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Characterising the existence of an odd period cycle in price dynamics for an exchange economy

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Abstract

In this paper, we study a standard exchange economy model with Cobb-Douglas type consumers and give a necessary and sufficient condition for the existence of an odd period cycle in the Walras-Samuelson (tatonnement) price adjustment process. This is a new application of a recent result characterising the existence of a topological chaos for a unimodal interval map by Deng, Khan, Mitra (2022). Moreover, we show that starting from any positive initial price, the price is eventually attracted to a chaotic region. Using our characterisation, we obtain a numerical example of an exchange economy where the price dynamics exhibit an odd period cycle but no period three cycle.

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1 Introduction

In this paper, we study a standard exchange economy model with two consumers of Cobb-Douglas type and two goods.[1] Denote the two consumers by $i = 1, 2$ and the two goods by x and y. Let $\alpha, \beta \in (0, 1)$. We assume that the consumer 1 has the utility function $u^1(x, y) = x^\alpha y^{1-\alpha}$ and the consumer 2 has the utility function $u^2(x, y) = x^\beta y^{1-\beta}$ in the consumption space $\mathbb{R}^2_+$. Also, we assume that the consumer 1 has the initial endowment $w^1 = (\bar{x}, 0)$ and the consumer 2 has the initial endowment $w^2 = (0, \bar{y})$ where $\bar{x}, \bar{y} > 0$. Fix the price of good y as 1 and that of good x as $p > 0$. Then by the standard optimisation result under the budget constraints, we obtain the excess demand function $z(p)$ for good x, that is given by $z(p) = \bar{y}\beta/p - \bar{x}(1 - \alpha)$. Now we define the Walras-Samuelson (tatonnement) price adjustment process by

$$p_{t+1} = f(p_t) = p_t + \lambda z(p_t) = p_t + \lambda[\bar{y}\beta/p_t - \bar{x}(1 - \alpha)] \text{ where } \lambda \in \mathbb{R}_{++}. \tag{1.1}$$

Note that λ denotes the speed of adjustment and p_j denote the price of good x at time j.

Let $E = (0, 1) \times (0, 1)$. Then the following (that is a slight extension of (Bhattacharya & Majumdar, 2007, Prop. 9.10)) is not difficult to show (see Section 3 for a proof):

Proposition 1.1. *There exist open sets $A \subset E$, $B \subset \mathbb{R}^2_{++}$, and $C \subset \mathbb{R}_{++}$ such that if $(\alpha, \beta) \in A$, $(\bar{x}, \bar{y}) \in B$, and $\lambda \in C$ then the process (1.1) has a period three cycle (hence exhibits a Li-Yorke chaos).*

It is well-known that the existence of a period three cycle implies a Li-Yorke chaos (by the famous Li-Yorke theorem (Li & Yorke, 1975, Thm. 1)), and this argument has been used a lot in economic literature, see Benhabib and Day (1980), Benhabib and Day (1982), Day and Shafer (1985), Nishimura and Yano (1996) for example. However, this is a bit overkill: by (Block & Coppel, 1992, Chap. II), we know that the existence of a cycle of *any* odd length (not necessarily of period three) implies a Li-Yorke chaos. The main result of this paper extends Proposition 1.1 in the following way (see Theorem 1.5 for a precise statement): 1. Replace a period three cycle with an odd period cycle (or a turbulence for the second iterate), 2. Give a specific (algebraic) form of the sets A, B, and C, 3. Give a *necessary and sufficient* condition for the existence of an odd period cycle (or a turbulence for the second iterate) rather than giving a sufficient condition only (that is usually done in many economic literature such as Benhabib and Day (1980), Benhabib and Day (1982), Day and Shafer (1985), Nishimura and Yano (1996)). Now we clarify what we mean by a Li-Yorke chaos, a turbulence, and a topological chaos. (There are several definitions of a chaos in the literature.) The following definitions are taken from (Ruette, 2017, Def. 5.1) and (Block & Coppel, 1992, Chap. II). Let g be a continuous map of a closed interval I into itself:

Definition 1.2. We say that g exhibits a *Li-Yorke chaos* if there exists an uncountable *scrambled set* $S \subset I$, that is, for any $x, y \in S$ we have

$$\limsup_{n\to\infty} \mid g^n(x) - g^n(y) \mid > 0 \text{ and } \liminf_{n\to\infty} \mid g^n(x) - g^n(y) \mid = 0,$$

and for $x \in S$ and y being a periodic point of g,

$$\limsup_{n\to\infty} \mid g^n(x) - g^n(y) \mid > 0.$$

Definition 1.3. We call g *turbulent* if there exist three points, x_1, x_2, and x_3 in I such that $g(x_2) = g(x_1) = x_1$ and $g(x_3) = x_2$ with either $x_1 < x_3 < x_2$ or $x_2 < x_3 < x_1$. Moreover, we call g *(topologically) chaotic* if some iterate of g is turbulent.

