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A production function with variable elasticity of substitution greater than one

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Abstract

The idea of this paper comes from the famous remark of Piketty and Zuckman: "It is natural to imagine that the elasticity of substitution was much less than one in the eighteenth and nineteenth centuries and became larger than one in the twentieth and twenty-first centuries. One expects a higher elasticity of substitution in high-tech economies where there are lots of alternative uses and forms for capital." The main aim of this paper is to prove the existence of a production function of variable elasticity of substitution with values greater than one.

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1 Introduction

The elasticity of substitution is an essential macroeconomic concept and plays a crucial role in a multitude of theoretical economics topics. Due to its relevance, a long debate has emerged in the literature, not only for its value - lower or higher than one, but also on the two possible alternatives - constant elasticity or variable elasticity. So far, there has been no consensus on these issues. More than that, some empirical studies confirm one of the hypotheses and others confirm exactly the opposite hypothesis, in particular that concerning its value in relation to one. As was pointed out by Grossman et al. (2017), the size of the elasticity of substitution is much debated and still controversial. For example Chirinko (2008), who surveyed and evaluated a large number of studies, suggests a value of the elasticity of substitution in the range of 0.4 to 0.6, Knoblach et al. (2020) provide evidence that for the economy as a whole, the elasticity most likely falls in the range of 0.45 to 0.87. On the other hand, Karabarbounis and Neiman (2014) estimate an elasticity of substitution greater than one, but Chirinko, Fazzari, and Meyer (2011), Chirinko and Mallick (2017) and Lawrence (2015), all estimate elasticities below one.

The problem that the elasticity is constant or variable, is more or less, a theoretical approach. To decide between the two alternatives, we need a model, built on solid assumptions and then, if possible, analyzed via econometric methods to estimate its parameters.

One of the first empirical study on the elasticity of factor substitution, Arrow et al. (1961) observed that factor substitutability can vary between industries and across countries. More recently Miyagiwa and Papageorgiou (2007), concluded that the elasticity of factor substitution could be related to the level of economic development of a country. Duffy and Papageorgiou (2000) use cross-country data to show that generally, richer countries have an elasticity of factor substitution above one, while poorer countries have an elasticity of factor substitution below one.

An interesting analysis on this issue is found in Piketty and Saez's paper (2014). They try to analyze the effect of the average annual real rate of return (r) and the capital income ratio β , on the share of capital income in the national income, defined as $\alpha = r\beta$. In the standard economic model with perfectly competitive markets, r is equal to the marginal product of capital. As the volume of capital β rises, the marginal product r tends to decline. The important question is whether r falls more or less rapidly than the rise in β . If the standard hypothesis in economics are assumed, that is a unitary elasticity, then the fall in r , exactly offsets the rise in β , so that the capital share is a technological constant. However, historical variations in capital shares are far from negligible. In recent decades, rich countries have experienced both a rise in β and a rise in α , which suggests that σ is somewhat larger than one. This hypothesis has been suggested by Sato and Hoffman (1968), who argued that as time passes and new technologies become available the opportunities for factor substitution are increased.

As noted above, there is a wide variety of estimates of the elasticity of substitution and this one can be interpreted as a missing specification of the production function and, allows us to consider other production functions that present a variable elasticity. Although there are a large number of papers devoted to production functions with a constant elasticity of substitution, the number of those which analysed production functions with variable elasticity is relatively limited and, unfortunately, written in the early 60 and 70. The first production function with the variable elasticity of substitution was proposed by Liu and Hildebrand (1965) and a few years later by Lu (1967). The elasticity of substitution of this function has an interesting property, being a decreasing function from a bounded upper limit, to one. Unfortunately, this property is true only for some k greater than a given value $k_0 > 0$, and not for all $k > 0$. Two other production functions were developed by Hoffman and Sato (1968) and by Revankar (1971). Unfortunately, for these latter functions, the elasticity of substitution increases infinitely when k tends to infinity. The latest attempt to provide a production function with variable elasticity of substitution is provided by the paper of Chilarescu (2019). The domain of variation of this elasticity of substitution is limited to the unit interval $(0, 1)$.

The paper is organized as follows. The first section is this introduction In the next section we prove the existence of a new production function with variable elasticity of substitution, but having the property that its elasticity is greater than one for all $k > 0$. In the third section we point out some interesting properties of this new production function. The forth section presents some numerical results and the final section is dedicated to conclusions and some recommendations.

