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# Price competition in a mixed duopoly with discontinuous marginal cost function

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## Abstract

In this paper, we study price competition in a mixed duopoly with discontinuous marginal cost function. We obtain that the set of pure strategy Nash equilibrium is a range of prices. We show that when the market size is sufficiently big, the set of Nash equilibrium in the mixed duopoly is the same as the Bertrand competition with two firms. Hence, social welfare is also similar in this case. However, the upper bound of the set of pure strategy Nash equilibrium can be lower in the case of mixed duopoly than Bertrand competition with two firms, when the market size is not sufficiently big. Therefore, the nature of ownership is relevant in the case of smaller markets and not in the case of big markets. When the cost function of one of the firms is only discontinuous as specified in section 2 and the marginal cost of the other firm is constant, we get unique pure strategy Nash equilibrium in certain cases.

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# 1 Introduction

In this paper we study price competition in a mixed duopoly where the strictly increasing , piecewise linear with a point of discontinuity. One of the ways the firms may face such kind of marginal cost functions is when the price of atleast one input rises to a higher level beyond a threshold level of that input[1]. Roy Chowdhury (2009) have studied Bertrand competition with discontinuous marginal cost. The discontinuity arises because of non-rigid capacity constraint. The paper shows that there exists a range of prices that constitute pure strategy Nash equilibrium. This paper is similar to Roy Chowdhury (2009) in terms of price competition with discontinuous cost function. However, the exact specification of discontinuous cost functions are not same and the type of price competition is also not same since we consider a mixed duopoly market.

There are number of examples of mix market in the real world. Some of them are; the automobile industry in Germany, oil refining industry in India, telecommunication sector in India , iron and steel industry in India. In most of these mix markets, there is only one publicly owned or partially publicly owned firm and other firms are privately owned firms. In case of telecommunication sector in India, we see price competition among the firms.

In the literature on mixed market, the presence of public firm is considered as a means to regulate the oligopoly industry. De-Fraja and Delbono (1989) show that nationalization is always welfare improving than Stackelberg leadership of public firm. And Stackelberg leadership of public firm is better than the firms choosing quantity simultaneous in mixed oligopoly. In case of quantity competition where firms choose quantity simultaneously and the number of firms in the market is close to the optimal number, the public firm should maximize profit rather then social welfare. Cremer, Marchand and Thisse (1989) show that nationalization of a single private firm in Cournot oligopoly leads to higher social welfare due to increase in aggregate output provided the public firm satisfies the constraint of non-negative profit. Fershtman (1990) shows that privatization of inefficient public firm leads to fall in social welfare in quantity competition. Delbono and Denicolo (1993) show that the public sector firm can be used as an instrument to reduce the wastage in resources because of duplication of effort in R&D investment, thus increasing social welfare.

There are works showing that ownership is neutral when the firms are getting output subsidy. White (1996) finds that when firms are getting output subsidy and continued after privatization also, the social welfare remains same in both pre and post privatization of the public firm in quantity competition. Poyago-Theotoky (2001) also finds that ownership is irrelevant in quantity competition and in Stackelberg leadership of public firm in the presence of output subsidy. Myles (2002) extends the above result to general downward sloping demand function and increasing cost function. Hashimzade, Khovadaisi and Myles (2007) find irrelevance of ownership in differentiated product market when output subsidy is provided to the firms. Fjell and Heywood (2004) show that when the firms are getting output subsidy, the social welfare falls with the privatization of the public firm if privatization leads to private leadership in the market. Tomaru (2006) obtains the

[1]In several contexts such types of cost functions may arise. One instance can be when the price of capital is higher after certain amount, the cost function become discontinuous.

irrelevance result in case of output subsidy when there is partial privatization in quantity competition. Tomaru and Saito (2010) find that the irrelevance result as in White (1996) does not hold, if the outcome is private leadership in endogenous timing game with privatization. Matsumura and Okumura (2013) obtain that in quantity competition with firms choosing quantity simultaneously and in Stackelberg quantity competition the ownership is irrelevant when the firms have to produce a minimum level of output. Scrimitore (2014) also extends the results on irrelevance of ownership in both quantity and price competition in differentiated product market.