It is known that a map g is topologically chaotic if and only if g has a periodic point whose period is not a power of 2, see (Block & Coppel, 1992, Chap. II). This implies that a map g is topologically chaotic

[1] Since a Cobb-Douglas utility function is a special case of a CES-type utility function, one might try to use a general CES utility function, that is $u(x, y) = \left(a_i x^{b_i} + (1 - a_i) y^{b_i}\right)^{1/b_i}$. However, in that case, it becomes very difficult (if not impossible) to obtain a precise algebraic characterisation for the existence of an odd cycle (as we did in this paper). If one still wants to use a general CES utility function, then it would be necessary to rely on some numerical computations to obtain a similar characterisation.

if and only if the topological entropy of g is positive, see (Block & Coppel, 1992, Chap. VIII). So, in this paper, we focus on a special type of a topological chaos (or a positive topological entropy) in the context of price dynamics. See Ruette (2017) for more characterisations and (subtle) mutual relations of various kinds of chaos.

To state our main result (Theorem 1.5), we need some preparation. In particular, we recall the following mathematical result characterising the existence of a topological chaos for a unimodal interval map (Deng, Khan, & Mitra, 2022, Cor. 3). Our main result is a direct application of this result. Let $\mathfrak{G}$ be the set of continuous maps from a closed interval $[a, b]$ to itself so that an arbitrary element $g \in \mathfrak{G}$ satisfies the following two properties:

1. there exists $m \in (a, b)$ with the map g strictly decreasing on $[a, m]$ and strictly increasing on $[m, b]$.
2. $g(a) > a$, $g(b) \leq b$, and $g(x) < x$ for all $x \in [m, b)$.

For $g \in \mathfrak{G}$, let $\Pi := \{x \in [a, m] \mid g(x) \in [a, m] \text{ and } g^2(x) = x\}$. Now we are ready to state (Deng et al., 2022, Cor. 3):

Proposition 1.4. *Let $g \in \mathfrak{G}$. The map g has an odd-period cycle if and only if $g^2(m) > m$ and $g^3(m) > \max\{x \in \Pi\}$ and the second iterate g^2 is turbulent if and only if $g^2(m) > m$ and $g^3(m) \geq \min\{x \in \Pi\}$.*

Finally, we are ready to state our main result. We keep the same notation f, $\bar{x}$, $\bar{y}$, α, β, and λ from Equation (1.1). We write $f|_E$ for the restriction of f to the closed interval $E := [f(\sqrt{\bar{y}\lambda\beta}), f^2(\sqrt{\bar{y}\lambda\beta}) + \sqrt{\bar{y}\lambda\beta}]$. (We will explain the significance of the number $\sqrt{\bar{y}\lambda\beta}$ and the reason for the choice of E in the next section.) Note that in the next section we will show that $f(\sqrt{\bar{y}\lambda\beta}) < f^2(\sqrt{\bar{y}\lambda\beta}) + \sqrt{\bar{y}\lambda\beta})$ (so, E is a non-degenerate closed interval) and that $f|_E$ is a map to E.

Theorem 1.5. *Let $\frac{\bar{y}\beta}{\bar{x}^2(1-\alpha)^2} < \lambda < \frac{4\bar{y}\beta}{\bar{x}^2(1-\alpha)^2}$. Then the map f in Equation (1.1) has the following properties:*

1. *$f|_E \in \mathfrak{G}$.*
2. *$f|_E$ has an odd period cycle if and only if $\frac{25\bar{y}\beta}{9\bar{x}^2(1-\alpha)^2} < \lambda < \frac{4\bar{y}\beta}{\bar{x}^2(1-\alpha)^2}$.*
3. *The second iterate $(f|_E)^2$ is turbulent if and only if $\frac{25\bar{y}\beta}{9\bar{x}^2(1-\alpha)^2} \leq \lambda < \frac{4\bar{y}\beta}{\bar{x}^2(1-\alpha)^2}$.*
4. *If $\frac{25\bar{y}\beta}{9\bar{x}^2(1-\alpha)^2} \leq \lambda < \frac{4\bar{y}\beta}{\bar{x}^2(1-\alpha)^2}$, then the closed interval E is attracting for f, that is, $f^n(p) \in E$ for some $n \in \mathbb{N}$.*

