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The distributions of annual earnings and interest rates in a continuous time Markov chain economy

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Abstract

In this note, we show that two-state continuous time Markov processes can generate empirically plausible distributions of annual earnings and interest rates. Annual earnings and interest rates in continuous time models are functions of a path integral over the instantaneous values and therefore continuously distributed. We develop an algorithm computing the cross-sectional distributions of annual earnings and interest rates. This algorithm can be used to simulate annual income and interest rates in continuous time models or calibrate the fundamental parameters of the continuous time Markov processes.

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1 Introduction

Large-scale macroeconomic models that attempt to discover non-linear relations between the macroeconomy and financial markets are becoming increasingly important. However, solving these models is often challenging in respect of both analytical and numerical manner. This gives rise to the need for methods that help solve these large-scale models efficiently. To this end, continuous time models are often chosen as they often entail insightful closed-form characterizations of equilibrium outcomes. Even if the closed-form solutions are not available, continuous time models can be numerically solved faster than their discrete-time representation counterpart. However, when these continuous time models are brought to the data by means of calibration or estimation, it is important to match the time frequency of flow variables in the model with that in the data.[1] For example, the distributions of annual earnings and interest rates must be constructed from the model to match the empirical distributions of annual earnings and interest rates in the data. In other words, one would need to compute annual earnings and interest rates in the model and then compare their moments with those in the data. This is important but sometimes is not well-developed in the economics literature. Though there have been a number of papers that attempt to match the first and second moment of annual wages in continuous time models with those in the data (Benhabib et al., 2011; Lise, 2013; Gabaix et al., 2016; Aoki and Nirei, 2017; Cao and Luo, 2017, Khieu and Wälde, 2023), very little is understood about the method of computing annual earnings and interest rates and their distributions. Kaplan et al. (2018) attempt to match higher moments of income changes but not the income level; they use the Simulated Method of Moments to simulate income from the model at a high frequency and then aggregate to annual income. To the best of our knowledge, Khieu et al. (2020) are the first that match higher moments of annual wages using a path integral and its moment generating function.

In continuous time models, randomness in earnings and interest rates is often modelled using finite state Markov chains (Elliott and van der Hoek, 2013). At each point in time, the model generates instantaneous values of earnings and interest rates; when one would like to obtain monthly or annual earnings and interest rates, a path integral over a certain length of time is needed. As the continuous time earnings and interest rate processes follow a distribution, so does the path integral. While the instantaneous values are not observed, we observe the path integral in the data. In this paper, we show that annual earnings and the logarithm of the gross annual interest rate are path integrals of the instantaneous values. We then provide a formal proof that the path integral and hence annual earnings and interest rates generated by two-state continuous time Markov processes are continuously distributed. We develop an algorithm computing the distribution of the path integral, which can be used to (i) simulate the distributions of annual earnings and interest rates when the parameters of continuous time Markov processes are given or (ii) calibrate the parameters when they are unknown. In the calibration process, our algorithm allows for simulating the path integral and therefore computing theoretical moments of annual earnings and interest rates as functions of fundamental parameters of continuous time Markov processes. These parameters are then to be chosen to minimize the (weighted) sum of the squared differences between theoretical and empirical moments. Our method simulates the path integral exactly from the continuous time process and does not imply any discretization as the Simulated Method of Moments. This is an important advantage to calibrate continuous time models.

The rest of this paper is structured as follows. Section 2 presents the wealth accumulation

[1]The matching exercise is not relevant for stock variables like wealth, capital, or default times since they are accumulated.

process and derives an expression for annual interest rates. Section 3 describes the continuous time income process and the corresponding path integral. Section 4 presents a formal approach and an algorithm computing the distribution of the path integral followed by Section 5, which discusses two applications of the algorithm. Finally, Section 6 concludes.

