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Individual rationality conditions of identifying matching costs in transferable utility matching games

Suguru Otani
University of Tokyo Market Design Center

Abstract

The widely applied method for measuring assortativeness in a transferable utility matching game is the matching maximum score estimation proposed by Fox (2010). This article reveals that by combining unmatched agents, transfers, and individual rationality conditions with sufficiently large penalty terms, it's possible to identify the coefficient parameter of a single common constant, i.e., matching costs in the market.

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Contact: Suguru Otani - suguru.otani@e.u-tokyo.ac.jp.
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1 Introduction

One of the most well-known methods for measuring matching assortativeness was developed by Fox (2010, 2018). Although the method allows researchers to flexibly incorporate additional inequalities from equilibrium conditions, the additional information that helps identification is unknown. This article focuses on pairwise stability in a transferable utility (TU) matching model and investigates the identification by adding unmatched agents, transfer data, and individual rationality (IR) conditions, which are available in many empirical applications. This article theoretically and numerically reveals that using unmatched agents, transfers, and individual rationality conditions with sufficiently large penalty terms makes it possible to identify the coefficient parameter of a single common constant, i.e., matching costs in the market.

First, transfer data is known to improve identification power practically if it is available. The working paper version of Fox and Bajari (2013) uses bidding price data in FCC Spectrum Auction in a single market setting. They numerically confirm that if the number of agents is small or the variance of errors is large, bidding price data generated from a tatonnement process drastically reduces the bias and RMSE (the root of mean squared error) of an estimated single parameter. Akkus et al. (2015) use acquisition prices between buyer firms and target firms as transfer data. Pan (2017) uses CEO compensations between CEOs and firms. Appendix A of Akkus et al. (2015) shows Monte Carlo evidence that adding transfers not only improves the accuracy of estimation but also enables researchers to identify the coefficient of a non-interacted term. In another strand, there is some theoretical and empirical evidence that using unmatched agents helps identification. Fox et al. (2018) show nonparametric identification results in two-sided matching model. Section 5 of their paper proves that the distribution of unobserved complementarities conditional on observed characteristics can be recovered if each firm on each side is part of exactly one match or is unmatched in each feasible assignment. Otani (2021) uses unmatched firms to identify and estimate matching costs in a one-sided one-to-many coalitional mergers in Japanese shipping industry with Monte Carlo simulations. This article shows how these data affect construction of inequalities for identification and estimation.

Second, individual rationality (IR, henceforth) conditions, i.e., the binary choice information are necessary for identification of a fixed cost (Bresnahan and Reiss, 1990). In the TU matching game, IR conditions capture whether agents choose to stay matched or unmatched. This article shows under what conditions matching fixed costs can be identified and estimated.

2 Model

2.1 Baseline matching model

I consider a two-sided one-to-one TU merger matching game in a single market. Let $\mathcal{N}_b$ and $\mathcal{N}_s$ be the sets of potential finite buyers and sellers respectively. Let $b = 1, \cdots, |\mathcal{N}_b|$ be buyer firms and let $s = 1, \cdots, |\mathcal{N}_s|$ be seller firms where $|\cdot|$ is cardinality. Let $\mathcal{N}_b^m$ denote the set of ex-post matched buyers and $\mathcal{N}_b^u$ denote that of ex-post unmatched buyers such that

$\mathcal{N}_b = \mathcal{N}_b^m \cup \mathcal{N}_b^u$ and $\mathcal{N}_b^m \cap \mathcal{N}_b^u = \emptyset$. For the seller side, define $\mathcal{N}_s^u$ and $\mathcal{N}_s^m$ as the set of ex-post matched and unmatched sellers such that $\mathcal{N}_s = \mathcal{N}_s^m \cup \mathcal{N}_s^u$ and $\mathcal{N}_s^m \cap \mathcal{N}_s^u = \emptyset$. Let $\mathcal{M}^m$ be the sets of all ex-post matched pairs $(b,s) \in \mathcal{N}_b^m \times \mathcal{N}_s^m$. Let $\mathcal{M}$ denote the set of all ex-post matched pairs $(b,s) \in \mathcal{M}^m$ and unmatched pairs $(\tilde{b}, \emptyset)$ and $(\emptyset, \tilde{s})$ for all $\tilde{b} \in \mathcal{N}_b^u$ and $\tilde{s} \in \mathcal{N}_s^u$ where $\emptyset$ means a null agent generating unmatched payoff.