Now, several comments are in order. First, in the next section, we will show that the condition $\frac{\bar{y}\beta}{\bar{x}^2(1-\alpha)^2} < \lambda < \frac{4\bar{y}\beta}{\bar{x}^2(1-\alpha)^2}$ above is the weakest condition for f to be economically meaningful (to keep $p > 0$) and also for $f|_E$ to be in $\mathfrak{G}$. In particular, the upper bound for λ directly follows from the requirement $p > 0$. So, the upper bound in parts 2, 3, and 4 for λ is not really interesting. The real meat is in the lower bound.

Second, it is well-know that the vertical stretch the graph of f controls the existence of a chaos: if we stretch the graph further, we are likely to obtain a chaos. Now, looking at Equation (1.1), we see that if we make α, β, $\bar{y}$ small, or $\bar{x}$ large, the "valley" of the graph of f goes deep down. Also, it is clear that a small λ makes the graph of f "flat". Our lower bound $L := \frac{25\bar{y}\beta}{9\bar{x}^2(1-\alpha)^2}$ agrees with these observations: L is an increasing function of α, β, $\bar{y}$ and a decreasing function of $\bar{x}$. In other words, it is easy to generate a chaos (small λ gives a chaos) if α, β, or $\bar{y}$ is small or $\bar{x}$ is large. We give an economic explanation for this. By looking at the form of the excess demand function $z(p) = \bar{y}\beta/p - \bar{x}(1-\alpha)$ (or the demand function of each consumer for good x, that is, $x = \alpha\bar{x}$ for consumer 1 and $x = \frac{\beta\bar{y}}{p}$ for consumer 2 respectively), we see that p goes up very sharply (almost irrespective of the parameter values) after p gets close to 0. To generate a chaos, p needs to drop sharply afterwords, that is possible (or at least easy) if α, β, or $\bar{y}$ is small or $\bar{x}$ is large for the following reasons: 1. if α or β is small, the demand for good 1 is weak (thus p drops sharply), 2. if $\bar{y}$ is small, then the demand of consumer 2 for good x (that was excessively strong when p is close to 0) drops sharply (since the budget for consumer 2 is tight), 3. if $\bar{x}$ is large, when p is very high, a large excess supply happens.

Third, part 4 of Theorem 1.5 shows that if λ is sufficiently large, for any initial $p > 0$, our price dynamics eventually trap p inside a (Li-Yorke) chaotic region E. This is interesting since parts 1, 2, and 3 say nothing about what is going on outside of E. This sort of analysis is not done in Deng et al. (2022). We would like to obtain a similar characterisation of this behaviour (in terms of the iterates of $g(m)$ as in Proposition 1.4) for a general continuous unimodal map in the future.

Fourth, our results are sensitive to the parameter values chosen (i.e., α, β, $\bar{x}$, $\bar{y}$). Thanks to the precise algebraic form of the lower/upper bound for λ, we can measure the sensitivity with respect to each parameter value: just take the partial derivatives of the lower/upper bound.

Finally, we give an application of Theorem 1.5. By the Sharkovsky order in Sharkovsky (1964), we know that if the map f has a cycle of period three, then it also has a cycle of any odd order. Thus, it is natural to guess that if λ is close to the lower bound for λ in Theorem 1.5 (but still above the lower bound), the map f has an odd period cycle but no period three cycle. We give one example where this is actually the case. Using our concrete characterisation of the existence of an odd period cycle, we obtain:

Proposition 1.6. *Let $\bar{x} = 4, \bar{y} = 2, \alpha = 0.75, \beta = 0.5$. If $2.77 < \lambda \leq 3.00$, then the map f in Equation (1.1) has an odd period cycle but no period three cycle.*

Note that in this case, the lower bound is $\frac{25\bar{y}\beta}{9\bar{x}^2(1-\alpha)^2} = 2.77$ and the upper bound is $\frac{4\bar{y}\beta}{\bar{x}^2(1-\alpha)^2} = 4$. Our numerical computation (see Section 3 for details) shows that Proposition 1.6 is almost an if and only if statement: for $\lambda \geq 3.01$, we get a period three cycle. Although many examples of this sort can be obtained by the same method, a complete characterisation for the existence of a period three cycle for a unimodal map is not known. (Thus it is not possible to obtain the precise λ value where a bifurcation happens.) We leave it for a future work.