2 Wealth accumulation and the interest rate process

Let us consider the following wealth accumulation process

$$da(t) = r(t)\, a(t)\, dt, \tag{1}$$

where $a(t)$ and $r(t)$ represent the stock of wealth and instantaneous interest rate at time t. The change in wealth equals the instantaneous value of interest $r(t)\, a(t)$. The instantaneous interest rate fluctuates between a low value r^{low} and a high value r^{high} according to

$$dr(t) = \left[r^{\text{high}} - r(t)\right] dq_{\text{low}}(t) + \left[r^{\text{low}} - r(t)\right] dq_{\text{high}}(t), \tag{2}$$

where $r^{\text{high}} > r^{\text{low}} > 0$, and $q_{\text{low}}(t)$ and $q_{\text{high}}(t)$ are two Poisson processes with corresponding arrival rates $\lambda^{\text{low}} > 0$ and $\lambda^{\text{high}} > 0$.[2] The interest rate jumps from its current level $r(t)$ to the new level r^{low} or r^{high} when the appropriate increment, $dq_{\text{high}}(t)$ or $dq_{\text{low}}(t)$, equals unity. The arrival rate λ^{low} describes how often the interest rate jumps from the state r^{low} while λ^{high} captures how quickly the interest rate jumps from the state r^{high}. A high λ^{low} (or a high λ^{high}) means the interest rate jumps from r^{low} to r^{high} (or from r^{high} to r^{low}) relatively quickly. Intuitively, when an individual leaves the state with a high return relatively quickly over her life cycle, it implies the time she spends in the low-return state is relatively long. λ^{high} therefore is a falling function of λ^{low}. Solving this differential equation (1) for wealth at any future point t_1 in time yields

$$a(t_1) = a_0 e^{\int_0^\tau r(t)dt}, \tag{3}$$

where a_0 is initial wealth at time $t = t_0$ and $t_1 := t_0 + \tau, \tau > 0$. Let us denote $\tilde{r}(t_0)$ as the per-period interest rate over the time interval $\mathbb{T} \equiv [t_0, t_1]$. A simple accounting principle implies that wealth at time t_1 is equal to initial wealth a_0 plus interest accrued over the time interval $\mathbb{T}$, $\tilde{r}(t_0)\, a_0$, i.e.

$$a(t_1) = a_0 (1 + \tilde{r}(t_0)). \tag{4}$$

Equating the wealth levels from (3) and (4) yields

$$\tilde{r}(t_0) = e^{\int_0^\tau r(t)dt} - 1. \tag{5}$$

Per-period interest rate is constituted by an exponential of a path integral over instantaneous values. Without loss of generality, let us assume $t_0 = 0$.

Lemma 1 *The lower and upper bounds of the per-period interest rate $\tilde{r}$ are respectively given by*

$$\tilde{r}^L = e^{\tau r^{low}} - 1, \tag{6}$$

$$\tilde{r}^H = e^{\tau r^{high}} - 1. \tag{7}$$

[2] See Cox and Miller (1977) for more background on Poisson processes.

The proof is straightforward. The lowest possible value of the per-period interest rate materializes when the instantaneous interest rate remains low over the time interval $\mathbb{T}$. This is given from (6). When the instantaneous interest rate remains high for the entire time interval, the per-period interest rate is given by (7), which is the highest possible value of $\tilde{r}$, i.e. the upper bound of $\tilde{r}$.

Lemma 2 *The per-period interest rate $\tilde{r}$ is a continuous random variable.*

Proof. Trivially, $\tilde{r}$ is random due to randomness in the instantaneous interest rate r. Equation (5) implies that the logarithm of the gross per-period interest rate is a path integral of instantaneous interest rate, i.e.

$$R(0) \equiv \ln(1 + \tilde{r}(0)) = \int_0^{\tau} r(t)\, dt. \tag{8}$$

To show $\tilde{r}$ is a continuous variable, it suffices to show that the logarithm of the gross per-period interest rate R is continuous. First note that the lower and upper bounds of R are τr^{low} and τr^{high}, respectively. It now suffices to show that

$$\Pr\left(R(0) \in [\tau r^{\text{low}}, \tau r^{\text{low}} + \varepsilon]\right) > 0, \tag{9}$$

and that

$$\Pr\left(R(0) \in [\tau r^{\text{high}} - \varepsilon, \tau r^{\text{high}}]\right) > 0, \tag{10}$$

for some arbitrary positive $\varepsilon \in \Xi \equiv \left[0, \tau(r^{\text{high}} - r^{\text{low}})\right]$. Consider two cases.