Each firm can match at most one agent on the other side, so $|\mathcal{N}_b^m| = |\mathcal{N}_s^m|$. The matching joint production function is defined as $f(b,s) = V_b(b,s) + V_s(b,s)$ where $V_b : \mathcal{M} \to \mathbb{R}$ and $V_s : \mathcal{M} \to \mathbb{R}$. The net matching values for buyer b and seller s are defined as $V_b(b,s) = f(b,s) - p_{b,s}$ and $V_s(b,s) + p_{b,s}$, where $p_{b,s} \in \mathbb{R}_+$ is the equilibrium merger price paid to seller firm s by buyer firm b and $p_{b\emptyset} = p_{\emptyset s} = 0$. For scale normalization, I assume $V_b(b,\emptyset) = 0$ and $V_s(\emptyset, s) = 0$ for all $b \in \mathcal{N}_b$ and $s \in \mathcal{N}_s$.

The matching allocation is incentive compatible given equilibrium merger prices if each agent maximizes its profit. The allocation is feasible if agents' excess demand for counterpart agents equals zero. Under some technical assumptions, Azevedo and Hatfield (2018) show that the matching allocation with equilibrium prices is a competitive equilibrium if the allocation is incentive compatible given the equilibrium prices and is feasible. The competitive equilibrium is equivalent to the stable matching.[1] At competitive equilibrium, each buyer maximizes $V_b(b,s)$ across seller firms, whereas each seller maximizes $V_s(b,s)$ across buyer firms.

Specifically, the stability conditions for buyer firm $b \in \mathcal{N}_b$ and seller firm $s \in \mathcal{N}_s$ are as follows:

$$\begin{aligned} V_b(b,s) &\geq V_b(b,s') \quad \forall s' \in \mathcal{N}_s \cup \emptyset, s' \neq s, \\ V_s(b,s) &\geq V_s(b',s) \quad \forall b' \in \mathcal{N}_b \cup \emptyset, b' \neq b. \end{aligned} \tag{1}$$

Based on Equation (1) and equilibrium price conditions $p_{b',s} \leq p_{b,s}$ and $p_{b,s'} \leq p_{b',s'}$ in Akkus et al. (2015), I construct the inequalities for matches $(b,s) \in \mathcal{M}$ and $(b',s') \in \mathcal{M}$, $(b',s') \neq (b,s)$ as follows:

$$\begin{aligned} f(b,s) - f(b,s') &\geq p_{b,s} - p_{b,s'} \geq p_{b,s} - p_{b',s'}, \\ f(b',s') - f(b',s) &\geq p_{b',s'} - p_{b',s} \geq p_{b',s'} - p_{b,s}, \\ V_s(b,s) - V_s(b',s) &\geq 0, \\ V_{s'}(b',s') - V_s(b,s') &\geq 0, \end{aligned} \tag{2}$$

where $p_{b',s}$ and $p_{b,s'}$ are unrealized equilibrium merger prices that cannot be observed in the data. The last two inequalities cannot be derived from the data because the researchers cannot observe how the total matching value $f(b,s)$ is shared between buyer b and seller s.

2.2 Transfer and unmatched data.

In many empirical applications, researchers do not obtain data on transfers and unmatched firms. The minimum number of possible inequalities is discussed in Result 1.

[1]See Proposition 3.3 of Galichon (2018) for reference.