2 Proof of Theorem 1.5

In the following proof, most results follow from direct (but slightly messy) algebraic calculations. We give some sketches of our manipulations while pointing some important steps out rather than writing all the details. All calculations can be easily checked by a computer algebra system, say, Magma Bosma, Cannon, and Playoust (1997), Python Rossum and Drake (2009), etc.

First, for the function f in Equation (1.1), we have $f'(p) = 1 - \frac{\bar{y}\lambda\beta}{p^2}$ and $f''(p) = \frac{2\bar{y}\lambda\beta}{p^3} > 0$ for any $p > 0$. So, f is strictly convex (unimodal) and takes its minimum at $p = \sqrt{\bar{y}\lambda\beta}$. We sometimes write s for $\sqrt{\bar{y}\lambda\beta}$ to ease the notation. First of all, since we assume that $p > 0$, we must have $f(p) > 0$ for any $p > 0$, hence $f(s) > 0$. This gives that $\lambda < \frac{4\bar{y}\beta}{\bar{x}^2(1-\alpha)^2}$.

Next, we want (a restriction of) f to be in $\mathfrak{G}$. Note that any function f in $\mathfrak{G}$ must have some m in its interior of the domain, say $[a, b]$, with f strictly decreasing on $[a, m]$ and strictly increasing on $[m, b]$. Since our f is unimodal, this forces m to be s. Moreover f needs to satisfy $f(p) < p$ for all $p \in [s, b)$. So in particular, we must have $f(s) < s$. This implies $\frac{\bar{y}\beta}{\bar{x}^2(1-\alpha)^2} < \lambda$. Note that $s < f^2(s)+s$ since $f^2(s) > 0$ (under the condition $\lambda < \frac{4\bar{y}\beta}{\bar{x}^2(1-\alpha)^2}$), so we have $f(s) < s < f^2(s) + s$. Now we set $a := f(s)$ and $b := f^2(s) + s$. (Now it is clear that $[a, b]$ is non-degenerate.)

Lemma 2.1. *If $\frac{\bar{y}\beta}{\bar{x}^2(1-\alpha)^2} < \lambda < \frac{4\bar{y}\beta}{\bar{x}^2(1-\alpha)^2}$, then $f|_E \in \mathfrak{G}$.*

Proof. We need to show three things: (1) $f(a) > a$ and $f(b) \leq b$. (2) $f(p) < p$ for all $p \in [s, b)$. (3) $f|_E$ is a map from E to itself. We begin with (2). Let $p \in [s, b)$. Then we have $p - f(p) = -\lambda[\frac{\bar{y}\beta}{p} - \bar{x}(1-\alpha)]$. Since $\lambda > 0$ and $\frac{\bar{y}\beta}{p} - \bar{x}(1-\alpha) \leq \frac{\bar{y}\beta}{s} - \bar{x}(1-\alpha)$, it is enough to show that $\frac{\bar{y}\beta}{s} - \bar{x}(1-\alpha) < 0$. Now $\frac{\bar{y}\beta}{s} - \bar{x}(1-\alpha) = \frac{\bar{y}\beta}{\sqrt{\bar{y}\lambda\beta}} - \bar{x}(1-\alpha)$, so $\frac{\bar{y}\beta}{s} - \bar{x}(1-\alpha) < 0$ is equivalent to $\lambda > \frac{\bar{y}\beta}{\bar{x}^2(1-\alpha)^2}$, that is certainly true (since this is our assumption). Now we show that the same argument gives $f(b) < b$ in (1). (This strict inequality is stronger than we need here, but we need it to prove part 4 of Theorem 1.5, see the proof of Lemma 2.6 below.) We have $b - f(b) = -\lambda[\frac{\bar{y}\beta}{f^2(s)+s} - \bar{x}(1-\alpha)]$. Since $\lambda > 0$ and $\frac{\bar{y}\beta}{f^2(s)+s} - \bar{x}(1-\alpha) < \frac{\bar{y}\beta}{s} - \bar{x}(1-\alpha)$, it is enough to show that $\frac{\bar{y}\beta}{s} - \bar{x}(1-\alpha) < 0$. We have shown that this is true. Next we show that $f(a) > a$. By a direct calculation, we have that $f(a) - a = f(f(s)) - f(s) = \lambda[\frac{\bar{y}\beta}{f(s)} - \bar{x}(1-\alpha)] > 0$ is equivalent to $\left(\sqrt{\lambda} - \frac{\sqrt{\bar{y}\beta}}{\bar{x}(1-\alpha)}\right)^2 > 0$. Now the assumption $\frac{\bar{y}\beta}{\bar{x}^2(1-\alpha)^2} < \lambda$ forces $f(a) - a > 0$.

