of ρ and ρ_1 larger than one. Nonetheless, even in those cases, the researcher can identify the underlying nesting structure by: i) looking at the median values of the point elasticities: in all the parametrizations considered, when $\rho \neq \rho_1$, the AES and HES partial elasticities point to the assumed ((E;K);L) structure; ii) evaluating graphically the distribution of the point elasticities.

5. Morishima elasticities of substitution

Most of the recent empirical literature estimating substitution elasticities with functions composed by three or more inputs has recoursed to Morishima partial elasticities (MES). This type of elasticity, originally concived by Morishima (1967) and then revived by Blackorby and Russell (1989), informs on the percentage change in a two inputs ratio given a percentage change in the price of one of the two inputs.[5] Being an asymmetric measure, it delivers more information than the Allen and Hichs partial elasticities because it allows looking at two degrees of substitutability for each pair of inputs. MES are related to AES by the following rule

$$\sigma_{ij}^{MES} = f_j x_j / f_i x_i \left(\sigma_{ij}^{AES} - \sigma_{jj}^{AES}\right)$$

factors that are AES substitutes are MES substitutes, factors that are AES complements might become MES substitutes. While Blackorby and Russell (1989) showed that, with non-nested CES functions, AES, HES and MES coincide (and are constant), with nested CES fuctions it is unknown how MES behave. In the following, we empirically investigate this by looking at MES estimates in the three cases presented in section 4.

Results showed in Table V provide several interesting insights. Firstly, we observe that in case i) all MES coincides and equate the AES and HES reported in Table IV, as predicted. Secondly, we see that in cases ii) and iii) EK_{MES} and KE_{MES} approximately coincide and equal EK_{HES}. Thirdly, we find that in cases ii) and iii) LE_{MES} equals LK_{MES} and that they differ from the corresponding AES. Lastly, we observe that prediction intervals are narrow overall, indicating that the point elasticities are not expected to vary much.

Table V – Estimated median translog MES and prediction intervals

	i)			ii)			iii)		
	ρ=0.1, σ = 0.909 ρ_1=0.1, σ_1 =0.909			ρ=0.9, σ = 0.526 ρ_1=0.1, σ_1 =0.909			ρ=0.1, σ = 0.909 ρ_1=0.9, σ_1 = 0.526		
	$\boldsymbol{\sigma}$	low	up	$\boldsymbol{\sigma}$	low	up	$\boldsymbol{\sigma}$	low	up
EK_{MES}	**0.901**	0.889	0.913	**0.899**	0.887	0.911	**0.537**	0.531	0.543
KE_{MES}	**0.901**	0.889	0.913	**0.898**	0.886	0.910	**0.536**	0.530	0.541
EL_{MES}	**0.909**	0.899	0.919	**0.531**	0.526	0.536	**0.909**	0.899	0.918
LE_{MES}	**0.905**	0.895	0.915	**0.715**	0.708	0.722	**0.728**	0.721	0.734
KL_{MES}	**0.909**	0.899	0.919	**0.533**	0.528	0.538	**0.908**	0.899	0.918
LK_{MES}	**0.905**	0.895	0.915	**0.715**	0.709	0.724	**0.728**	0.722	0.736

[5] Formally, MES are defined as:

$$\sigma_{ij}^{MES} = \frac{f_j}{x_j}\frac{|D_{ij}|}{|D|} - \frac{f_j}{x_i}\frac{|D_{jj}|}{|D|}$$

6. Conclusions

This paper describes a new empirical approach for discriminating among alternative nested structures for CES production functions based on the graphical analysis of translog point elasticities and their prediction intervals. Furthremore, it provides some insights on the rules followed by Morishima elasticities of substitution in a nested CES framework. A Monte Carlo simulation shows that the approach is robust to different parametrizations.

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