Case 1: $r(0) = r^{\text{low}}$

Consider the following path integral

$$R(0) = \int_0^{\tau_1} r^{\text{low}} dt + \int_{\tau_1}^{\tau} r^{\text{high}} dt = \tau r^{\text{low}} + \varepsilon_1^{\text{low}} = \tau r^{\text{high}} - \varepsilon_1^{\text{high}},$$

where $\tau_1 \leq \tau$, $\varepsilon_1^{\text{low}} \equiv (\tau - \tau_1)\left(r^{\text{high}} - r^{\text{low}}\right)$ and $\varepsilon_1^{\text{high}} \equiv \tau_1\left(r^{\text{high}} - r^{\text{low}}\right)$. Note that $r(t)$ can jump between r^{low} and r^{high} many times between 0 and τ and that τ_1 is the total amount of time that the instantaneous interest rate is in state r^{low} between 0 and τ. $\tau_1 = \tau$ if $r(t)$ does not jump between 0 and τ.

Since the instantaneous interest rate jumps from r^{low} to r^{high} with the arrival rate λ^{low}, the amount of time that $r(t)$ remains at r^{low}, given that $r(0) = r^{\text{low}}$, is exponentially distributed with parameter λ^{low}. As long as τ is finite, there *always* exists $\tau_1 \leq \tau$ such that $\varepsilon_1^{\text{low}} \in \Xi$ and $\varepsilon_1^{\text{high}} \in \Xi$. This implies (9) and (10).

Case 2: $r(0) = r^{\text{high}}$

In this case, let us consider

$$R(\tau) = \int_0^{\tau_2} r^{\text{high}} dt + \int_{\tau_2}^{\tau} r^{\text{low}} dt = \tau r^{\text{low}} + \varepsilon_2^{\text{low}} = \tau r^{\text{high}} - \varepsilon_2^{\text{high}},$$

where $\tau_2 \leq \tau$, $\varepsilon_2^{\text{low}} \equiv \tau_2\left(r^{\text{high}} - r^{\text{low}}\right)$ and $\varepsilon_2^{\text{high}} \equiv (\tau - \tau_2)\left(r^{\text{high}} - r^{\text{low}}\right)$. τ_2 is the total amount of time between 0 and τ that the instantaneous interest rate is in state r^{high}. $\tau_2 = \tau$ if $r(t)$ does not jump between 0 and τ.

The amount of time that $r(t)$ remains at r^{high}, given that $r(0) = r^{\text{high}}$, is exponentially distributed with parameter λ^{high}, which is the arrival rate at which the instantaneous interest rate jumps from r^{high} to r^{low}. Given that τ is finite, there *always* exists $\tau_2 \leq \tau$ such that $\varepsilon_2^{\text{low}} \in \Xi$ and $\varepsilon_2^{\text{high}} \in \Xi$. This proves (9) and (10). ■

Lemmas 1 and 2 imply that the per-period interest rate $\tilde{r}$ is continuously distributed over the support $\mathbb{S}= \left[\tilde{r}^L, \tilde{r}^H\right]$. As the instantaneous interest rate r follows a distribution, the per-period interest rate $\tilde{r}(\tau)$ given from (5) obeys a distribution as well. There are two interpretations for the distribution of $\tilde{r}(\tau)$. First, it is the distribution of the per-period interest rate at time τ of one individual. Second, when there are sufficiently many individuals that draw the instantaneous interest rate $r(t)$ independently from an identical distribution, the distribution of $\tilde{r}(\tau)$ is the cross-sectional distribution of the per-period interest rate at time τ. We adopt the second interpretation.

