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Dividing the expected payoff resulting from joint actions

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Abstract

We consider situations where players hit targets with known probabilities and are rewarded according to given rules. The division of the expected payoff resulting from their joint actions is studied in the context of transferable utility games, using the Shapley value as the allocation rule.

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1 Introduction

Consider a group of actors (or factors) and a target that each actor is able to hit with a given probability. In this framework, the question of the allocation among the agents of the resulting overall probability of success comes in. Framing this problem as a cooperative game with transferable utility – *a probability game* – Hou et al. (2018) have proposed to use the Shapley value. Probability games and their duals have been further studied by Dehez (2023) together with a proper axiomatisation of the Shapley value on probability games.

The present paper extends the previous setting by considering a situation where there are several targets to which payoffs are associated. The question then concerns the allocation of the resulting *expected payoff*. This is an ex-ante allocation that provides a key to the apportionment of actual payoffs.

We consider different scenarios, assuming that each player hit at most one target. If a coalition forms, payoffs could be allocated depending simply on the targets hit by its members. Alternatively, if some of its members hit a same target, the payoff could be allocated to that coalition only once. And a payoff could go to a coalition only if the associated target has not been hit by the complementary coalition. The games emerging under the first scenario are additive and, as a consequence, the Shapley value allocates to players their individual expected payoffs. It turns out that the games emerging from the two other scenarios lead to games that are dual to each other. As a consequence, they both rise to the same Shapley value, a remarkable result.

Potential domains of applications are many. Here are two examples. In portfolio management, where rewards are the feasible returns and players are securities, the Shapley value allows for the identification of the relative contribution of each financial investment. In health matters where rewards are health indices and players are risk factors, the Shapley value allows for the identification of the relative importance of each factor. Insurance management is another possible example.

In all three scenarios, payoffs are being added. In contrast, we consider a situation where, if a player joins a coalition, its payoff is being multiplied by a coefficient that is specific to that player: a payoff for a player is associated to his ability to increase the payoff of a coalition. The coefficient is assumed to be a random variable with known probability distribution. The associated transferable utility game is a cooperative product game, a concept introduced by Rosales (2014) in an unpublished memo and studied later in more detail in Dehez (2024).

The paper is organised as follows. Probability games and their duals are defined in Section 2. Targets are introduced in Section 3 and three scenarios are considered, assuming that payoffs are added. Section 4 is devoted to the case where payoffs are multiplied. The last section covers concluding remarks.

2 Probabilistic situations

Consider a set $N = \{1,\ldots,n\}$ of players facing a target. By his actions, a player *alone* can hit the target with a given and known probability, $p_i \in [0,1]$ for player i. Probabilistic independence prevails: the success of a player is independent of the success of the other players. A pair (N, p) defines a *probabilistic situation*.

A transferable utility game (TU-game for short) is a pair (N,v) where N is the set of players and v is a (characteristic) function that associates a real number to every subsets (coalitions) of N. By convention, $v(\emptyset) = 0$.

Notation: Finite sets are denoted by upper-case letters. Lower-case letters are used to denote their cardinals: $t = |T|$, $s = |S|$, … Set inclusion (non-strict) is denoted by $\subset$. For a vector x, $x(S)$ denotes the sum of its coordinates over the subset S. Set inclusion, non-strict and strict, are denoted by $\subset$ and $\subsetneq$. Braces are sometimes omitted for coalitions, for instance $v(i,j)$ replaces $v(\{i,j\})$. By convention, a sum over an empty set is zero and a product over an empty set is equal to 1.

2.1 Probability games

Given a probabilistic situation (N,p), Hou et al. (2018) define the transferable utility game (N,v) whose characteristic function v is given by:

$$v(S) = 1 - \prod_{i\in S}(1-p_i) \text{ for all } S \subset N. \tag{1}$$

$v(S)$ is the probability that *at least one* member of S succeeds. The game (N,v) is *concave* and thereby *subadditive*. The *dual* game (N,v^d) is defined by:

$$v^d(S) = v(N) - v(N\setminus S) = \prod_{i\in N\setminus S}(1-p_i) - \prod_{i\in N}(1-p_i) = \left(1-\prod_{i\in S}(1-p_i)\right)\prod_{i\in N\setminus S}(1-p_i). \tag{2}$$

$v^d(S)$ is the probability that at least one player in S succeeds, assuming that players outside S *all* fail. The game (N,v^d) is *convex* (and thereby *superadditive*) as dual of a concave game.[1]

By definition of the dual, the probability to be allocated among the players is the same in both cases: the question concerns the division of the collective probability of success that is given by $v(N) = v^d(N)$.