In a mixed duopoly with price competition, the private firm chooses price to maximize profit and the public firm chooses price to maximize social welfare. The payoff function is not same. In this context, it is natural to ask whether this difference in payoff function in a mixed competition leads to different set of Nash equilibrium outcomes than the Bertrand competition. The literature on price competition in mixed duopoly has number of results showing that ownership is irrelevant, it means that the Nash equilibrium prices are similar to Bertrand competition . Ogawa and Kato (2006) and Dastidar and Sinha (2011) have shown that ownership is irrelevant in price competition with strictly convex, increasing and differentiable cost function in a duopoly market. In the similar set up, Mahanta(2016) shows that when the number of public firms is above a threshold level, the pure strategy Nash equilibrium in the Mixed oligopoly is a subset of the set of pure strategy Nash equilibrium obtained in the Bertrand competition. Roy Chowdhury and Roy Chowdhury (2018) finds that in a mixed duopoly with price competition when the private firm can ration the demand but the public firm cannot ration, Coalition Proof Nash equilibrium price is same as the perfectly competitive price. When the public firm is privatized the welfare does not increase. These works on price competition in mixed market take strictly increasing continuous and twice differentiable cost function. In this paper we study price competition in homogeneous good mixed duopoly market. We show that ownership is not completely irrelevant in the set-up considered in the paper. We find that ownership is not relevant when the size of the market is sufficiently big. When the size of the market is sufficiently big, the privatization will not have any effect on the social welfare.

This paper extends the line of research in price competition in a mixed duopoly considering marginal cost to be discontinuous and stepwise increasing. This paper is the first work on mixed duopoly with discontinuous and step wise increasing marginal cost function. The total cost function is strictly increasing, piecewise linear with a point of discontinuity. The structure of the paper is, we present the model in Section 2. In Subsection 3.1, we show the results when both the firms are private firm. In Subsection 3.2, we show the results when firm 1 is a public firm and firm 2 is a private firm. In Section 4, we conclude.

# 2 Model

There are two firms in the market, firm 1 is a public sector firm that maximizes social welfare and firm 2 is a private firm that maximizes profit. The demand function is $A - p = q$, where $p$ = price, $q$ = output and $A$ is a positive real number. The firm which sets the lowest price gets all the demand at that price. If both the firms set the same price, the market demand is equally shared[2]. The demand curve of each firm $i$ is

$$q_i = \begin{cases} A - p_i, \text{ if } p_i < p_{-i} \\ \frac{A-p_i}{2}, \text{ if } p_i = p_{-i} \\ 0, \text{ if } p_i > p_{-i} \end{cases}$$

The cost function of each firm $i$ is similar and it is

$$cq = \begin{cases} c_l q, \text{if } q \leq q_o \\ c_h q, \text{ if } q > q_o \end{cases}$$

The total cost function is strictly increasing, piecewise linear and discontinuous at $q = q_o$. When the input price increases after certain level the marginal cost increases and the corresponding level of output level at which marginal cost increases is $q_o$. When the production is more than $q_o$ units of output, the price of the inputs increases to a higher level. So the total cost becomes $c_h q$ for $q > q_o$ rather than $c_l q_o + c_h(q - q_o)$ for $q > q_o$ . The marginal cost curve is discontinuous at the point $q = q_o$. It is a step wise increasing function because marginal cost is $c_l$ for $q \leq q_o$ and $c_h$ for $q > q_o$ where $c_l < c_h$. We get this kind of cost function when the input supplier can discriminate price ( charge different price) based on the amount of input purchased. Another situation in which we can get this kind of cost function is to produce more than a certain amount of output more expansive technology is required. It leads to increase in marginal cost from $c_l$ to $c_h$. We assume that the point of discontinuity is same for both the firms. It means that technology is same for both the firms and they also face similar input market.

We get a price $p_o$ such that $p_o = A - q_o$. Further, we assume that the firms have to meet the demand at each price, they cannot ration. We also assume that $A > c_h > c_l$.

Based on the assumptions stated above the profit function of each firm $i$ is,

$$\pi_i = \begin{cases} (A - p_i)(p_i - c_l), \text{ if } p_i < p_{-i} \text{ and } (A - p_i) \leq q_o \\ (A - p_i)(p_i - c_h), \text{ if } p_i < p_{-i} \text{ and } (A - p_i) > q_o \end{cases}$$

$$\hat{\pi}_i = \begin{cases} \left(\frac{A-p_i}{2}\right)(p_i - c_l), \text{ if } p_i = p_{-i} \text{ and } \left(\frac{A-p_i}{2}\right) \leq q_o \\ \left(\frac{A-p_i}{2}\right)(p_i - c_h), \text{ if } p_i = p_{-i} \text{ and } \left(\frac{A-p_i}{2}\right) > q_o \\ 0, \text{ if } p_i > p_{-i} \end{cases}$$

We take the social welfare as the sum of consumer surplus and producers surplus[3]. The payoff function of the public sector firm is to maximize social welfare. The payoff of firm $i$, if it wants to maximize social welfare is,

$$sw_i = \begin{cases} \int_{p_i}^{A} (A - x)dx + (A - p_i)(p_i - c_l), \text{ if } p_i < p_{-i} \text{ and } (A - p_i) \leq q_o \\ \int_{p_i}^{A} (A - x)dx + (A - p_i)(p_i - c_h), \text{ if } p_i < p_{-i} \text{ and } (A - p_i) > q_o \end{cases}$$