To prove (3), we need to show that the maximum value of $f|_E$ does not exceed $b = f^2(s) + s$ (that the minimum value of $f|_E$, that is $f(s)$, is in E is clear). Since f is unimodal, the maximum of f on E is taken either at a or at b. For the first case, we have $f(a) = f(f(s)) = f^2(s) < f^2(s) + m = b$. For the second case, we need $f(b) \leq b$, but this is true by part (1) above.

$\square$

We have proved part 1 of Theorem 1.5. We assume $\frac{\bar{y}\beta}{\bar{x}^2(1-\alpha)^2} < \lambda < \frac{4\bar{y}\beta}{\bar{x}^2(1-\alpha)^2}$ for the rest of the paper. Now we are ready to prove parts 2 and 3 of the theorem. In view of Proposition 1.4 and Lemma 2.1, we only need to translate two conditions $f^2(m) > m$ and $f^3(m) > \max\{x \in \Pi\}$ (or $f^3(m) \geq \min\{x \in \Pi\}$) in terms of $\bar{x}$, $\bar{y}$, α, β, and λ. First we show that

Lemma 2.2. *$f^2(s) > s$ if and only if $\frac{9\bar{y}\beta}{4\bar{x}^2(1-\alpha)^2} < \lambda$.*

Proof. Under the condition $\frac{\bar{y}\beta}{\bar{x}^2(1-\alpha)^2} < \lambda$, a direct computation shows that $f^2(s) > s$ is equivalent to $2\bar{x}^2(1-\alpha)^2\lambda - 5\bar{x}(1-\alpha)\sqrt{\bar{y}\beta\lambda} + 3\beta\bar{y} > 0$. Now the statement follows. $\square$

Next we show that

Lemma 2.3. *If $\frac{9\bar{y}\beta}{4\bar{x}^2(1-\alpha)^2} < \lambda$, then the set $\Pi = \{x \in [a, s] \mid f(x) \in [a, s] \text{ and } f^2(x) = x\}$ is a singleton, namely $\Pi = \{\frac{\bar{y}\beta}{\bar{x}(1-\alpha)}\}$ (the unique fixed point for $f|_E$).*

Proof. First, we compute the fixed points of $f|_E$. Solving $f(p) = p$ for $p > 0$, we obtain $p = \frac{\bar{y}\beta}{\bar{x}(1-\alpha)}$. So, f has the unique fixed point, which we name z. It is clear that $z < s$ (this follows from $\frac{\bar{y}\beta}{\bar{x}^2(1-\alpha)^2} < \lambda$) and $a \leq z$ (since $a = f(s)$ is the minimum of f). So, we have $z \in \Pi$. Next, we compute the period 2 points for f on $[a, s]$. Solving $f^2(p) = p$ for p (with Python), we get $p = \frac{\bar{y}\beta}{\bar{x}(1-\alpha)}, -\frac{\bar{x}\alpha\lambda}{2} + \frac{\bar{x}\lambda}{2} - \frac{1}{2}\sqrt{\bar{x}^2\alpha^2\lambda^2 - 2\bar{x}^2\alpha\lambda^2 - 2\bar{y}\beta\lambda + \bar{x}^2\lambda^2}, -\frac{\bar{x}\alpha\lambda}{2} + \frac{\bar{x}\lambda}{2} + \frac{1}{2}\sqrt{\bar{x}^2\alpha^2\lambda^2 - 2\bar{x}^2\alpha\lambda^2 - 2\bar{y}\beta\lambda + \bar{x}^2\lambda^2}$. The first p is z (the fixed point), and the other two points are the period 2 points. We name the last two points as w_1 and w_2 respectively ($w_1 \leq w_2$). If we show that $s < w_2$, we are done. A direct calculation (with Python) shows that $s < w_2$ is equivalent to $\frac{9\bar{y}\beta}{4\bar{x}^2(1-\alpha)^2} < \lambda$ (this is our assumption). $\square$

Now we assume $\frac{9\bar{y}\beta}{4\bar{x}^2(1-\alpha)^2} < \lambda$. (So Π is a singleton.) Finally we show

Lemma 2.4. *$f^3(s) > \max\{x \in \Pi\}$ if and only if $\frac{25\bar{y}\beta}{9\bar{x}^2(1-\alpha)} < \lambda$.*