2.2 The Shapley value of a probability game

Given a set of players N, the set $G(N)$ of all characteristic functions on N coincides with the real vector space $\mathbb{R}^{2^n-1}$. In proving the uniqueness of his value, Shapley (1953) shows that the collection of *unanimity games* $(u_T)_{T\subset N, T\neq\emptyset}$ defined by

$$\begin{aligned} u_T(S) &= 1 \quad \text{if } T \subset S, \\ &= 0 \quad \text{if } T \not\subset S, \end{aligned}$$

is a basis of $G(N)$: given a characteristic function v in $G(N)$, there exists a unique $(2^n-1)-$dimensional vector $(\alpha_T)_{T\subset N, T\neq\emptyset}$ such that:

$$v(S) = \sum_{T\subset N}\alpha_T u_T(S) = \sum_{T\subset S}\alpha_T. \tag{3}$$

The coefficients α_T can be defined recursively, starting with $\alpha_\emptyset = 0$, as follows:

$$\alpha_T = v(T) - \sum_{S\subsetneq T}\alpha_S \;\Rightarrow\; \alpha_T = \sum_{S\subset T}(-1)^{t-s}v(S) \text{ for all } T \subset N. \tag{4}$$

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[1] A game (N,v) is *superadditive* if $v(S\cup T) \geq v(S) + v(T)$ for all disjoint subsets S and T. A game (N,v) is *convex* if $v(S\cup T) + v(S\cap T) \geq v(S) + v(T)$ for all subsets S, T. It is *subadditive* (resp. *concave*) if the game $(N,-v)$ is superadditive (resp. convex). For more on convexity of TU-games, refer to Shapley (1971).

Following Harsanyi (1959), $\alpha_T(N,v)$ is the *dividend* accruing to coalition T once all sub-coalitions have received their dividends. By (3), the sum of all dividends is equal to $v(N)$. An allocation can then be obtained by distributing the dividends of each coalition among its members. Harsanyi shows that the *Shapley value* gives to each player the sum of the *per capita* dividend of the coalitions of which he is a member. Hence, we have:

$$SV_i(N,v) = \sum_{T \subset N: i \in T} \frac{1}{t} \alpha_T(N,v). \tag{5}$$

Using (4), the Harsanyi dividends of the probability game (N,v) associated to the probabilistic situation (N,p) are given by:

$$\alpha_T(N,v) = (-1)^{t-1} \prod_{i \in T} p_i. \tag{6}$$

They alternate in sign according to coalition size and have no particular interpretation. The dividends of the dual game (N,v^d) are given by:

$$\alpha_T(N,v^d) = \prod_{j \in T} p_j \prod_{j \in N \setminus T} (1-p_j). \tag{7}$$

It is the probability that players in T all succeed while players outside T all fail. Dividends of dual probability games are nonnegative: dual probability games are *positive*, a class of games on which solution concepts converge: the core coincides with the set of weighted Shapley values and the Harsanyi set that collects all possible distributions of dividends (see Dehez, 2017 for a survey).

The Shapley value is a *self-dual* allocation rule: the value of a game coincides with the value of its dual. Introducing successively (6) in (5) and (7) in (5), we obtain two *equivalent* formulations of the Shapley value associated to a probabilistic situation (N,p):

$$SV_i(N,p) = \sum_{T \subset N: i \in T} \frac{(-1)^{t-1}}{t} \prod_{j \in T} p_j = p_i \sum_{T \subset N: i \in T} \frac{(-1)^{t-1}}{t} \prod_{j \in T \setminus i} p_j \tag{8}$$

and

$$SV_i(N,p) = \sum_{T \subset N: i \in T} \frac{1}{t} \prod_{j \in T} p_j \prod_{j \in N \setminus T} (1-p_j) = p_i \sum_{T \subset N: i \in T} \frac{1}{t} \prod_{j \in T / i} p_j \prod_{j \in N \setminus T} (1-p_j) \tag{9}$$

where $i = 1,\ldots,n$. Hence, whatever is the definition of the probability of success of a coalition, (1) or (2), the Shapley value defines the same allocation: (8) and (9) coincide.