2 A new production function with variable elasticity of substitution

As in the paper of Sato and Hoffman (1968), we consider the case of a production function assumed to be homogeneous of degree one and denote by: $y = f(k) = \frac{F(K,L)}{L}$, the output per-capita and by $k = \frac{K}{L}$, the capital labor ratio. Under this hypothesis, the marginal rate of substitution and the elasticity of substitution can be expressed in the following representation

$$r(k) = \frac{y}{y'} - k \quad (1)$$

and

$$\sigma(k) = \frac{y' (ky' - y)}{ky y''}, \quad (2)$$

and thus they obtained the following result:

$$\frac{dy}{y} = \frac{dk}{k + \zeta \exp \int \frac{dk}{k\sigma(k)}}, \quad (3)$$

If we express the elasticity of substitution in terms of the marginal rate of substitution,

$$\sigma(k) = \frac{r(k)}{kr'(k)} = \frac{\frac{dk}{k}}{\frac{dr}{r}}, \quad (4)$$

then, the equation (3) becomes

$$\frac{dy}{y} = \frac{dk}{k + \zeta r(k)}, \quad \zeta > 0. \quad (5)$$

As it is well-known, the marginal rate of substitution is a positive increasing function. In order to obtain an explicit solution for the above differential equation, we assume that the function $r(k)$ takes the following form:

$$r(k) = \frac{k [\alpha (1 - \eta) k^\psi + \beta (1 - \theta)]}{\alpha \eta k^\psi + \beta \theta}, \quad \alpha, \beta, \eta, \theta \in (0, 1). \quad (6)$$

Without loss of generality, we can suppose that $\zeta = 1$ and thus the equation (5) will be written

$$\frac{dy}{y} = \frac{dk}{k} \frac{\alpha \eta k^\psi + \beta \theta}{\alpha k^\psi + \beta}. \quad (7)$$

If we denote $\eta = \theta + \omega\psi$ then we obtain

$$\frac{dy}{y} = \frac{dk}{k} \frac{\alpha [\theta + \omega\psi] k^\psi + \beta \theta}{\alpha k^\psi + \beta}, \quad (8)$$

whose solution is given by

$$y = f(k) = \gamma k^\theta [\alpha k^\psi + \beta]^\omega, \quad \gamma > 0. \quad (9)$$

A remark is absolutely necessary here. The above form of the marginal rate of substitution follows immediately from the more general expression

$$r(k) = \frac{k [\alpha \delta k^\psi + \beta \varphi]}{\alpha \eta k^\psi + \beta \theta}.$$

In order to obtain an explicit solution of the differential equation (3) we need to choose $\delta = 1 - \eta$ and $\varphi = 1 - \theta$. The explicit form of the parameter η is suggested by this new property of the elasticity of substitution, that is, its values are greater than one.

It is just a simply exercise to prove that $y_0 = f(0) = 0$. The first derivative is given by

$$f'(k) = \gamma \frac{\alpha (\theta + \omega\psi) k^\psi + \beta \theta}{k^{1-\theta} (\alpha k^\psi + \beta)^{1-\omega}}, \quad (10)$$

and it is a positive function for all $k > 0$.

$$\lim_{k \rightarrow 0} f'(k) = \gamma \lim_{k \rightarrow 0} \frac{\alpha(\theta + \omega\psi)k^\psi + \beta\theta}{k^{1-\theta}(\alpha k^\psi + \beta)^{1-\omega}} = +\infty.$$

$$\lim_{k \rightarrow +\infty} f'(k) = \gamma \lim_{k \rightarrow +\infty} \frac{\alpha(\theta + \omega\psi)k^\psi + \beta\theta}{k^{1-\theta}(\alpha k^\psi + \beta)^{1-\omega}} = 0, \quad \psi > 0, \quad \omega \in (0, 1), \quad \theta + \omega\psi < 1.$$

The second derivative is given by

$$f'' = -f \frac{\alpha^2(\theta + \omega\psi)(1 - \theta - \omega\psi)k^{2\psi} + \alpha\beta[\psi\omega(1 - \psi) + 2\theta(1 - \theta - \omega\psi)]k^\psi + \beta^2\theta(1 - \theta)}{k^2(\alpha k^\psi + \beta)^2}, \quad (11)$$

and it is negative for all $k > 0$. Substituting the results obtained from the equations (9), (10) and (11) into the equation (2) we get the following relation for the elasticity of substitution

$$\sigma(k) = \frac{[\alpha(\theta + \omega\psi)k^\psi + \beta\theta][\alpha(1 - \theta - \omega\psi)k^\psi + \beta(1 - \theta)]}{[\alpha(\theta + \omega\psi)k^\psi + \beta\theta][\alpha(1 - \theta - \omega\psi)k^\psi + \beta(1 - \theta)] - \alpha\beta\omega\psi^2 k^\psi} \geq 1. \quad (12)$$