3 Earnings process and the path integral for earnings

Consider an individual whose instantaneous earnings $w(t)$ evolves according to

$$dw(t) = \left[w^h - w(t)\right] dq_l(t) + \left[w^l - w(t)\right] dq_h(t). \tag{11}$$

Equation (11) shows that instantaneous earnings that the individual receives at any point t in time can be either w^l or w^h. The jumps between these two states are described by the increments $dq_h(t)$ and $dq_l(t)$, where $q_h(t)$ and $q_l(t)$ are two Poisson processes with constant arrival rates $\lambda_h > 0$ moving the individual from high income w^h to low income w^l and $\lambda_l > 0$ moving the individual from low income w^l to high income w^h. Let $W(0)$ be the per-period income for the individual over the time interval $\mathbb{T}$. Per-period income $W(0)$ is a path integral (see Khieu et al., 2020)

$$W(0) = \int_0^{\tau} w(t)dt. \tag{12}$$

As instantaneous earnings $w(t)$ follow a distribution governed by two Poisson processes $q_h(t)$ and $q_l(t)$, the path integral (12) implies that per-period income $W(0)$ also obeys a distribution. Note that the path integrals (8) and (12) are of analogy. $W(0)$ is therefore a continuous random variable. There are two interpretations for the distribution of $W(0)$. First, it is the distribution of per-period income over the time interval $\mathbb{T}$ of one individual. Second, when there are sufficiently many individuals that draw instantaneous earnings $w(t)$ independently from an identical distribution, the distribution of $W(0)$ is the cross-sectional distribution of per-period income over the time interval $\mathbb{T}$. The second interpretation is adopted.

Despite the analogy of the path integrals (8) and (12), per-period interest rate and per-period income are distinct from each other. Per-period interest rate (5) is derived from the wealth accumulation process and therefore associated with the exponential of a path integral. Per-period income (12) is simply a time aggregation. This is an important point when it comes to simulation or calibration.

4 Computing the distribution of the path integrals

The analogy of the path integrals (8) and (12) allows us to work with either one. Let us now work with (12) since it allows us to compute the distribution of per-period income over the time interval $\mathbb{H} \equiv [0, \Delta]$[3]. Assume at time $t_k^i \in \mathbb{H}$ instantaneous income jumps to state

[3]Computing the path integral (8) only gives us the logarithm of the gross per-period interest rate. To derive the distribution of the per-period interest rate, a transformation from logarithm to level is needed.

w^i for the k-th time, $i \in \mathbb{S} = \{l, h\}$, $k \in \mathbb{K} = \{1, 2, ..., K\}$, where K is the total times that instantaneous income jumps to state w^i. K is unknown. Let Δ_k^i be the holding time for state w^i starting from time t_k^i. This holding time Δ_k^i is exponentially distributed according to parameter λ^i. Per-period income over the time interval $[t_k^i, t_k^i + \Delta_k^i]$ is given by

$$W_k^i = \int_{t_k^i}^{t_k^i + \Delta_k^i} w^i dt, \ i \in \mathbb{S}, \ k \in \mathbb{K}. \tag{13}$$

As w^i is independent of time, (13) is written

$$W_k^i = w^i \Delta_k^i.$$

As we are interested in per-period income over the interval $\mathbb{H}$ only, let us set $t_k^i + \Delta_k^i = \Delta$ when $t_k^i + \Delta_k^i > \Delta$ holds. Therefore, W_k^i is equal to zero when $t_k^i = \Delta$ holds. Per-period income over the time interval $\mathbb{H}$ is therefore given by

$$W(0) = \sum_{\mathbb{S}} \sum_{\mathbb{K}} W_k^i. \tag{14}$$

As an illustration, Figure 1 shows that the individual receives low income w^l at time $t = 0$ and her income remains low for a length of time Δ_1^l. Then her income jumps to w^h at time t_1^h and remains high for a length of time Δ_1^h. At time t_3^l the individual's income jumps from high to low for the third time and remains low for a length of time $\bar{\Delta}_3^l$. As $t_3^l + \bar{\Delta}_3^l > \Delta$, let us set $\Delta_3^l = \Delta - t_3^l$ and $\Delta_3^h = 0$. Given this realization of instantaneous income, per-period income over the time interval $\mathbb{H}$ is given by

$$W(0) = \left(W_1^l + W_2^l + W_3^l\right) + \left(W_1^h + W_2^h + W_3^h\right),$$

which is the area under the instantaneous income schedule $w(t)$ and above the horizontal axis in the time interval $[0, \Delta]$.[4]