[2]It is a standard assumption in the literature on price competition.
[3]This is the standard practise in the literature on mixed oligopoly.

$$\hat{sw}_i = \begin{cases} \int_{p_i}^{A}(A-x)dx + 2\big(\frac{A-p_i}{2}\big)\big(p_i - c_l\big), \text{ if } p_i = p_{-i} \text{ and } \big(\frac{A-p_i}{2}\big) \leq q_o \\ \int_{p_i}^{A}(A-x)dx + 2\big(\frac{A-p_i}{2}\big)\big(p_i - c_h\big), \text{ if } p_i = p_{-i} \text{ and } \big(\frac{A-p_i}{2}\big) > q_o \\ \int_{p_{-i}}^{A}(A-x)dx + (A-p_{-i})(p_{-i}-c_l), \text{ if } p_i > p_{-i} \text{ and } (A-p_{-i}) \leq q_o \\ \int_{p_{-i}}^{A}(A-x)dx + (A-p_{-i})(p_{-i}-c_h), \text{ if } p_i > p_{-i} \text{ and } (A-p_{-i}) > q_o \end{cases}$$

Next we define a few notations. The monopoly price and quantity when marginal cost is $c_l$ is $p^{mon}_{c_l} = \dfrac{A+c_l}{2}$, $q^{mon}_{c_l} = \dfrac{A-c_l}{2}$. The monopoly price and quantity when marginal cost is $c_2$ is $p^{mon}_{c_h} = \dfrac{A+c_h}{2}$, $q^{mon}_{c_h} = \dfrac{A-c_h}{2}$ .

# 3 Results

First we derive the results when firm 1 and firm 2 maximize profit that is both are private firms. The results in Subsection 3.1 pertains to a slightly different version of cost function compared to Roy Choudhary (2009). The detail proofs are presented so that comparison becomes easier and it helps to derive some of the results on mixed duopoly.

## 3.1 When both the firms are private firms

It is clear from the profit function that $\pi^{c_l}_i(p) = \hat{\pi}^{c_l}_i(p) = 0$ at $p = c_l$ and $p = p^{Max}$ and $\pi^{c_h}_i(p) = \hat{\pi}^{c_h}_i(p) = 0$ at $p = c_h$ and $p = p^{Max}$ for each firm $i, i = 1, 2$. First we compare the profits of each firm at two different marginal costs when a firm serves the market as a single seller and when it shares the market.

**Lemma 3.1.** $\pi^{c_l}_i(p) > \pi^{c_h}_i$ *and* $\hat{\pi}^{c_l}_i(p) > \hat{\pi}^{c_h}_i, i = 1, 2$ *at each* $p \in [c_h, p^{Max}]$, .

*Proof.* We know $\pi^{c_l}_i(p) = (A-p)(p-c_l)$ and $\pi^{c_h}_i = (A-p)(p-c_h)$. Since $c_h > c_l$, it is obvious that $\pi^{c_l}_i(p) > \pi^{c_h}_i$ for each $p \in [c_l, p^{Max}]$ and for each $i = 1, 2$.
We have $\hat{\pi}^{c_l}_i(p) = \big(\dfrac{A-p}{2}\big)\big(p-c_l\big)$ and $\hat{\pi}^{c_h}_i(p) = \big(\dfrac{A-p}{2}\big)\big(p-c_h\big)$. Using $c_h > c_l$ we get that $\hat{\pi}^{c_l}_i(p) > \hat{\pi}^{c_h}_i$ for each price $p \in [c_l, p^{Max}]$ and for each $i = 1, 2$. □

We get that the curve of $\pi^{c_l}_i(p)$ will always lie above the curve of $\pi^{c_h}_i(p)$ whenever the profit is positive. Note that $\pi^{c_l}_i(p) > \hat{\pi}^{c_l}_i(p)$ for each $p \in [c_l, p^{Max}]$ and each $i = 1, 2$ and $\pi^{c_h}_i(p) > \hat{\pi}^{c_h}_i(p)$ for each $p \in [c_h, p^{Max}]$ and for each $i = 1, 2$.

**Lemma 3.2.** *There exists a unique price* $\hat{p}$, $\hat{p} \in [c_h, p^{Max})$ *such that* $\hat{\pi}^{c_l}_i(\hat{p}) = \pi^{c_h}_i(\hat{p})$.

*Proof.* We have $\pi^{c_h}_i(p) = (A-p)(p-c_h)$ and $\hat{\pi}^{c_l}_i(p) = \big(\dfrac{A-p}{2}\big)\big(p-c_l\big)$. If $p < c_h$ then $(A-p)(p-c_h) < 0$. But for $p < c_h, \big(\dfrac{A-p}{2}\big)\big(p-c_l\big) > 0$ . So $\hat{\pi}^{c_l}_i(p) > \pi^{c_h}_i(p)$ for $p < c_h$.

