Economics Bulletin

Volume 45, Issue 1

Revisiting the ‘group-size paradox' in the private provision of a pure public good

Tilak Sanyal
Department of Economics, Shibpur Dinobundhoo College, Howrah, India

Abstract

In the presence of multiple varieties of a private good and a single pure public good, which is provided by the voluntary contribution of the individuals, Bag and Mondal (2014) shows that if the private and public goods are gross complement, the total amount of public good increases as the group size increases. But, if they are gross substitute, the relation between the two exhibits an inverted U pattern. In this paper, we assume a single variety of a private good and a pure public good. Our framework differs from that of Bag and Mondal (2014) in two aspects - all the individuals have general quasi- concave utility function instead of CES utility function, and the private good industry is competitive instead of monopolistic competitive. We show that if the two goods are gross substitute, the amount of public good increases with the group size under a unique sufficient condition which requires the labor demand in private good industry to be almost inelastic. But, if the two goods are gross complement, the relation is ambiguous. We also consider some possible extensions of the basic framework and intuitively discuss the relation between group size and level of provision of the public good.

I am grateful to an anonymous Reviewer of the journal for his/her insightful comments and suggestions which have enriched the paper immensely. However, the usual disclaimer applies.
Citation: Tilak Sanyal, (2025) "Revisiting the ‘group-size paradox' in the private provision of a pure public good", *Economics Bulletin*, Volume 45, Issue 1, pages 653-667
Contact: Tilak Sanyal - sanyal.tilak@gmail.com.
Submitted: March 18, 2024. **Published:** March 30, 2025.

1 Introduction

In the situation where a pure public good is provided by the voluntary contribution of the individuals, the conventional wisdom is that an increase in the number of the contributors will increase the amount of public good. But, it was Olson (1965) who first pointed out that free riding escalates in a large group which leads to reduction in the provision of the public good. Clearly, this led to a paradox. However, Chamberlin (1974), McGuire (1974) and Andreoni (1988) partly countered this view by showing that as the number of contributors increases, total amount of public good also increases and approaches a finite upper bound. Thus, a debate emerged as whether the 'group-size paradox' exists or not.

Several other papers have contributed to this debate either by showing the existence of the paradox or by refuting it under different circumstances. For instance, while Esteban and Ray (2001) finds no evidence of the paradox; Pecorino and Temimi (2008), Pecorino (2009) and Mondal (2015) show the prevalence of the paradox.

The purpose of this paper is to analyze theoretically the 'group-size paradox' in a general equilibrium framework which is closest to that of Bag and Mondal (2014), and also examine the robustness of their results. Bag and Mondal (2014) consider a non-empty set of identical individuals, each consuming a pure public good and multiple varieties of a private good. Hence, the private good industry is monopolistic competitive in nature. The public good is provided by the voluntary contribution of the individuals, each having CES utility function. Moreover, each individual inelastically supplies one unit of labor. Using a general equilibrium framework, they show that if the private and public goods are gross complement, the total amount of public good increases with the group size i.e. there is no paradox. But, if they are gross substitute, the relation between total amount of public good and group size follows an inverted U pattern. This means, the paradox exists beyond a threshold level of group size.

Our framework differs from that of Bag and Mondal (2014) in two aspects. *First*, we consider a general quasi-concave utility function of the individuals. *Second*, we consider a single variety of a private good and a competitive private good industry. We find that in general, the relation between group size and amount of public good provided is ambiguous, irrespective of whether the public and private goods are gross substitute or complement. But, if they are substitute, there exists a sufficient condition which ensures that as the group size increases, total amount of the public good also increases. More specifically, the sufficient condition requires the labor demand in private good industry to be almost inelastic. On the other hand, if the public and private goods are complement, there is neither any unique necessary nor sufficient condition that can ensure a monotonic relation between the group size and amount of the public good. Clearly, this result is different from that of Bag and Mondal (2014). We also consider some possible extensions and using the economic intuition of the above result, we find that the relation between the two is, in general, ambiguous.