Result 1. *Suppose that researchers obtain (A) transfer data p_{bs} for matched buyer firm b and seller firm s for all $b \in \mathcal{N}_b^m$ and $s \in \mathcal{N}_s^m$, and (B) unmatched data about $\mathcal{N}_b^u$ and $\mathcal{N}_s^u$ with their observed characteristics. Let ${}_nC_k$ be the number of combinations of k objects from n objects, and ${}_nP_k$ be the number of permutations, representing the different ordered arrangements of a k-element subset of an n-set.*[2] *Then, the followings hold:*

(i) Researchers can construct the following inequalities for matches $(b, s) \in \mathcal{M}$ and $(b', s') \in \mathcal{M}, (b', s') \neq (b, s)$:

$$f(b, s) - f(b, s') \geq p_{b,s} - p_{b',s'}. \tag{3}$$

The minimum number of total inequalities is ${}_{|\mathcal{M}^m|+|\mathcal{N}_b^u|+|\mathcal{N}_s^u|}P_2$.

(ii) If the data does not contain (A), researchers can construct the following inequalities for matches $(b, s) \in \mathcal{M}$ and $(b', s') \in \mathcal{M}, (b', s') \neq (b, s)$:

$$f(b, s) + f(b', s') \geq f(b, s') + f(b', s). \tag{4}$$

The minimum number of total inequalities is ${}_{|\mathcal{M}^m|+|\mathcal{N}_b^u|+|\mathcal{N}_s^u|}C_2$.

(iii) If the data does not contain (B), researchers can construct Inequality (4) for matches $(b, s) \in \mathcal{M}^m$ and $(b', s') \in \mathcal{M}^m, (b', s') \neq (b, s)$. The minimum number of total inequalities is ${}_{|\mathcal{M}^m|}P_2$.

(iv) If the data does not contain (A) and (B), researchers can construct Inequality (3) for matches $(b, s) \in \mathcal{M}^m$ and $(b', s') \in \mathcal{M}^m, (b', s') \neq (b, s)$. The minimum number of total inequalities is ${}_{|\mathcal{M}^m|}C_2$.

2.3 Individual rationality conditions

IR conditions are implicitly included in Inequality (3) when transfer data is available because $\mathcal{M}$ restores unmatched buyer $\tilde{b}$ and seller $\tilde{s}$ as $(\tilde{b}, \emptyset)$ and $(\emptyset, \tilde{s})$, respectively. For matched pair $(b, s) \in \mathcal{M}$ and unmatched buyer $(\tilde{b}, \emptyset) \in \mathcal{M}$, Inequality (3) gives $f(b, s) - f(b, \emptyset) = f(b, s) \geq p_{b,s} - p_{\tilde{b},\emptyset} \geq p_{b,s}$, i.e.,

$$f(b, s) - p_{b,s} \geq 0. \tag{5}$$

Importantly, unlike pairwise inequalities, the IR condition holds for each matched firm on each side regardless of the availability of transfer data. To distinguish IR conditions from pairwise inequalities, let $IR(\mathcal{M}^m) = 2|\mathcal{M}^m| = |\mathcal{M}_b^m| + |\mathcal{M}_s^m|$ be the number of inequalities from IR conditions.

Table I summarizes the above results and illustrates a useful configuration of pairwise inequalities. Note that ${}_{|\mathcal{M}^m|+|\mathcal{N}_b^u|+|\mathcal{N}_s^u|}P_2$ and ${}_{|\mathcal{M}^m|}P_2$ include the same inequalities in $IR(\mathcal{M}^m)$. Section 3 uses the result to demonstrate a necessary modification for identifying matching costs.

[2] For reference, ${}_nC_k = \frac{n!}{k!(n-k)!}$ and ${}_nP_k = \frac{n!}{(n-k)!}$ where ! is a factorial function.

Table I: **The minimum number of total inequalities.**

		Unmatched	
		Available	Unavailable
Transfer	Available	$\lvert\mathcal{M}^m\rvert + \lvert\mathcal{N}_b^u\rvert + \lvert\mathcal{N}_s^u\rvert P_2$	$\lvert\mathcal{M}^m\rvert P_2$
	Unavailable	$\lvert\mathcal{M}^m\rvert + \lvert\mathcal{N}_b^u\rvert + \lvert\mathcal{N}_s^u\rvert C_2 + IR(\mathcal{M}^m)$	$\lvert\mathcal{M}^m\rvert C_2 + IR(\mathcal{M}^m)$