Again at $p = p^{Max}$, $(A - p) = 0$, so $\hat{\pi}_i^{c_l}(p) = \pi_i^{c_h}(p)$ .
Now solve for $p$ such that $(A-p)(p-c_h) = \big(\frac{A-p}{2}\big)\big(p-c_l\big)$. In the equation $(A-p)(p-c_h) = \big(\frac{A-p}{2}\big)\big(p-c_l\big)$, we can cancel $(A-p)$ term from both sides because $(A-p) > 0$ for all $p \in [c_h, p^{Max})$. We are left with a linear equation in a single unknown. We get $\hat{\pi}_i^{c_l}(p) = \pi_i^{c_h}(p)$ at $p = 2c_h - c_l = \hat{p}$ when $\hat{p} \in [c_h, p^{Max})$. Therefore, $\hat{p}$ is the unique price in the range $p \in [c_h, p^{Max})$.

□

Lemma 3.2 says that there exists a unique price $\hat{p}$ at which the profit of a firm while serving the market alone by producing at the higher marginal cost is equal to the profit when the market is shared by producing at the lower marginal cost. The immediate corollary from Lemma 3.2 is that when $p$ is $p < \hat{p}$, we have $\hat{\pi}_i^{c_l}(p) > \pi_i^{c_h}(p)$. When $p$ is $p > \hat{p}$, we have $\hat{\pi}_i^{c_l}(p) < \pi_i^{c_h}(p)$. This means that the curve of $\hat{\pi}_i^{c_l}(p)$ will lie above the curve of $\pi_i^{c_h}(p)$ for $p < \hat{p}$ and the curve of $\hat{\pi}_i^{c_l}(p)$ will lie below the curve of $\pi_i^{c_h}(p)$ for $\hat{p} < p < p^{Max}$. The two curves intersect at the point $p = \hat{p} = 2c_h - c_l$. It is obvious that $\hat{p} > c_h$, since $c_h > c_l$.

We have another price $p_o$ at which the profit of a firm $i$ is $\pi_i(p_i) = (A - p_i)(p_i - c_h)$, if $p_i < p_{-i}$ and $p_i < p_o$. We assume $p_o > c_h$. So we get three possibilities, $p_o > \hat{p}$, $p_o < \hat{p}$, and $p_o = \hat{p}$. Suppose $p_o < \hat{p}$ and firm 2 sets price $p_2 = p_o$, if firm 1 sets $p_1$ such that $p_1 < p_2$ then the profit of firm 1 is $\pi_1^{c_h}(p_1)$. From lemma 3.2, we know that $\pi_1^{c_h}(p_1) < \hat{\pi}_1^{c_l}(p_o)$, since $p_1 < p_o < \hat{p}$. So firm 1 should match the price set by firm 2. Firm 1 has no tendency to under cut the price of firm 2. This argument applies to firm 2 also when firm 1 sets price $p_1 = p_o$ under the condition $p_o < \hat{p}$.

Again suppose $p_o < \hat{p}$ and firm 2 sets price $p_2$ such that $p_o < p_2 < \hat{p}$. If firm 1 sets $p_1$ such that $p_1 = p_2 - \epsilon$ then the profit of firm 1 is $\pi_1^{c_l}(p_2 - \epsilon)$. We know $\pi_1^{c_l}(p_2 - \epsilon) > \hat{\pi}_1^{c_l}(p_2)$ from Lemma 3.2. So firm 1 has the tendency to under cut the price of firm 2. Therefore, if firm 2 sets price $p_2$ higher than $p_o$ when $p_o < \hat{p}$, firm 1 has the tendency to under cut the price of firm 2. So firm 2 will not set price $p_2$ such that $p_2 > p_o$ when $p_o < \hat{p}$. Similar argument holds for firm 2 also when firm 1 sets price higher than $p_o$ when $p_o < \hat{p}$.

When $p_o < \hat{p}$, firm 1 and firm 2 will set the same price in the price range $[c_l, p_o]$. There is no tendency among firm 1 and firm 2 to under cut each others price. Thus, each price in the range $[c_l, p_o]$ constitutes a pure strategy Nash equilibrium under the condition $p_o < \hat{p}$.

Suppose $p_o > \hat{p}$, firm 2 sets $p_2$ such that $p_2 \leq \hat{p}$. If firm 1 sets $p_1 = p_2 - \epsilon$, the profit of firm 1 is $\pi_1^{c_h}(p_1)$ since $p_1 < p_o$ and also because it serves the whole market alone. If firm 1 sets $p_1 = p_2$, the profit of firm 1 is $\hat{\pi}_1^{c_l}(p_1)$. We know from Lemma 3.2 that $\pi_1^{c_h}(p_2 - \epsilon) < \hat{\pi}_1^{c_l}(p_2)$. So firm 1 will not under cut the price of firm 2. This holds true for firm 2 also.