The Shapley value follows a natural decomposition of the collective probability of success. In the 2-player case, the collective probability $p_1 + p_2 - p_1 p_2$ is decomposed as follows:

$$x_1 = p_1 - \frac{1}{2} p_1 p_2 \text{ and } x_2 = p_2 - \frac{1}{2} p_1 p_2.$$

This decomposition is easily extended to any number of players. In the 3-player case, the collective probability is given by $p_1 + p_2 + p_3 - p_1 p_2 - p_1 p_3 - p_2 p_3 + p_1 p_2 p_3$ and the amount allocated to player 1 is given by:

$$x_1 = p_1 - \frac{1}{2} p_1 p_2 - \frac{1}{2} p_1 p_3 + \frac{1}{3} p_1 p_2 p_3 = p_1 \left(1 - \frac{1}{2} p_2 - \frac{1}{2} p_3 + \frac{1}{3} p_2 p_3\right).$$

Looking at (8), we observe that the Shapley value allocates to a player a share that is *proportional* to his probability of success: $SV_i(N,p) = p_i\, f(p_{-i})$ where p_{-i} denotes the vector of probabilities from which the coordinate i has been eliminated and $f:\mathbb{R}_+^{n-1}\to\mathbb{R}_+$ is the *player-independent* and *symmetric*[2] function given by:

$$f(z)=1+\sum_{T\subset\{1,...,n-1\}}\frac{(-1)^t}{t+1}\prod_{j\in T}z_j. \tag{10}$$

In Dehez (2023), it is shown that *proportionality,* together with *efficiency* and *symmetry*, characterises the Shapley value of a probability game: there is one and only one function f verifying these three properties.[3]

3 Associating payoffs to targets

We consider situations where there may be several targets. Probabilities and a payoff are associated to each target enabling the definition of a transferable utility game (N,w) where $w(S)$ is the expected payoff resulting the coordinated actions of the player in S. The resulting Shapley value $SV(N,w)$ then defines a fair allocation of the expected payoff $w(N)$ and a distribution key $(\alpha_1,...,\alpha_n)$ where $\alpha_i=SV_i(N,w)/w(N)$ measures the relative contribution of player i. That key can then be used to allocate the realised payoff.

We first consider the case of a single target, extending the previous section. The case of several targets is analysed thereafter within different scenarios.

3.1 The case of a single target

Let r be the payoff associated to the target, $r>0$. There are two possible scenarios. In the first scenario, a coalition obtains the payoff if it succeeds, independently of the success or failure of the complementary coalition. If coalition S forms, its expected payoff is given by:

$$w_0(S)=\left(1-\prod_{i\in S}(1-p_i)\right)r \tag{11}$$

Referring to the probability game (N,v), we have $w_0(S)=v(S)\,r.$ In the second scenario, a coalition obtains the payoff if it succeeds while the complementary coalition fails. If coalition S forms, its expected payoff is given by:

$$w_0^d(S)=\left(1-\prod_{i\in S}(1-p_i)\right)\prod_{i\in N\setminus S}(1-p_i)\,r. \tag{12}$$

Duality applies: $w_0^d(S)=v^d(S)\,r.$ In both cases, the question concerns the allocation of the collective expected payoff given by:

$$w_0(N)=w_0^d(N)=(1-\prod_{i\in N}(1-p_i))\,r.$$

By self-duality, the Shapley value can be computed indifferently from (11) or (12). By linearity of the Shapley value, it is simply given by:

$$SV_i(N,p,r)=SV_i(N,p)\,r$$

where $SV_i(N,p)$ is given, by (8) or (9).