2.4 Matching maximum score estimator

Fox (2010) proposes a maximum score estimator using Inequality (3) or (4). The maximum score estimator is consistent if the model satisfies a rank order property, i.e., the probability of observing matched pairs is larger than the probability of observing swapped matched pairs. I specify $f(b,s)$ as a parametric form $f(b,s|X,\beta)$ where X is a vector of observed characteristics of all buyers and sellers and β is a vector of parameters. Given X, one can estimate β without IR conditions by maximizing the following objective function:

$$Q(\beta) = \begin{cases} \sum_{(b,s)\in\mathcal{M}} \sum_{(b',s')\in\mathcal{M},(b',s')\neq(b,s)} \mathbb{1}[f(b,s|X,\beta) - f(b,s'|X,\beta) \geq p_{b,s} - p_{b',s'}] \\ \qquad\qquad \text{if transfer data is available,} \\ \sum_{(b,s)\in\mathcal{M}} \sum_{(b',s')\in\mathcal{M},(b',s')\neq(b,s)} \mathbb{1}[f(b,s|X,\beta) + f(b',s'|X,\beta) \geq f(b,s'|X,\beta) + f(b',s|X,\beta)] \\ \qquad\qquad \text{otherwise,} \end{cases} \tag{6}$$

where $\mathbb{1}[\cdot]$ is an indicator function. If unmatched data is unavailable, $\mathcal{M}$ is replaced with $\mathcal{M}^m$. If IR conditions are included, $Q(\beta)$ is modified to the following:

$$\tilde{Q}(\beta) = Q(\beta) + \lambda \cdot \sum_{(b,s)\in\mathcal{M}^m} \mathbb{1}[f(b,s|X,\beta) \geq 0], \quad \lambda \geq 1, \tag{7}$$

where λ is the importance weight of IR conditions. If λ is larger, the importance of the IR condition term is larger for the evaluation of $\tilde{Q}(\beta)$. If the transfer data are available, the correction term is redundant as in Table I, but it does not affect the search for the maximizer of $\tilde{Q}(\beta)$. Section 3 investigates the importance of λ.

3 IR conditions identify the coefficient parameter of a constant

The estimation of fixed costs is one of the fundamental tasks in structural empirical studies. Suppose that researchers want to estimate an additive separable matching cost in a single market. Let c be the matching cost and assume $c < 0$ for exposition.[3] Following the

[3] If researchers expect $\beta X_b X_s$ to be negative, i.e., the market to generate negative assortative matchings and want to investigate a subsidy effect inducing matchings, they can assume $c > 0$ as the subsidy effect. The logic is the same in the main text.

literature, I specify $f(b, s)$ as follows:

$$f(b, s) = \beta X_b X_s + c \cdot \mathbb{1}[b \neq \emptyset \text{ or } s \neq \emptyset], \tag{8}$$

where X_b and X_s are vectors of continuous observable characteristics for buyer b and seller s and β is a vector of parameters. Note that hypothetical matching cost c exists for unmatched pairs. Then, Result 2 holds.

Result 2. *Suppose that the matching joint production function is specified as (8). Then,*

(i) by using pairwise inequalities based only on matched pairs, c cannot be identified via $Q(\cdot)$;

(ii) in addition to (i), even if transfer data is available, c cannot be identified via $Q(\cdot)$;

(iii) in addition to (i), even if unmatched data is available, the lower bound of c cannot be identified without IR conditions;

(iv) in addition to (iii), c can be identified only when IR conditions are used with a sufficiently large importance weight via $\tilde{Q}(\beta)$, whether the transfer data is included or not.

Note that *(iv)* of Result 2 does not reveal how large the importance weight λ should be. The appropriate weight of λ depends on the sample sizes of matched and unmatched agents as a tuning parameter. The working paper version provides detailed numerical experiments to show the necessity of the weight.

4 Conclusion

This article investigates the identification of a common fixed cost of matching in a TU matching game. For identification, IR conditions with a sufficiently large importance weight are necessary unless you have data of both unmatched agents and transfers.