Again suppose $p_o > \hat{p}$, firm 2 sets $p_2$ such that $\hat{p} < p_2 < p_o$. If firm 1 sets $p_1 = p_2 - \epsilon$, the profit of firm 1 is $\pi_1^{c_h}(p_2 - \epsilon)$. If firm 1 sets $p_1 = p_2$, the profit of firm 1 is $\hat{\pi}_1^{c_l}(p_2)$. From Lemma 3.2, we know that $\pi_1^{c_h}(p_2 - \epsilon) > \hat{\pi}_1^{c_l}(p_2)$ for $p > \hat{p}$. So firm 1 will under cut the price $p_2$ of firm 2. Therefore, firm 2 will never set price $p_2 > \hat{p}$ when $\hat{p} < p_o$. This holds for firm 1 also. Thus, when $\hat{p} < p_o$, each price in the range $[c_l, \hat{p}]$ is a pure strategy Nash equilibrium.

When $p_o = \hat{p}$ from the above discussion it is clear that each price in the range $[c_l, \hat{p} = p_o]$ is a pure strategy Nash equilibrium. We present the above result in the following proposition.

**Proposition 3.3.** *Given the set-up, each price in the range* $[c_l, \min\{p_o, \hat{p}\}]$ *is a pure strategy Nash equilibrium.*

Proposition 3.3 gives the pure strategy Nash equilibrium when both the firms maximizes profit. The set of pure strategy Nash equilibrium strategy is not unique, instead it is a range of prices. The profit increases as the price increases in this price range under the condition $A > 4c_h - 3c_l$. Dastidar(2001) shows the ordering of the profit of firms in this price range.

## 3.2 Mixed Duopoly

Next we find the set of pure strategy Nash equilibrium in the mixed duopoly market where firm 1 maximizes social welfare ( public firm) and firm 2 maximizes profit ( private firm). Social welfare is,

$sw^{c_l}(p) = \int_p^{p^{Max}} (A - x)dx + \pi_1^{c_l}(p),\ sw^{c_h}(p) = \int_p^{p^{Max}} (A - x)dx + \pi_1^{c_h}(p),$

$\hat{sw}^{c_l}(p) = \int_p^{p^{Max}} (A-x)dx+\hat{\pi}_1^{c_l}(p)+\hat{\pi}_2^{c_l}(p) = \int_p^{p^{Max}} (A-x)dx+2\hat{\pi}_1^{c_l}(p) = \int_p^{p^{Max}} (A-x)dx+\pi_1^{c_l}(p).$

The first equality is due to similarity in the cost function of both the firms. The second equality is due to linear cost function. The second inequality shows that the social welfare is same when a single seller serves the whole market and when the market is shared. Note social welfare is such that $sw^{c_l} > sw^{c_h}$ as $c_h > c_l$.The social welfare is always higher when the marginal cost is lower. We know that the social welfare $sw^{c_l}(p)$ is maximized at $p = c_l$. And the social welfare $sw^{c_h}(p)$ is maximized at $p = c_h$. By comparing the payoff function of firm 1 that is social welfare when it shares the market and when it is the single seller in the market, we get the following result.

**Lemma 3.4.** *There exists a unique price* $p'$ *such that* $sw^{c_l}(p) = sw^{c_h}(c_h)$ *and* $p > c_h$.

*Proof.* As defined $sw^{c_l}(p) = \int_p^{p^{Max}} (A - x)dx + (A - p)(p - c_l)$ and $sw^{c_h} = \int_{c_h}^{p^{Max}} (A - x)dx$ at $p = c_h$.

We solve for $p$ taking the equation $\int_p^{p^{Max}} (A - x)dx + (A - p)(p - c_l) = \int_{c_h}^{p^{Max}} (A - x)dx$.

Simple manipulation of this expression gives, $(A - p)(p - c_l) = \int_{c_h}^{p}(A - x)dx$. We get a quadratic equation, $p^2 - 2c_l p - (2A(c_h - c_l) - c_h^2) = 0$.

The solution is $p = c_l \pm \sqrt{c_l^2 + (2A(c_h - c_l) - c_h^2)}$.

Next we show that $p = c_l + \sqrt{c_l^2 + (2A(c_h - c_l) - c_h^2)}$ is greater than $c_h$.

Suppose $c_l + \sqrt{c_l^2 + (2A(c_h - c_l) - c_h^2)} > c_h$. It implies $2A(c_h - c_l) > (c_h - c_l)^2 + c_h^2 - c_l^2$. It implies $A > c_h$. It is always true. So $c_l + \sqrt{c_l^2 + (2A(c_h - c_l) - c_h^2)} > c_h$. It is obvious that $p = c_l + \sqrt{c_l^2 + (2A(c_h - c_l) - c_h^2)}$ is the unique price such that $sw^{c_l}(p) = sw^{c_h}(c_h)$ and $p > c_h$. □

The implication of Lemma 3.4 is that $sw^{c_h}(c_h) > sw^{c_l}(p)$ for $p > p'$ and $sw^{c_h}(c_h) < sw^{c_l}(p)$ for $p < p'$. Now we compare the price $p'$ and $\hat{p}$.