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[2] A function is symmetric if permuting its arguments leaves its value unchanged.

[3] Efficiency requires that the value of the game $v(N)$ is exactly distributed. Symmetry requires that an identical amount is allocated to pairs of players who contribute identically to coalitions to which they both belong.

3.2 The case of several targets

Consider a situation involving a finite set M of targets, $m \geq 2$. Target h is associated with a real number $r_h > 0$ expressed in terms of some "money". We denote by $p_{ih} \in [0,1]$ the probability that player i hits target h and by P the corresponding probability matrix $[p_{ih}]$.

We assume that each player can hit one (and only one) of the m targets with probability one: the p_{ih} sum up to one for all $i \in N$. Three scenarios will be considered.

We denote by X_i the (discrete) random variable associated to player i. It is defined by the payoffs $(r_1,...,r_m)$ and corresponding probabilities $(p_{i1},..., p_{im})$. For example, if $n = m = 3$, there are $3^3 = 27$ possible events. For example, the probabilities are associated to the event (r_1, r_3, r_1). Table 1 lists the events and probabilities for the case where n = 3, $m = 2$, and $p_{i2} = 1 - p_{i1}$.

The first scenario is the simplest: once a player hits a target, he obtains the corresponding payoff. The expected payoffs then define the *additive* game (N, w_1):

$$w_1(S) = \sum_{i \in S} E[X_i] = \sum_{h \in M} r_h \sum_{i \in S} p_{ih}.$$

We have indeed $w_1(S) = \sum_{i \in S} w_i(S)$ for all $S \subset N$. The Shapley value being an additive allocation rule, it simply allocates to each player his individual expected payoff:

$$SV_i(N, w_1) = E[X_i] = \sum_{h \in M} p_{ih} r_h \quad (i = 1,\ldots,n).$$

Example 1 Consider the case of 3-player case and two targets, with the individual probabilities $(p_{11}, p_{21}, p_{31}) = (0.6, 0.3, 0.7)$. The resulting probability distribution is given in Table 1. Assuming that $(r_1, r_2) = (5,7)$ are the payoffs, the resulting game (N, w_1) is given by $w_1 = (5.8, 6.4, 5.6 \,|\, 12.2, 11.4, 12 \,|\, 17.8)$ and its Shapley value is given by (5.8, 6.4, 5.6), giving the distribution key (0.326, 0.359, 0.315).

event	probabilities	
r_1, r_1, r_1	$p_{11} p_{21} p_{31}$	0.126
r_1, r_1, r_2	$p_{11} p_{21} p_{32}$	0.054
r_1, r_2, r_1	$p_{11} p_{22} p_{31}$	0.294
r_1, r_2, r_2	$p_{11} p_{22} p_{32}$	0.126
r_2, r_1, r_1	$p_{12} p_{21} p_{31}$	0.084
r_2, r_1, r_2	$p_{12} p_{21} p_{32}$	0.036
r_2, r_2, r_1	$p_{12} p_{22} p_{31}$	0.196
r_2, r_2, r_2	$p_{12} p_{22} p_{32}$	0.084

Table 1: Events and probabilities

In a second scenario, a coalition obtains a payoff once the corresponding target has been hit by one of its members, independently of the number of its members that have succeeded. This is illustrated in Table 2. The expected payoffs define the game (N, w_2):

$$w_2(S) = \sum_{h \in M} v_h(S)\, r_h$$

where $v_h(S)$ is the probability that at least one member of *S* hits target *h*, as given by (1):