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A Proof of results

Proof of Result 1

Proof. I demonstrate (i) because other parts are analogously proven. $\mathcal{M}$ restores unmatched buyer $\tilde{b}$ and seller $\tilde{s}$ as $(\tilde{b}, \emptyset)$ and $(\emptyset, \tilde{s})$, so the size of the set of unmatched pairs is $(|\mathcal{N}_b^u| + |\mathcal{N}_s^u|)$. The number of possible combinations of $(b, s) \in \mathcal{M}$ and $(b', s') \in \mathcal{M}, (b', s') \neq (b, s)$

is ${}_{|\mathcal{M}^m|+|\mathcal{N}_b^u|+|\mathcal{N}_s^u|}C_2$ and each combination gives two inequalities as in Inequality (3). This gives

$$2 \cdot_{|\mathcal{M}^m|+|\mathcal{N}_b^u|+|\mathcal{N}_s^u|} C_2 = 2\frac{(|\mathcal{M}^m|+|\mathcal{N}_b^u|+|\mathcal{N}_s^u|)(|\mathcal{M}^m|+|\mathcal{N}_b^u|+|\mathcal{N}_s^u|-1)}{2!} =_{|\mathcal{M}^m|+|\mathcal{N}_b^u|+|\mathcal{N}_s^u|} P_2. \tag{9}$$

□

Proof of Result 2

Proof. To prove (i), substituting (8) into (3) gives

$$\begin{aligned} &\beta X_b X_s + c \cdot \mathbb{1}[b \neq \emptyset \text{ or } s \neq \emptyset] + \beta X_{b'} X_{s'} + c \cdot \mathbb{1}[b' \neq \emptyset \text{ or } s' \neq \emptyset] \\ &\quad \geq \beta X_b X_{s'} + c \cdot \mathbb{1}[b \neq \emptyset \text{ or } s' \neq \emptyset] + \beta X_{b'} X_s + c \cdot \mathbb{1}[b' \neq \emptyset \text{ or } s \neq \emptyset], \end{aligned}$$

where the indicator functions must be 1 for matched pair (b, s) and unrealized matched pair (b, s') so that matching cost c is canceled out. Similarly, (ii) is also proved in the same way.

To prove (iii), it is sufficient to consider pairwise inequalities based on an unmatched pair as the pairwise inequalities of matched pairs are shown above. With transfer data, by substituting (8) into (3), the pairwise inequalities based on unmatched pairs denoted by $(\tilde{b}, \emptyset)$ and $(\emptyset, \tilde{s})$ can be reduced to a single inequality as follows:

$$0 \geq \beta X_{\tilde{b}} X_{\tilde{s}} + c - p_{\tilde{b},\tilde{s}},$$

where $p_{\tilde{b},\tilde{s}} = 0$. This only provides the upper bound of c as $-\beta X_b X_s \geq c$. Without transfer data, the same inequality is derived using (4).

To prove (iv), when transfer data is available, IR condition (5) for matched pair (b, s) gives

$$\beta X_b X_s + c - p_{b,s} \geq 0,$$

whereas, when transfer data is not available, IR condition (5) for matched pair (b, s) gives

$$\beta X_b X_s + c \geq 0.$$

Thus, IR condition (5) provides the lower bound of c as $c \geq -\beta X_b X_s + p_{b,s}$ or $c \geq -\beta X_b X_s$.

Finally, I demonstrate the necessity of λ. In the modified objective function in (7), the additional term

$$\lambda \cdot \sum_{(b,s) \in \mathcal{M}^m} 1[\beta X_b X_s + c \geq 0],$$

takes up to $\lambda \cdot |\mathcal{M}^m|$. However, $Q(\beta)$ of (7) can take up to either of the numbers in Table I which are much larger than $|\mathcal{M}^m|$. In finite samples, the part does not need to achieve its perfect score, so some fractions of pairwise inequalities are not satisfied even at the maximizer of the objective function. This implies that, if λ and $|\mathcal{M}^m|$ are small, the evaluation of IR conditions is dominated by the number of unsatisfied pairwise inequalities, which can be larger than $|\mathcal{M}^m|$. Thus, λ must be large enough, corresponding to $|\mathcal{M}^m|$. □

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