The rest of the paper is organized as follows. Section 2 presents the basic model. Section 3 provides the comparative static result. Section 4 discusses some extensions of the basic model and section 5 concludes the paper.

2 The Model

Let $\mathcal{L} = \{1, 2, \ldots, L\}$ be a non-empty and finite set of identical individuals. Each individual consumes a private good and voluntarily contributes towards a public good. Both the goods are normal in consumption. Suppose an individual $i \in \mathcal{L}$ consumes $x_i \geq 0$

amount of private good and contributes $g_i \geq 0$ towards the provision of the public good. Denoting $\sum_{i \in \mathcal{L}} g_i$ by G and assuming \$1 of contribution transforms into one unit of public good, G is also the amount of public good provided in the economy. We define the utility function of $i \in \mathcal{L}$ as below:

$$u_i = u(G, x_i) \tag{1}$$

where $u_G, u_x > 0$; $u_{GG}, u_{xx} < 0$ and $u_{Gx} = u_{xG} > 0$.

Each individual inelastically supplies one unit of labor which is used either in the production of the private good or the public good or both. If the wage rate is $w > 0$, then each individual has an income of w. Also, let the market price of private good be $p > 0$. Thus, the total expenditure of $i \in \mathcal{L}$ is $(px_i + g_i)$. Then, the budget constraint of i becomes:

$$g_i + px_i \leq w \tag{2}$$

Let us now set up the demand side of the economy.

2.1 The Demand Side

The individual $i \in \mathcal{L}$ chooses $(g_i, x_i) \in \mathbb{R}^2_+$ to maximize equation 1 subject to equation 2. Since this is a static model, there is no scope for lending or borrowing and hence, the budget constraint holds with equality. Substituting $x_i = \frac{w - g_i}{p}$ from equation 2 into equation 1 we get:

$$u_i = u\left(g_i + \sum_{j \in \mathcal{L}-\{i\}} g_j, \frac{w - g_i}{p}\right) \tag{3}$$

Maximizing equation 3 with respect to $g_i \geq 0$ we obtain the following F.O.C.:

$$u_G\left(g_i + \sum_{j \in \mathcal{L}-\{i\}} g_j, \frac{w - g_i}{p}\right) - \frac{1}{p} u_x\left(g_i + \sum_{j \in \mathcal{L}-\{i\}} g_j, \frac{w - g_i}{p}\right) \leq 0 \tag{4}$$

Assumption 1. (Bergstrom et al (1986)) *$w > w^* = \phi(G^*) - G^*$, where $\phi(.)$ is the inverse of the demand function for public good and G^* is the equilibrium amount of public good provided.*

This assumption follows from Bergstrom et al (1986) which shows that individuals having wealth higher than the critical level w^* will contribute strictly positive amount for the provision of public good. In this paper we assume that all $i \in \mathcal{L}$ have income higher than the critical level and hence, all are contributors in equilibrium. Thus, equation 4 holds with equality.

Assumption 2. $w \geq 1$

This assumption is trivial and will be required later for the derivation of results.

Assumption 1 ensures that in equilibrium $g_i > 0$ for all $i \in \mathcal{L}$. Again, as all the individuals are identical, we have a symmetric equilibrium where $g_i = \frac{G}{L}$. Thus, replacing g_i by $\frac{G}{L}$ and $\left(g_i + \sum_{j \in \mathcal{L}-\{i\}} g_j\right)$ by G, we rewrite equation 4 as below:

$$u_G\left(G, \frac{w - \frac{G}{L}}{p}\right) - \frac{1}{p} u_x\left(G, \frac{w - \frac{G}{L}}{p}\right) = 0 \tag{5}$$

From equation 5 we obtain the demand function for the public good as:

$$G^D = G\left(w, p, L\right) \tag{6}$$

From equation 5 we have the following lemma.

Lemma 1. *The public good and the private good are gross substitute (complement) if and only if* $px_i\left(u_{Gx} - \frac{u_{xx}}{p}\right) < (>)\, u_x$.