**Lemma 3.5.** *$\hat{p} < p'$ if and only if $(c_h + c_l) < \frac{2}{3}A$*

*Proof.* We have $\hat{p} = 2c_h - c_l$ from Lemma 3.2. We have $p' = c_l + \sqrt{c_l^2 + (2A(c_h - c_l) - c_h^2)}$ from Lemma 3.4. Suppose $\hat{p} < p'$. It implies,
$2c_h - c_l < c_l + \sqrt{c_l^2 + (2A(c_h - c_l) - c_h^2)}$
$\implies 4(c_h - c_l)^2 + c_h^2 - c_l^2 < 2A(c_h - c_l)$
$\implies (c_h + c_l) < \frac{2}{3}A$.
The sufficiency part can also be demonstrated in the similar way. □

Lemma 3.5 gives us which p is greater of $\hat{p}$ and $p'$. When $A$ is sufficiently high that is the choke price of the market demand is sufficiently high, $\hat{p}$ is less than $p'$. This ordering of price plays a crucial role in determining the upper bound of the set of the pure strategy Nash equilibrium. It is also clear from Lemma 3.5 that $\hat{p} > p'$ if and only if $(c_h + c_l) > \frac{2}{3}A$.

Suppose firm 2 sets price $p_2$ such that $p_2 > p'$. Firm 1 maximizes social welfare so if firm 1 matches the price of firm 2, the social welfare is $sw^{c_l}(p_2)$. If firm 1 sets $p_1 = c_h$, the social welfare is $sw^{c_h}(c_h)$. From Lemma 3.4 we know $sw^{c_h}(c_h) > sw^{c_l}(p_2)$ since $p_2 > p'$. So firm 1 will set $p_1 = c_h$ and firm 2 can anticipate this action of firm 1. Firm 2 will never set price $p_2$ greater than $p'$.

These $p'$ and $p_o$ can be $p' < p_o$ or $p' \geq p_o$. Suppose $p' < p_o$ and firm 2 sets price $p_2$ such that $p' < p_2 < p_o$. Using the above argument we get that firm 1 will set $p_1 = c_h$ since $sw^{c_h}(c_h) > sw^{c_l}(p_2)$. This optimal strategy of firm 1 is true for any price $p_2$ of firm 2 such that $p_2 > p'$. Suppose $p' < p_o$ and firm 2 sets $p_2 \leq p'$. In this case if firm 1 sets $p_1 = p_2$, the social welfare is $sw^{c_l}(p_2)$. If firm 1 sets $p_1 = p_2 - \epsilon$, the social welfare is $sw^{c_h}(p_2 - \epsilon)$ since $p_2 \leq p' < p_o$. From Lemma 3.4 we know that $sw^{c_h}(p_2 - \epsilon) < sw^{c_l}(p_2)$ since $p_2 \leq p'$. When $p' < p_o$ and firm 2 sets $p_2 \leq p'$, the optimal strategy of firm 1 is to match the price of firm 2.

Now suppose firm 1 sets price $p_1$ such that $p_1 \leq p' < p_o$. If firm 2 matches the price of firm 1, the profit of firm 2 is $\hat{\pi}_2(p_1)$. If firm 2 undercuts the price of firm 1 and sets $p_2 = p_1 - \epsilon$, the profit of firm 2 is $\pi_2^{c_h}(p_1 - \epsilon)$. Further consider that $\hat{p} < p_1 \leq p' < p_o$, this implies that $\pi_2^{c_h}(p_1 - \epsilon) > \hat{\pi}_2(p_1)$ from lemma 3.2. So firm 2 has the tendency to undercut the price of firm 1 when $\hat{p} < p_1 \leq p' < p_o$. So firm 1 will never set price $p_1$ such that $\hat{p} < p_1 \leq p' < p_o$.

Next, suppose that firm 1 sets price $p_1$ such that $p_1 \leq \hat{p}$, when $p' < p_o$. From the discussion in proposition 3.3, firm 2 will match the price of firm 1. Therefore, in the range $[c_l, \hat{p}]$, the optimal strategy of each firm is to match the prices when $p' < p_o$. From Lemma 3.5 we know that $\hat{p} < p'$ under the condition that $\frac{2A}{3} > c_l + c_h$. Each price in the range $[c_l, \hat{p}]$ is a pure strategy Nash equilibrium when $p' < p_o$.