Proof. This calculation is a bit involved, so we give some details. Let z be the fixed point of f. Since we know that $\max\{x \in \Pi\} = \{z\}$, we have

$$\begin{aligned}
f^3(s) - \max\{x \in \Pi\} &= f(f^2(s)) - z \\
&= f(f^2(s)) - f(z) \\
&= \left(f^2(s) + \lambda\left[\frac{\bar{y}\beta}{f^2(s)} - \bar{x}(1-\alpha)\right]\right) - \left(z + \lambda\left[\frac{\bar{y}\beta}{z} - \bar{x}(1-\alpha)\right]\right) \\
&= f^2(s) - z + \lambda\left[\frac{\bar{y}\beta}{f^2(s)} - \frac{\bar{y}\beta}{z}\right] \\
&= f^2(s) - z - \bar{y}\beta\lambda\left[\frac{f^2(s) - z}{zf^2(s)}\right] \\
&= (f^2(s) - z)\left(1 - \frac{\bar{y}\beta\lambda}{f^2(s)z}\right)
\end{aligned}$$

We consider the first term of the last expression. A direct calculation shows that $s > z$ if and only if $\lambda > \frac{\bar{y}\beta}{\bar{x}^2(1-\alpha)^2}$ (which we already assumed). So, we have $f^2(s) - z > f^2(s) - s > 0$. (The last inequality followed from Lemma 2.2.) Next, a direct calculation shows that the second term of the last expression is positive if and only if $3\bar{x}^2(1-\alpha)^2\lambda - 8\bar{x}(1-\alpha)\sqrt{\bar{y}\beta\lambda} + 5\bar{y}\beta > 0$. Now we see that under the condition $\frac{\bar{y}\beta}{\bar{x}^2(1-\alpha)^2} < \lambda$, this is equivalent to $\frac{25\bar{y}\beta}{9\bar{x}^2(1-\alpha)^2} < \lambda$. $\square$

It is clear that $\min\{x \in \Pi\} = \{z\}$. So by the same argument, we obtain

Lemma 2.5. $f^3(s) \geq \min\{x \in \Pi\}$ *if and only if* $\frac{25\bar{y}\beta}{9\bar{x}^2(1-\alpha)} \leq \lambda$.

Note that $\frac{9\bar{y}\beta}{4\bar{x}^2(1-\alpha)^2} \approx \frac{2.25\bar{y}\beta}{\bar{x}^2(1-\alpha)^2} < \frac{2.78\bar{y}\beta}{\bar{x}^2(1-\alpha)^2} \approx \frac{25\bar{y}\beta}{9\bar{x}^2(1-\alpha)^2}$. By Proposition 1.4 and Lemmas 2.1, 2.2, 2.3, 2.4, 2.5, we have proved parts 2 and 3 of Theorem 1.5. Finally, we are left to show (part 4 of Theorem 1.5):

Lemma 2.6. *If* $\frac{25\bar{y}\beta}{9\bar{x}^2(1-\alpha)^2} \leq \lambda < \frac{4\bar{y}\beta}{\bar{x}^2(1-\alpha)^2}$, *then the closed interval* E *is attracting for* f, *that is,* $f^n(p) \in E$ *for some* $n > 0$.

Proof. Since $E = [a, b]$, we need to prove: 1. if $0 < p < a$, then $f^{n_1}(p) \in E$ for some $n_1 \in \mathbb{N}$, 2. if $b < p$, then $f^{n_2}(p) \in E$ for some $n_2 \in \mathbb{N}$. Note that if $0 < p < a$ (the first case), then $f(p) \geq f(s) = a$ since $f(s)$ is a global minimum for f, thus we just need to consider the second case. Let $p > b$. Then we have $f(p) - p = \lambda[\frac{\bar{y}\beta}{p} - \bar{x}(1-\alpha)] \leq \lambda[\frac{\bar{y}\beta}{b} - \bar{x}(1-\alpha)] = f(b) - b < 0$. (The last strict inequality follows from $\frac{\bar{y}\beta}{\bar{x}^2(1-\alpha)^2} < \lambda$, see the proof of Lemma 2.1) This shows that if $p > b$, in each iteration of f, the value of p drops by $b - f(b) > 0$ at least. So, to prove that p is attracted to E, it is enough to show that the size of the drop is not too big (thus p does not jump over E). Therefore, it is sufficient to have $p - f(p) \leq b - a$. Since this is equivalent to $\lambda[\bar{x}(1-\alpha)] \leq f^2(s) + s - f(s)$ and $s - f(s) > 0$ under our assumption on λ, it suffices to show that $\lambda[\bar{x}(1-\alpha)] \leq f^2(s)$. After some computations, we obtain

$$f^2(s) - \lambda[\bar{x}(1-\alpha)] = \frac{\lambda\left[3\bar{x}^2(1-\alpha)^2\lambda - 8\bar{x}\sqrt{\bar{y}}(1-\alpha)\sqrt{\beta\lambda} + 5\bar{y}\beta\right]}{\bar{x}(\alpha-1)\lambda + 2\sqrt{\bar{y}\beta\lambda}}.$$