Proof. See the Appendix.

Lemma 1 defines some restrictions on the utility function for which the private good and the public good become gross substitute (complement). We can interpret the conditions in the following way. Suppose $u_{Gx} \approx 0$. This means, a change in consumption of the private good marginally affects the valuation of the public good. Now, a fall in the price of the private good will increase its demand. But as the valuation of public good remains *almost* unchanged, the individual can reduce the consumption of public good to increase the consumption of private good. In this case, the two goods are gross substitute. Alternatively, assume that the value of u_{Gx} is substantially high. Then, a change in consumption of private good changes the valuation of public good by a large amount. Hence, if we consider a fall in the price of private good, increase in its demand increases the valuation for public good as well (since, $u_{Gx} > 0$). This induces the individual to increase the demand for public good. In this case, the private good is complementary to the public good.

From equation 5 we have another lemma.

Lemma 2. $\frac{dG}{dw} = \frac{\frac{1}{p}\left(\frac{u_{xx}}{p} - u_{Gx}\right)}{u_{GG} - \frac{u_{Gx}}{pL} - \frac{u_{Gx}}{p} + \frac{u_{xx}}{p^2 L}} > 0$

Proof. See the Appendix.

It is very easy to understand the intuition of lemma 2. As the public good is a normal good having positive income effect, an increase in wage income of the individual raises the demand for the public good.

Now, we derive the demand function of the private good. From equation 2 we have $x_i = \frac{w - g_i}{p}$. If we call X^D as the aggregate demand for private good, in symmetric equilibrium we can write $x_i = \frac{X^D}{L}$. We have already seen that in equilibrium, $g_i = \frac{G(w,p,L)}{L}$. Using these two we have the following:

$$X^D = \frac{wL - G(w,p,L)}{p} = X(w,p,L) \tag{7}$$

In the next section, we set up the supply side of the economy.

2.2 The Supply Side

Let $\mathcal{N} = \{1, 2, \ldots, N\}$ be a non-empty and finite set of identical firms competitively producing the private consumption good. We assume that labor is the only input. The production technology of each firm is given by the production function $x = f(l)$, where $f'(l) > 0$ and $f''(l) < 0$. Suppose firm $k \in \mathcal{N}$ employs l_k amount of labor to produce x_k amount of private good. Thus, its profit function becomes:

$$\pi_k = pf(l_k) - wl_k$$

The firm maximizes the above profit function by choosing $l_k > 0$. It gives the following F.O.C.:

$$pf'(l_k) - w = 0$$

Let L_x be the total amount of labor employed by all the firms. As the firms are identical, in symmetric equilibrium we must have $l_k = \frac{L_x}{N}$. Using this in the above equation, we obtain:

$$pf'\left(\frac{L_x}{N}\right) - w = 0 \tag{8}$$

From equation 8 we derive the aggregate labor demand function for the private good industry as:

$$L_x^D = L_x(w, p, N) \tag{9}$$

Let X^S denote the aggregate amount of private good produced in the economy. Then, using the production function, we write the aggregate supply function of the private good as:

$$X^S = f(L_x^D) = f(L_x(w, p, N)) \tag{10}$$

From equation 8 we have the following lemma.

Lemma 3. *(i)* $\frac{dL_x}{dw} = \frac{N}{pf''} < 0$
(ii) $\frac{dL_x}{dp} = -\frac{f'N}{pf''} > 0$

Proof. See the Appendix.

The interpretation of lemma 3 is very simple. Part (i) shows that the labor demand function in the private good industry is standard negatively sloped. Part (ii) shows that as the price of private good increases, the real wage paid to the workers decline, resulting in an increase in labor demand.

Following Bag and Mondal (2014), we assume that one unit of labor is used to produce one unit of the public good. Thus, total amount of labor demanded for public good production is obtained from equation 6 as:

$$L_G^D = G(w, p, L) \tag{11}$$

After describing the demand and supply sides of the economy, we proceed to compute the general equilibrium in the next section.