Passing to the limit for $k \rightarrow 0$ and $k \rightarrow +\infty$, we obtain

$$\lim_{k \rightarrow 0} \sigma(k) = 1, \quad \text{and} \quad \lim_{k \rightarrow +\infty} \sigma(k) = \begin{cases} 1 & \text{if } 2\psi \geq 1 \\ 1 + \frac{\omega\psi^2}{2\theta(1 - \theta - \omega\psi) + \omega\psi(1 - \psi)} & \text{if } 2\psi < 1 \end{cases}$$

The first derivative of the elasticity of substitution is given by

$$\sigma'(k) = \frac{\alpha\beta\omega\psi^3 k^{\psi-1} [\beta^2\theta(1 - \theta) - \alpha^2(\theta + \omega\psi)(1 - \theta - \omega\psi)k^{2\psi}]}{\{[\alpha(\theta + \omega\psi)k^\psi + \beta\theta][\alpha(1 - \theta - \omega\psi)k^\psi + \beta(1 - \theta)] - \alpha\beta\omega\psi^2 k^\psi\}^2}. \quad (13)$$

If

$$k \in \left(0, \left[\frac{\beta^2\theta(1 - \theta)}{\alpha^2(\theta + \omega\psi)(1 - \theta - \omega\psi)}\right]^{\frac{1}{2\psi}}\right),$$

then $\sigma'(k)$ is positive and therefore $\sigma(k)$ will be an increasing function. For all

$$k > \left[\frac{\beta^2\theta(1 - \theta)}{\alpha^2(\theta + \omega\psi)(1 - \theta - \omega\psi)}\right]^{\frac{1}{2\psi}}$$

$\sigma'(k)$ is negative and therefore $\sigma(k)$ will be a decreasing function.

The relative share of capital (π_k) and the relative share of labor (π_l), are given by

$$\pi_k = \frac{kr}{f} = \frac{\alpha(\theta + \omega\psi)k^\psi + \beta\theta}{\alpha k^\psi + \beta}, \quad \pi_l = 1 - \frac{kr}{f} = 1 - \frac{\alpha(\theta + \omega\psi)k^\psi + \beta\theta}{\alpha k^\psi + \beta}. \quad (14)$$

It is just a simple exercise to prove that the relative share of capital is a bounded increasing function, the relative share of labor is a bounded decreasing function and,

$$\lim_{k \rightarrow \infty} \pi_k = \theta + \omega\psi < 1 \quad \text{and} \quad \lim_{k \rightarrow \infty} \pi_l = 1 - \theta - \omega\psi > 0.$$

3 Some properties of the new production function

The new production function has the following properties:

- a. $f(k) \geq 0$, $f'(k) \geq 0$, $f''(k) \leq 0$ for all $k \geq 0$, $\lim_{k \rightarrow 0} f'(k) = +\infty$ and $\lim_{k \rightarrow +\infty} f'(k) = 0$.

b. The elasticity of substitution is, first an increasing function up to a bounded upper limit and then a decreasing function.

c. If $\omega = 0$, then the new production function reduces to the Cobb-Douglas function,

$$f(k) = \gamma k^\theta.$$

d. If $\theta = 0$ and $\omega = \frac{1}{\psi}$, then the new production function reduces to the CES function,

$$f(k) = \gamma [\alpha k^\psi + \beta]^\frac{1}{\psi}.$$

Consequently, for the production function given by (9), we may assume that $\alpha + \beta = 1$.