We check that in the last expression, the denominator is strictly positive if $0 < \lambda < \frac{4\bar{y}\beta}{\bar{x}^2(1-\alpha)^2}$ (this follows from our assumption). Also, we see that the numerator is positive if $\frac{25\bar{y}\beta}{9\bar{x}^2(1-\alpha)^2} \leq \lambda$.

□

3 Proofs of Propositions 1.1 and 1.6

Proof of Proposition 1.1. The first part of the following argument is a replicate of (Bhattacharya & Majumdar, 2007, Proof of Prop. 9.10). We include this to make the paper self-contained. Let $(\alpha, \beta) = (0.75, 0.5)$, $(\bar{x}, \bar{y}) = (4, 2)$, and $\lambda = 3.61$. Then we have $f(3.75) = 1.9$, $f(1.9) = 0.19$, $f(0.19) = 15.58 > 3.75$. So, by the Li-Yorke theorem, there exists a period three cycle. Also, by (Bhattacharya & Majumdar, 2007, Prop. 9.10 and its proof), we know that the choices of (α, β) and λ are robust (for this fixed $(\bar{x}, \bar{y})$), so it is clear that there exist open sets A and C as in Proposition 1.1. Thus, the only thing we need to show is that the choice of $(\bar{x}, \bar{y})$ is also robust. This follows from the following numerical/graphical argument.

In Figure 1, we plot the graphs of $p_{t+1} = f(p_t)$ (blue) and $p_{t+3} = f^3(p_t)$ (orange) together with the 45° line (green). We see that the orange curve crosses the green line at p^* (the red dot) between $p = 2$ and $p = 3$. A numerical computation gives $p^* \cong 2.66$, and it is clear that p^* is a point of period three. (From the picture we see that p^* is not a fixed point.) Since the orange curve crosses (but not touching) the green line at p^* and f^3 is continuous in $\bar{x}$ and $\bar{y}$, a small perturbation of $\bar{x}$ and $\bar{y}$ does not affect the existence of a period three cycle. So, there exists an open set B as required. (Actually, this graphical argument gives the existence of open sets A and C as well.) □

Proof of Proposition 1.6. Let $(\bar{x}, \bar{y}) = (4, 2)$, $(\alpha, \beta) = (0.75, 0.5)$, and $2.77 < \lambda \leq 3.00$. Then by Theorem 1.5 it is clear that the map f has an odd period cycle. Only thing we need to show now is that f does not have a period three cycle. We use a graphical/numerical argument. In Figures 2, 3, and 4 we plot the graphs of f (blue) and f^3 (orange) with the 45° line (green). A general pattern is that if we increase λ, the graph of f gets slightly deeper down, and f^3 becomes more "wavy". (Compare Figure 1 ($\lambda = 3.61$) to Figure 2 ($\lambda = 2.78$) for example.) For Figure 2 ($\lambda = 2.78$), using a numerical computation, we have checked that the orange curve does not touch/cross the green line except at $p = 1$ (the fixed point of f). Thus, there is no period three cycle in this case. Likewise, for Figure 3 ($\lambda = 3$), there is no period three cycle (although the orange curve seems touching the green line but a numerical computation shows that it is not touching). However, if $\lambda = 3.01$, a numerical computation shows that there exists a period three cycle. In Figure 4, we