$$v_h(S) = 1 - \prod_{i \in S} (1 - p_{ih}).$$

event	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
r_1, r_1, r_1	r_1	r_1	r_1	r_1	r_1	r_1	r_1
r_1, r_1, r_2	r_1	r_1	r_2	r_1	$r_1 + r_2$	$r_1 + r_2$	$r_1 + r_2$
r_1, r_2, r_1	r_1	r_2	r_1	$r_1 + r_2$	r_1	$r_1 + r_2$	$r_1 + r_2$
r_1, r_2, r_2	r_1	r_2	r_2	$r_1 + r_2$	$r_1 + r_2$	r_2	$r_1 + r_2$
r_2, r_1, r_1	r_2	r_1	r_1	$r_1 + r_2$	$r_1 + r_2$	r_1	$r_1 + r_2$
r_2, r_1, r_2	r_2	r_1	r_2	$r_1 + r_2$	r_2	$r_1 + r_2$	$r_1 + r_2$
r_2, r_2, r_1	r_2	r_2	r_1	r_2	$r_1 + r_2$	$r_1 + r_2$	$r_1 + r_2$
r_2, r_2, r_2	r_2	r_2	r_2	r_2	r_2	r_2	r_2

Table 2: A same payoff is not repeated within a coalition.

Being a positive linear combination of concave games, the game (N, w_2) is concave.

In the preceding scenario, if a coalition *S* and its complementary $N \setminus S$ both hit the same target, they obtain the corresponding payoff. We exclude this in a third scenario where a payoff can be obtained only once: if a coalition forms, it obtains the payoff associated to a target only if no player outside the coalition has hit the same target. This is illustrated in Table 3.

event	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
r_1, r_1, r_1	0	0	0	0	0	0	r_1
r_1, r_1, r_2	0	0	r_2	r_1	r_2	r_2	$r_1 + r_2$
r_1, r_2, r_1	0	r_2	0	r_2	r_1	r_2	$r_1 + r_2$
r_1, r_2, r_2	r_1	0	0	r_1	r_1	r_2	$r_1 + r_2$
r_2, r_1, r_1	r_2	0	0	r_2	r_2	r_1	$r_1 + r_2$
r_2, r_1, r_2	0	r_1	0	r_1	r_2	r_1	$r_1 + r_2$
r_2, r_2, r_1	0	0	r_1	r_2	r_1	r_1	$r_1 + r_2$
r_2, r_2, r_2	0	0	0	0	0	0	r_2

Table 3: A same payoff does not go to a coalition and its complement.

The expected payoffs are computed on the basis of the dual probability games (N, v_h^d):

$$w_3(S) = \sum_{h \in M} v_h^d(S)\, r_h$$

where

$$v_h^d(S) = \left(1 - \prod_{i \in S}(1 - p_{ih})\right) \prod_{i \in N \setminus S}(1 - p_{ih}) = \left(1 - \prod_{i \in S}(1 - p_{ih})\right) \prod_{i \in N \setminus S}(1 - p_{ih})$$

is the probability that at least one member of *S* hit target *h* while those outside *S* all fail, as given by (2). $w_3 = w_2^d$ and, being a positive linear combination of convex games, the game (N, w_3) is convex. By self-duality and linearity, the two scenarios lead to the same Shapley value. Using (8), it is given by:

$$SV_i(N, P, r) = \sum_{h \in M} SV_i(N, p_h)\, r_h = \sum_{h \in M} p_{ih}\, r_h \sum_{T \subset N : i \in T} \frac{(-1)^{t-1}}{t} \prod_{j \in T \setminus i} p_{jh} = \sum_{h \in M} p_{ih}\, r_h\, f(p_{-ih})$$

where *f* is the function defined by (10).

Example 1 (continued) The games (N, w_2) and (N, w_3) associated with the probabilities $(p_{11}, p_{21}, p_{31}) = (0.6, 0.3, 0.7)$ and payoffs $(r_1, r_2) = (5, 7)$ are defined respectively by:

$$w_2 = (5.8,\ 6.4,\ 5.6 \mid 9.34,\ 8.46,\ 9.48 \mid 10.698) \text{ and}$$

$$w_3 = w_2^d = (1.218,\ 2.238,\ 1.358 \mid 5.098,\ 4.298,\ 4.898 \mid 10.698).$$

The (common) Shapley value and the associated distribution key are respectively given by (3.306, 4.116, 3.276) and (0.309, 0.385, 0.306).

4 Multiplicative effects

In the preceding section, payoffs were added. Now, consider a situation where adding a player to a coalition multiplies its payoff by a coefficient that is specific to that player: a payoff for a player is associated to his ability to increase the payoff of a coalition.