We are interested in the cross-sectional distribution of annual income for N individuals. When we assume the sample size N is sufficiently large, the fractions of individuals receiving low and high income at time $t = 0$ are respectively given by

$$\zeta_l = \frac{\lambda_h}{\lambda_l + \lambda_h},$$
$$\zeta_h = \frac{\lambda_l}{\lambda_l + \lambda_h} \equiv 1 - \zeta_l.$$

Thus, the cross-sectional distribution is made of $\zeta_l N$ individuals starting their income path with a low value and $\zeta_h N$ individuals starting their income path with a high value. The following algorithm describes a simulation method that generates per-period income of $\zeta_l N$ individuals receiving low income at time $t = 0$.

[4]Figure 1 is an illustration displaying an arbitrary realization of instantaneous income. It may show that Δ_k^is are equal for this realization, but it does not imply that Δ_k^is are equal for all realizations. With another realization, Δ_k^is might not be equal.

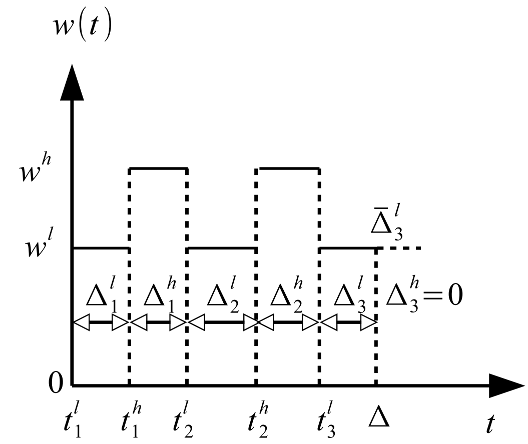

Figure 1 *A realization of instantaneous income and the corresponding per-period income*

Algorithm 1 ***(Individuals with low initial income)***

1. *Draw the length of time Δ^l that income remains low from time $t = 0$. Δ^l is exponentially distributed with parameter λ_l. There are two possibilities:*

   ⊛ *If $\Delta^l \geq \Delta$, set $\Delta_1^l = \Delta$ and $\Delta_1^h = 0$. The values of total income received during the time intervals Δ_1^l and Δ_1^h are respectively given by*

$$W_1^l = \int_0^{\Delta_1^l} w^l dt, \tag{15}$$

$$W_1^h = \int_{\Delta-\Delta_1^h}^{\Delta} w^h dt. \tag{16}$$

   Total income received over the time interval $\mathbb{H}$ is therefore given by

$$W(0) = W_1^l + W_1^h. \tag{17}$$

   ⊛ *If $\Delta^l < \Delta$, i.e. income jumps from low to high at time $t = \Delta^l$, then draw the length of time Δ^h that income remains high. Δ^h is exponentially distributed with parameter λ_h.*

   - *If $\Delta^l + \Delta^h \geq \Delta$, set $\Delta_1^l = \Delta^l$ and $\Delta_1^h = \Delta - \Delta_1^l$. Total income received over the time interval $\mathbb{H}$ is computed according to (17).*
   - *If $\Delta^l + \Delta^h < \Delta$, i.e. income jumps from high to low at time $t = \Delta^l + \Delta^h$, set $\Delta_1^l = \Delta^l$ and $\Delta_1^h = \Delta^h$. Denote $\Delta_1 := \Delta_1^l + \Delta_1^h$ and go to the next step.*

2. *Without loss of generality, assume we are now at step $n \geq 2$. Draw the length of time Δ^l that income remains low from time $t = \Delta_{n-1}$. There are two possibilities:*

⊛ *If* $\Delta^l + \Delta_{n-1} \geq \Delta$*, set* $\Delta_n^l = \Delta - \Delta_{n-1}$ *and* $\Delta_n^h = 0$*. The values of total income received during the time intervals* Δ_n^l *and* Δ_n^h *are respectively given by*

$$W_n^l = \int_{\Delta_{n-1}}^{\Delta_{n-1}+\Delta_n^l} w^l dt, \tag{18}$$

$$W_n^h = \int_{\Delta-\Delta_n^h}^{\Delta} w^h dt \tag{19}$$

Total income received over the time interval $\mathbb{H}$ *is therefore given by*

$$W(0) = \sum_{k=1}^{n} \left[W_k^l + W_k^h\right]. \tag{20}$$

⊛ *If* $\Delta^l + \Delta_{n-1} < \Delta$*, i.e. income jumps from low to high at time* $t = \Delta^l + \Delta_{n-1}$*, then draw the length of time* Δ^h *that income remains high.*