Suppose $p' \geq p_o$ and firm 2 sets price $p_2$ such that $p_o < p_2 < p'$. If firm 1 sets $p_1 = p_2 - \epsilon \geq p_o$, the social welfare is $sw^{c_l}(p_2 - \epsilon)$. Since $p_1 < p'$ and $p_1 > p_o$, we have $sw^{c_l}(p_2 - \epsilon) > sw^{c_l}(p_2)$. It is obvious from Lemma 3.4 and from the fact that $sw^{c_l}(p) = \hat{sw}^{c_l}(p)$. So firm 2 will never set the price $p_2 > p_o$, when $p_o \leq p'$. Again, suppose $p' \geq p_o$ and firm 2 sets $p_2$ such that $p_2 \leq p_o$. If firm 1 sets $p_1 = p_2 - \epsilon$, the social welfare is $sw^{c_h}(p_2 - \epsilon)$. If firm 1 matches the price of firm 2, the social welfare is $sw^{c_l}(p_2)$. From Lemma 3.4, we know that $sw^{c_l}(p_2) > sw^{c_h}(p_2 - \epsilon)$. So, firm 1 will match the price of firm 2 when $p_2 \leq p_o$ and $p_o \leq p'$.

Now, suppose firm 1 sets $p_1$ such that $\hat{p} < p_1 < p_o \leq p'$. It is possible for firm 1 to set such a price, Lemma 3.5 and Proposition 3.3 allows it. If firm 2 matches the price of firm 1, the profit is $\hat{\pi}_1^{c_l}(p_1)$. If firm 2 under cuts the price of firm 1 and sets $p_2 = p_1 - \epsilon$, the profit is $\pi_1^{c_h}(p_1 - \epsilon)$. From the implication of Lemma 3.2, we get that $\pi_1^{c_2}(p_1 - \epsilon) > \hat{\pi}_1^{c_1}(p_1)$. So firm 2 has the tendency to under cut the price of firm 1 when $\hat{p} < p_1 < p_o \leq p'$. But firm 1 wants to match the price of firm 2 in this range.

Suppose firm 2 sets price $p_2$ such that $p_2 < \hat{p} < p_o \leq p'$. In this case both the firms 1 and 2 matches the price. There is no tendency to under cut each others price.

We need to consider the strategies of the firms when $p_o < \hat{p} < p'$. It is possible from the discussion in Subsection 3.1 and Lemma 3.5. If any firm sets price $p_i$ such that $p_i \leq p_o$, the other firm will also match that price. The argument is similar to the proof of Proposition 3.3 and the discussion above. If firm 1 sets price in the range $p_0 < p_1 < \hat{p}$, firm 2 will have the tendency to under cut the price of firm 1. It is clear from Proposition 3.3. Therefore, we conclude that when $p_o \leq p'$, each firm matches the price of the other firm in the price range $[c_l, \min\{\hat{p}, p_o\}]$. Thus, each price in the range $[c_l, \min\{\hat{p}, p_o\}]$ is a pure strategy Nash equilibrium. We gather the above result in the following proposition.

**Proposition 3.6.** *Given the set-up, each price in the range* $[c_l, \min\{\hat{p}, p_o\}]$ *is a pure strategy Nash equilibrium if and only if* $(c_h + c_l) < \frac{2}{3}A$

We get multiple pure strategy Nash equilibrium because there is no tendency among the firms to undercut the price when the price lies in the range $[c_i, \min\{\hat{p}, p_0\}]$. If a firm under cuts the price of the other firm when the price lies in the range $[c_i, \min\{\hat{p}, p_0\}]$ , the market demand is sufficiently high greater than $q_0$, so the firm incurs higher marginal cost ($c_h$) to meet this demand. Due to increase in marginal cost, the profit of the firm is lower than sharing the market demand. In case of public firm the increase in consumer surplus when the price falls due to under cut is not sufficient to meet the fall in profit due to higher marginal cost. Thus, the publicly owned firm also cannot increase social welfare by under cutting price when the price lies in the range $[c_i, \min\{\hat{p}, p_0\}]$. Thus, there is no tendency in the public firm to under cut the price. Therefore, each price in the range $[c_i, \min\{\hat{p}, p_0\}]$ constitutes a pure strategy Nash equilibrium.

Proposition 3.3 and Proposition 3.6 gives that the set of pure strategy Nash equilibrium is similar in the Bertrand competition with two firms and the mixed duopoly with price competition under the set-up considered in this paper. However, the set of pure strategy Nash equilibrium in the case of mixed duopoly requires a further condition. The condition $\frac{2}{3}A > c_h + c_l$ means that the choke price of the market demand curve must be higher than a threshold level relative to the sum of the two marginal cost of the firms. We have interpreted it as the size of the market must be sufficiently big. It means the consumers in the market must be willing to pay sufficiently higher amount of price or at each price the market demand is sufficiently high. The importance of proposition 3.6 is that the market outcome in terms of equilibrium prices and social welfare are similar in mix duopoly and Bertrand competition under the set-up considered in the paper when the market size is sufficiently big. If the public firm is privatized when the market set-up is similar to the paper, the social welfare is going to remain same.