The coefficients are assumed to be independent random variables with known probability distribution function and finite mean. More precisely, the coefficient specific to player *i* is a random variable X_i defined on the interval $[1, +\infty)$. Hence, $\mu_i \geq 1$ and the case where $\mu_i = 1$ occurs only if $X_i = 1$ with probability one.

Given the assumption of independence, the associated transferable utility game is defined by:

$$w(S) = E\left[\prod_{i \in S} X_i\right] - 1 = \prod_{i \in S} \mu_i - 1. \tag{13}$$

Hence, a player contributes to a coalition once his expected coefficient is larger than 1. Indeed, $w(i) = \mu_i - 1$ and the marginal contributions of player *i* are given by:

$$w(S) - w(S \setminus i) = (\mu_i - 1) \prod_{j \in S \setminus i} \mu_j \quad \text{for all } S \text{ such that } i \in S.$$

The game (N, w) belongs to the class of *cooperative product games* introduced by Rosales (2014) and revisited in Dehez (2024).[4] Obviously, marginal contributions are increasing. As a

[4] Rosales defines $v(S)$ as the product of the coefficients. This would be consistent if he had assumed that $v(\varnothing) = 1$.

consequence, product games are convex (and thereby superadditive). They are *strictly* convex if $\mu_i > 1$ for all $i \in N$. Using (4), the Harsanyi dividends are given by:

$$\alpha_T(N,w) = \prod_{i\in T}(\mu_i - 1).$$

Consequently, using (5), the Shapley value is given by:

$$SV_i(N,\mu_1,...,\mu_n) = (\mu_i - 1)\sum_{T\subset N: i\in T}\frac{1}{t}\prod_{j\in T\setminus i}(\mu_j - 1).$$

We observe that proportionality applies to product games as well. The Shapley value can indeed be written as:

$$SV_i(N,\mu_1,...,\mu_n) = (\mu_i - 1)\, g(\mu_{-i})$$

where the function $g:\mathbb{R}^{n-1}\to\mathbb{R}$ is the player independent and symmetric function given by:

$$g(z) = 1 + \sum_{T\subset\{1,...,n-1\}}\frac{1}{t+1}\prod_{j\in T}(z_j - 1).$$

Dehez (2024) shows that efficiency, symmetry and proportionality characterise the Shapley value on the class of product games.

Example 2 The 3-player game associated to the means (1.3, 1.8, 2.1) is given by (0.3, 0.8, 1.1 | 1.34, 1.73, 2.78 | 3.914). The Shapley value and the associated distribution key are respectively given by (0.673, 1.448, 1.793) and (0.172, 0.370, 0.458).

5 Concluding remarks

The assumption of probabilistic independence is clearly a limitation. However, in the multiplicative case, the independence assumption can be dispensed with by considering the *geometric mean* instead of the linear mean. For an arbitrary random variable *X*, the geometric mean is defined by:

$$E_G[X] = \mathrm{e}^{E[\log(X)]}.$$

It is an alternative measure of the central tendencies of random variables defined on the positive reals. It is used in different contexts.[5] The geometric expectation of a discrete random variable *X* defined by $\{(x_1,q_1),...,(x_m,q_m)\}$, where $x_h > 0$ for all *h* and the q_h sum up to one, is then simply given by:

$$E_G[X] = \prod_{h=1}^{m} x_h^{q_h},$$

hence the name "geometric" mean. The geometric mean is linear and bounded above by the linear mean. The first property is immediate and the second property is a direct consequence of Jensen inequality. Furthermore, the geometric mean of a product of random variables defined on the positive reals is the product of their geometric means, a property that applies without requiring independence. Hence, we could define the game (N,w) in (13) by replacing μ_i by the geometric mean.

[5] There is a large literature on the geometric mean and its applications, in particular in finance and insurance. See for instance Jasiulewicz and Kordecki (2016).

We have assumed that each player can hit one but only one target. If instead, players can hit *simultaneously* several targets, the picture gets more complicated, except in the first scenario where players are rewarded independently of the performances of the other players. We have also limited our analysis to the discrete (and finite) case. Investigation of these situations could be the object of future research.

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