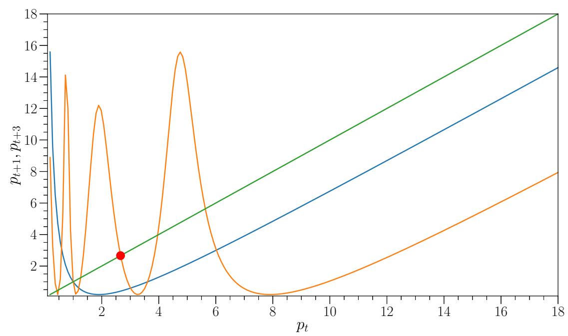

Figure 1: $\lambda = 3.61$

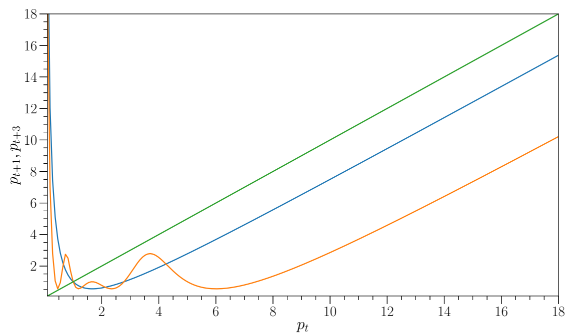

Figure 2: $\lambda = 2.78$

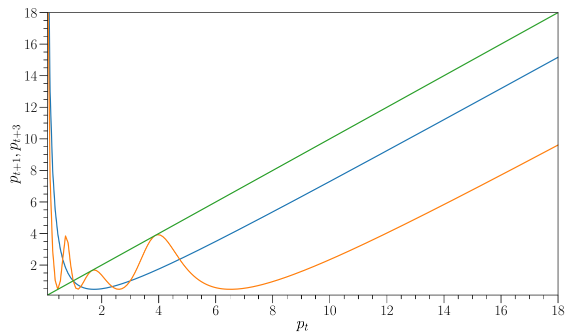

Figure 3: $\lambda = 3.0$

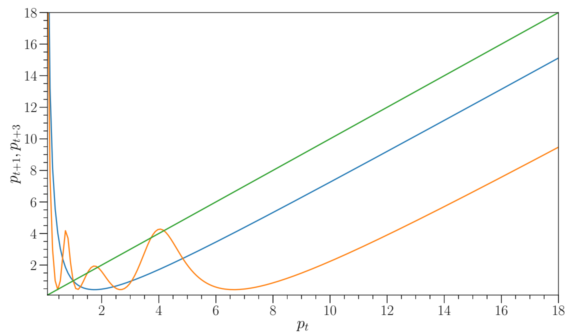

Figure 4: $\lambda = 3.05$

plotted the $\lambda = 3.05$ case since the $\lambda = 3.01$ case is indistinguishable from the $\lambda = 3$ case in picture. It is clear that the orange curve crosses at the green line near $p = 4$ (that is not a fixed point of f).

□

4 Related literature

We admit that our model in this paper is rather specific, say, compared to more generic models in Benhabib and Day (1980), Benhabib and Day (1982), or in Day and Shafer (1985). We believe that there is always a trade-off between using a generic model with a general result and using a specific model with a sharp result. In this paper, we focus on the latter by the following reasons: 1. Considering that obtaining a complete characterisation (a necessary and sufficient condition) for the existence of a chaos (or an odd period cycle to be more precise) is much more involved (as shown in this paper) than merely obtaining a sufficient condition using "Period three implies chaos". So if we use the generic model as in Benhabib and Day (1980), Benhabib and Day (1982), or in Day and Shafer (1985) to obtain a complete characterisation, the result would be very complicated and obscure (cannot be simplified much). 2. Our purpose to obtain a concrete (algebraic) characterisation is for applications, for example, in Uchiyama (2024a) and Uchiyama (2024b), after obtaining a complete characterisation using the same method (and some numerical computations), we have conducted a sensitivity analysis, that is, to investigate how each parameter value affects the existence of a chaos (with its economic interpretations). 2. In the followup paper Uchiyama (2024c), we investigate "ergodic properties" of our price dynamics model. Roughly speaking, we were able to show that we can predict the future price level "on average" even if a chaos exists. To conduct this analysis, it was necessary to specify the function form of f and to obtain the concrete algebraic form of the characterisation for the existence of a chaos.

It is true that if we use a discontinuous function (or a piecewise defined function), it is easy to generate a chaos. (There are such examples in Day & Shafer (1985) and in Deng et al. (2022) for example.) Also, there are examples in Deng et al. (2022) that use a well-studied "logistic map", that is a map of the form $f(x) = ax(1-x)$ for some $a > 0$. The point of this paper is to show that even with our innocent looking smooth continuous unimodal function (that is not of a logistic type and that naturally arises in basic economic theory), we still obtain a chaotic behaviour if certain conditions on the parameters are met. Further, putting some mild conditions, we obtain a complete characterisation (for the existence of an odd period cycle and of a turbulence) in terms of those parameters.

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Declaration

The author has no conflicts of interest to declare that are relevant to the content of this article.

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