e. If $\psi(\omega + \theta) = 1$, then the production function becomes

$$f(k) = \gamma k^\theta [\alpha k^\psi + 1 - \alpha]^\frac{1}{\psi} - \theta,$$

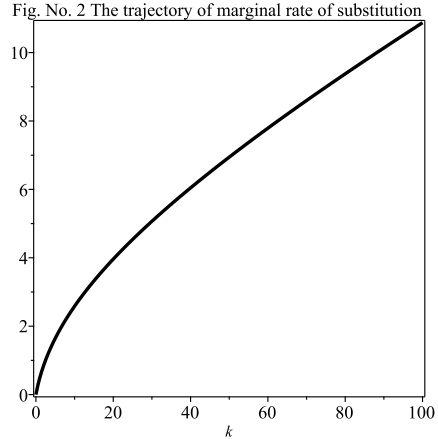
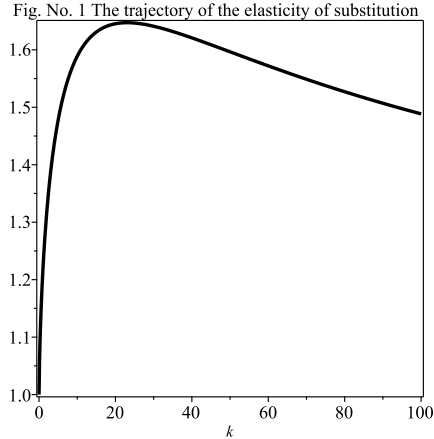
and thus we obtain a function with only four parameters to be estimated.

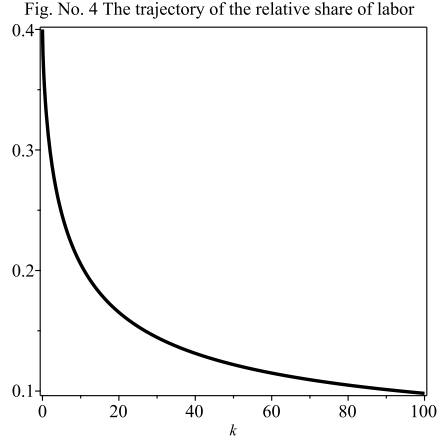
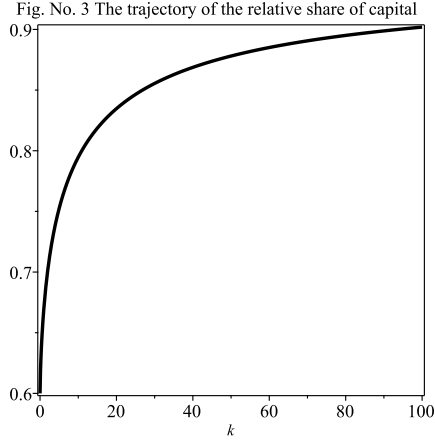
4 Numerical results

The parameters θ , ψ and ω can not be arbitrarily chosen. First of all they have to satisfy the restriction $\theta + \omega\psi < 1$. Also, we can easily observe that $\theta = \pi_k(0)$. In addition to these restrictions, in the case of this paper the choose of the parameters is obviously, more or less arbitrarily. Certainly, the most reasonable way in the choice of parameters is an econometric procedure, and this is the target of our future research. On this subject we will present some ideas and recommendations in the next section. Let us consider the following benchmark values for the economy:

$$\theta = 0.6, \quad \omega = 0.5, \quad \psi = 0.7, \quad \alpha = 0.2, \quad \beta = 0.8, \quad \gamma = 1.05$$

Substituting these values into the adequate above relations, and via Maple we obtain the following trajectories: the elasticity of substitution, the marginal rate of substitution and the two relative shares, capital and labor. These trajectories are presented in the following graphs.





We can observe that the elasticity of substitution increases with k to an upper bounded limit and then slowly decreases to the bounded inferior limit.

5 Conclusions and some recommendations

As we pointed out in the third section, the new production function has some interesting properties, not only because it contains as particular cases the Cob-Douglas or CEC production functions, but first of all because its elasticity of substitution is a variable function with values greater than one. More than this, the elasticity of substitution is first an increasing function up to a bounded upper limit, and then it decreases to a positive bounded lower limit.

Unfortunately, the new production function has a significant limitation, such as the CES function or other production functions with a variable elasticity of substitution. This function is non-linear in the parameters, and therefore there are many difficulties in the procedure of estimation. A possible alternative is to consider estimates obtained by fitting the production function to observations on the two variables y and k and write the function in the following form:

$$\ln(y) = \ln(\gamma) + \theta \ln(k) + \omega \ln(\alpha k^\psi + 1 - \alpha) + \varepsilon. \quad (15)$$

To estimate the parameters of the above equation, we can use nonlinear least squares methods, for which there exists some specialized softwares, or via the method developed by Kmenta (1967) for the CES production function. Some of these problems will be considered in a future paper.

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