When $\frac{2}{3}A < c_h + c_l$, the choke price of the market demand curve is not sufficiently big. We interpret is as market size is not sufficiently big when $\frac{2}{3}A < c_h + c_l$. From Lemma 3.5 we get $\hat{p} > p'$. Following the argument provided in the Proposition 3.6 we get that the each price in the range $[c_l, \min\{p_o, p'\}]$ is a pure strategy Nash equilibrium. When the market size is not big enough, the upper bound of the price range in case of mixed duopoly may be less than the two firm Bertrand competitions under the set-up considered in the paper. It means that the maximum price in mix duopoly may be less than the maximum price in Bertrand duopoly. So, the social welfare is mix duopoly is always going to be greater than equal to the social welfare in Bertrand Duopoly. Thus, the privatization of the public firms may lead to fall in social welfare when the market size is not sufficiently big. In other words when the maximum willingness to pay of the consumers is not sufficiently high or at each price the market demand is not high enough, the privatization of the public firm in a duopoly market may lead to fall in social welfare. Thus, the nature of ownership has effect on the market outcome and in social welfare.

Next we briefly look at the market outcome when the firms are not similar in terms of the cost function. Suppose the point of discontinuity is not same for the firms. The cost function of firm 1 ( public firm) becomes $c_h q_1$ when $q_1 > q_0^1$. The cost function of firm 2 (private firm) become $c_h q_2$ when $q_2 > q_0^2$. Suppose $q_0^1 > q_0^2$. We get two prices $p_0^1$ and $p_0^2$ such that we have $A - p_0^1 = q = q_0^1$ at $p_0^1$ and $A - p_0^2 = q = q_0^2$ at $p_0^2$. Since we assume $q_0^1 > q_0^2$, so we have $p_0^1 < p_0^2$. Following the argument similar to Proposition 3.6, we get that the new Nash equilibrium price range is $[c_l, \min\{\hat{p}, p_0^1\}]$ since $p_0^1 < p_0^2$. If $q_0^1 > q_0$ then $p_0^1 < p_0$, so the upper bound of the Nash equilibrium price range may be lower in this case. If $q_0^1 \geq q_0$ then $p_0^1 \geq p_0$, so the upper bound of the Nash equilibrium price range may be equal or higher. We get a situation where the upper bound of the Nash equilibrium price range may be lower. In this case the social welfare can always be higher than when the point of discontinuity of the cost functions is same. Note that it is not necessary that the upper bound of the Nash equilibrium price range is going to go down.

Suppose the cost function of firm 1 is discontinuous as specified in section 2. The cost function of firm 2 is $c_l q_2$ for all output. In this situation firm 2 always has the tendency to under cut the price if price is greater than $c_l$. Firm 1 is not going to under cut price below $c_h$. Therefore, we do not have pure strategy Nash equilibrium when the market demand is always shared equally. This is one of the standard results in Bertrand competition. Again, suppose the cost function of firm 1 is discontinuous as specified in section 2. The cost function of firm 2 is $c_h q_2$ for all output. In this case, firm 2 has no tendency to under cut price below $c_h$. However it under cuts any price above $c_h$. We know that $c_h < \min\{\hat{p}, p_o\}$ so firm 2 also has the tendency to under cut price when $p > c_h$. Therefore, we have a unique pure strategy Nash equilibrium at $p = c_h$. Thus, when the cost function of the public firm is discontinuous and the cost function of the private firm is continuous we get no pure strategy Nash equilibrium in one case and in the other there is a unique pure strategy Nash equilibrium. When the cost function of the private firm is discontinuous as specified in section 2 and the cost of function of the public firm is continuous, we get unique pure strategy Nash either at $p = c_l$ or $p = c_h$ depending the cost function of the public firm.

# 4 Conclusion

We conclude by saying that when the market size is sufficiently big, the set of pure strategy Nash equilibrium is same in the case of Mixed duopoly and the two firm Bertrand competitions under the type of cost functions considered. The difference in the nature of ownership of one of the firms do not make any difference to the equilibrium outcomes and hence to social welfare when the market size is big enough. However, when the market size is not big enough, the upper bound of the mixed duopoly can be less than the two firm Bertrand competitions. So the lower bound on the social welfare can be higher in case of mixed duopoly than Bertrand competition with two firms when the market size is not sufficiently high. The difference in the nature of ownership of one of the firms make difference to the set of pure strategy Nash equilibrium when the market size is not sufficiently big. The pure strategy Nash equilibrium in both the situations is given by a range of prices, so it is not unique. When the cost function of one of the firms is only discontinuous as specified in section 2 and the marginal cost of the other firm is constant, we get unique pure strategy Nash equilibrium in certain cases.

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