- *If* $\Delta_{n-1} + \Delta^l + \Delta^h \geq \Delta$*, set* $\Delta_n^l = \Delta^l$ *and* $\Delta_n^h = \Delta - \Delta^l - \Delta_{n-1}$*. Total income received over the time interval* $\mathbb{H}$ *is computed according to (20).*
- *If* $\Delta_{n-1} + \Delta^l + \Delta^h < \Delta$*, i.e. income jumps from high to low again at time* $t = \Delta_{n-1} + \Delta^l + \Delta^h$*, set* $\Delta_n^l = \Delta^l$ *and* $\Delta_n^h = \Delta^h$*. Denote* $\Delta_n := \Delta_{n-1} + \Delta_n^l + \Delta_n^h$ *and proceed to step* $n+1$*.*

The algorithm generating per-period income of $\zeta_h N$ individuals receiving high income at time $t = 0$ is quite analogous to Algorithm 1 and therefore presented in Appendix A. When we combine these individuals with those receiving low income at time $t = 0$, we obtain the cross-sectional distribution of per-period income.

5 Applications

There are two applications of our algorithm. First, the algorithm can be used to simulate the distributions of annual income and interest rates when the parameters of the continuous time Markov processes are known and given. Second, it can be used to calibrate the parameters to match empirical targets.

We now perform the first application assuming that the fundamental parameters of the labor income process are given. Specifically, we use the Algorithms 1 and 2 to generate the cross-sectional distribution of annual labor income. First, let us assume a unit of time is a year. As we are interested in computing annual labor income, let us set $\Delta = 1$. We set the low value and the high value of instantaneous labor income to 0 and 100, respectively. The sample size is ten millions, i.e. $N = 10^7$. To gain a clear insight into the shape of the distribution of annual labor income, let us run two simulations corresponding to two pairs of the arrival rates. First, the low-income exit rate is greater than the high-income exit rate, i.e. $\lambda_l > \lambda_h$. Second, the low-income exit rate is smaller than the high-income exit rate, i.e. $\lambda_l < \lambda_h$.[5] The parameters are shown in Table 1 below.[6]

[5]We do not expect that identical arrival rates are empirically plausible since they would imply a uniform distribution. Empirical distributions of annual income are skewed to the right (Cao and Luo, 2017; Khieu and Wälde, 2023).

[6]Since the parameters are assumed to be known, we set them on an ad hoc basis. The interpretation of the parameters is straightforward. For example, $\lambda_l = 0.8^{-1}$ means that the instantaneous probability (or the rate)

Table 1 *Parameters*

	Δ	w^l	w^h	λ_l	λ_h
Simulation 1	1	0	100	0.2^{-1}	0.8^{-1}
Simulation 2	1	0	100	0.8^{-1}	0.2^{-1}

The left panel of Figure 2 displays the histogram of annual labor income resulted from simulation 1.[7] The histogram shows that the distribution of annual labor income is skewed to the left. This is because instantaneous labor income jumps more often from low to high than from high to low. When instantaneous labor income leaves the high state more often than the low state, the realization of annual labor income is more likely to be low. This justifies the right-skewed histogram of annual labor income presented in the right panel of Figure 2. Kuhn and Ríos-Rull (2016) examine the Survey of Consumer Finances 2013 and find the distribution of annual income is indeed skewed to the right (see their Figure 6). The second simulation is therefore empirically relevant. This finding implies that a two-state continuous time Markov process can be used to model an empirically plausible distribution of annual labor income.[8]

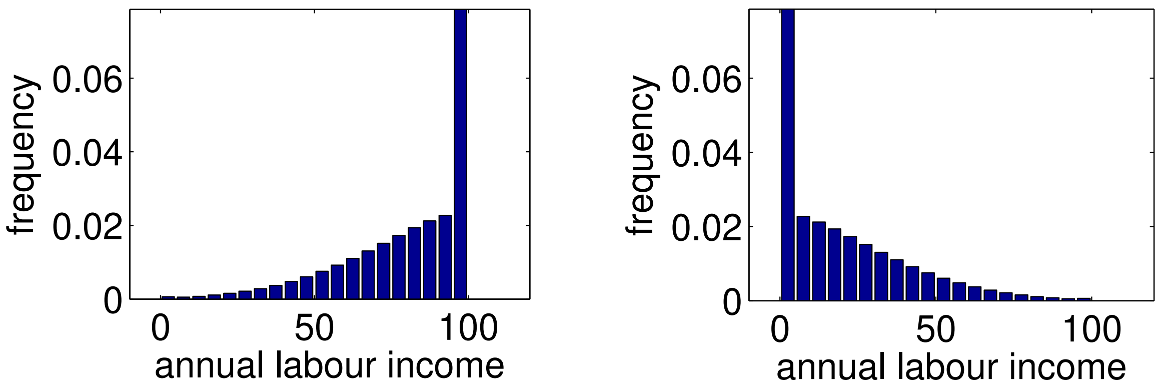

Figure 2 *Histograms of annual labor income in simulation 1 (left panel) and simulation 2 (right panel)*

Table 2 presents the moments of annual labor income generated by the model. Although w^l and w^h are identical in two simulations, the mean of annual labor income is larger in simulation 1 than in simulation 2. This is because $\lambda_l > \lambda_h$ holds in simulation 1 while the opposite holds in simulation 2. This also justifies why the skewness is negative in simulation 1 but turns positive in simulation 2. The variance and the kurtosis are identical in two simulations because of an identical support and the swap of the arrival rates.

We have shown that our algorithm can be used to simulate the distribution of annual labor income given the parameters. When one would like to match the moments of annual labor income in the model with some empirical counterparts, the parameters of the labor income process are to be calibrated. Our algorithm simulating the path integral allows for computing

that an individual jumps from low income to high income is 0.8^{-1}. Analogously, $\lambda_h = 0.2^{-1}$ means that the instantaneous probability that an individual jumps from high income to low income is 0.2^{-1}. These rates are larger than one due to a very small time interval.

[7] The simulation can be easily done using Matlab, Python, or Julia. Coleman et al. (2021) show that these three languages have little effect on performance.

[8] An extension to an N-state Markov process would be interesting, but it may come at the cost of computational burden and therefore efficiency.

the theoretical moments, which are functions of the parameters. The calibration exercise is to minimize the (weighted) sum of the squared differences between the theoretical moments and empirical moments choosing the parameters. As a right-skewed distribution of annual labor income is expected, the constraint $\lambda_h > \lambda_l$ should be taken into account in the calibration. This constitutes a constrained minimization problem, and the calibration exercise can easily be done by solving numerically a nonlinear equation. We leave the calibration exercise for future research because our primary objective is to provide an algorithm rather than calibrate a set of parameters for a specific data set.

Table 2 *Moments of annual labor income in the model*

	mean	*variance*	*skewness*	*kurtosis*
Simulation 1	80.0	430.3	−1.1	3.8
Simulation 2	20.0	430.3	1.1	3.8

6 Concluding remarks and outlook

Calibration of continuous time models is generally challenging since it requires matching the frequency of variables in the models with that in the data. We develop a rigorous algorithm that allows for computing the distributions of annual income and interest rates making use of a path integral. Higher moments can then be computed from the generated distributions and therefore are functions of the fundamental parameters of the continuous time Markov processes. Thus, this algorithm is useful for quantitative research using continuous time Markov processes since it allows for simulation of annual income and interest rates or calibration of the parameters of the Markov processes.

Another approach to calibrating the parameters of continuous time Markov processes is to employ the moment generating function of the path integral.[9] As the moments of the path integral are functions of the parameters, these parameters are chosen such that the moments in the data and model are as close as possible. With the calibrated parameters, our algorithm can be used to generate a histogram and a density of the path integral. Computing the moments of the path integral using the moment generating function is a promising research project.[10]

Disclosure of interest

There are no relevant financial or non-financial competing interests to report.

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[9] Albert and Brown (1991) present an interesting method estimating the arrival rates of Poisson processes using the design of a panel study. They use neither a path integral nor a moment generating function.

[10] Pollet and Stefanov (2002) presented a method computing the first moment of a path integral with random exit time. No attempt was made to compute higher moments. Khieu et al. (2020) offer a formal framework computing higher moments of the path integral using the moment generating function.

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Appendix

A Algorithm for individuals with high initial income

Algorithm 2 ***(Individuals with high initial income)***

1. *Draw the length of time Δ^h that income remains high from time $t = 0$. Δ^h is exponentially distributed with parameter λ_h. There are two possibilities:*

   ⊛ *If $\Delta^h \geq \Delta$, set $\Delta_1^h = \Delta$ and $\Delta_1^l = 0$. The values of total income received during the time intervals Δ_1^l and Δ_1^h are respectively given by*

$$W_1^h = \int_0^{\Delta_1^h} w^h dt, \tag{A.1}$$

$$W_1^l = \int_{\Delta-\Delta_1^l}^{\Delta} w^l dt. \tag{A.2}$$

   Total income received over the time interval $\mathbb{H}$ is therefore given by

$$W(0) = W_1^l + W_1^h. \tag{A.3}$$

   ⊛ *If $\Delta^h < \Delta$, i.e. income jumps from high to low at time $t = \Delta^h$, then draw the length of time Δ^l that income remains low. Δ^l is exponentially distributed with parameter λ_l.*

   - *If $\Delta^l + \Delta^h \geq \Delta$, set $\Delta_1^h = \Delta^h$ and $\Delta_1^l = \Delta - \Delta_1^h$. Total income received over the time interval $\mathbb{H}$ is computed according to (A.3).*
   - *If $\Delta^l + \Delta^h < \Delta$, i.e. income jumps from low to high at time $t = \Delta^l + \Delta^h$, set $\Delta_1^l = \Delta^l$ and $\Delta_1^h = \Delta^h$. Denote $\Delta_1 := \Delta_1^l + \Delta_1^h$ and go to the next step.*

2. *Without loss of generality, assume we are now at step $n \geq 2$. Draw the length of time Δ^h that income remains high from time $t = \Delta_{n-1}$. There are two possibilities:*

   ⊛ *If $\Delta^h + \Delta_{n-1} \geq \Delta$, set $\Delta_n^h = \Delta - \Delta_{n-1}$ and $\Delta_n^l = 0$. The values of total income received during the time intervals Δ_n^l and Δ_n^h are respectively given by*

$$W_n^h = \int_{\Delta_{n-1}}^{\Delta_{n-1}+\Delta_n^h} w^h dt, \tag{A.4}$$

$$W_n^l = \int_{\Delta-\Delta_n^l}^{\Delta} w^l dt \tag{A.5}$$

   Total income received over the time interval $\mathbb{H}$ is therefore given by

$$W(0) = \sum_{k=1}^{n} \left[W_k^l + W_k^h\right]. \tag{A.6}$$

   ⊛ *If $\Delta^h + \Delta_{n-1} < \Delta$, i.e. income jumps from high to low at time $t = \Delta^h + \Delta_{n-1}$, then draw the length of time Δ^l that income remains low.*

- *If* $\Delta_{n-1} + \Delta^l + \Delta^h \geq \Delta$*, set* $\Delta_n^h = \Delta^h$ *and* $\Delta_n^l = \Delta - \Delta^h - \Delta_{n-1}$*. Total income received over the time interval* $\mathbb{H}$ *is computed according to (A.6).*
- *If* $\Delta_{n-1} + \Delta^l + \Delta^h < \Delta$*, i.e. income jumps from low to high again at time* $t = \Delta_{n-1} + \Delta^l + \Delta^h$*, set* $\Delta_n^l = \Delta^l$ *and* $\Delta_n^h = \Delta^h$*. Denote* $\Delta_n := \Delta_{n-1} + \Delta_n^l + \Delta_n^h$ *and proceed to step* $n + 1$*.*