2.3 The General Equilibrium

From equations 7 and 10 we describe the equilibrium in the private good market as:

$$X(w, p, L) = f(L_x(w, p, N)) \tag{12}$$

Solving equation 12 we obtain:

$$p = p(w, N, L) \tag{13}$$

Using equations 9 and 11, the full employment condition in the labor market requires:

$$L = L_x(w, p, N) + G(w, p, L)$$

Substituting $p = p(w, N, L)$ from equation 13 in the above equation we get:

$$L = L_x(w, p(w, N, L), N) + G(w, p(w, N, L), L) \tag{14}$$

Solving equation 14 we get the equilibrium value of wage rate. We call it w^* and it is the following:

$$w^* = w^*(N, L) \tag{15}$$

Substituting this equilibrium value of wage rate in equation 13 we obtain the equilibrium value of price of the private good as:

$$p^* = p(w^*(N, L), N, L) = p^*(N, L) \tag{16}$$

Substituting the equilibrium values of wage rate and price of private good from equations 15 and 16 respectively into equations 6 and 7, we get the respective equilibrium amount of public good and private good produced in the economy as:

$$\begin{aligned} G^* &= G(w^*(N, L), p^*(N, L), L) = G^*(N, L) \qquad (17)\\ X^* &= X(w^*(N, L), p^*(N, L), L) = X^*(N, L) \end{aligned}$$

Making similar substitution in equations 9 and 11 we obtain the equilibrium amount of labor employed for production of private and public goods respectively as:

$$L_x^* = L_x(w^*(N, L), p^*(N, L), N) = L_x^*(N, L) \tag{18}$$

$$L_G^* = G(w^*(N, L), p^*(N, L), L) = G^*(N, L) \tag{19}$$

In the following section we present the comparative static result to see the impact of increase in group size on the level of provision of the public good.

3 Comparative Static

Suppose the number of individuals in the set $\mathcal{L}$ increases. This means we have $dL > 0$. Substituting the values of w^*, p^* and G^* from equations 15, 16 and 17 respectively into equation 5 and then differentiating both sides of it with respect to L we get:

$$\begin{aligned} &u_{GG}\frac{dG^*}{dL} + u_{Gx}\left[\frac{1}{p^*}\left(-\frac{1}{L}\frac{dG^*}{dL} + \frac{G^*}{L^2}\right) + \frac{1}{p^*}\frac{dw^*}{dL} - \frac{\left(w^* - \frac{G^*}{L}\right)}{p^{*2}}\frac{dp^*}{dL}\right] + \frac{u_x}{p^{*2}}\frac{dp^*}{dL}\\ &-\frac{1}{p^*}\left[u_{Gx}\frac{dG^*}{dL} + u_{xx}\left(\frac{1}{p^*}\left(-\frac{1}{L}\frac{dG^*}{dL} + \frac{G^*}{L^2}\right) + \frac{1}{p^*}\frac{dw^*}{dL} - \frac{\left(w^* - \frac{G^*}{L}\right)}{p^{*2}}\frac{dp^*}{dL}\right)\right] = 0 \end{aligned}$$

Collecting the coefficients of $\frac{dG^*}{dL}$, $\frac{dp^*}{dL}$ and $\frac{dw^*}{dL}$ and rearranging them on the L.H.S. of the equation we obtain:

$$\frac{dG^*}{dL} - \frac{\frac{u_{Gx}\left(w^* - \frac{G^*}{L}\right)}{p^{*2}} - \frac{u_x}{p^{*2}} - \frac{u_{xx}\left(w^* - \frac{G^*}{L}\right)}{p^{*3}}}{u_{GG} - \frac{u_{Gx}}{p^*L} - \frac{u_{Gx}}{p^*} + \frac{u_{xx}}{p^{*2}L}}\frac{dp^*}{dL} - \frac{\frac{u_{xx}}{p^{*2}} - \frac{u_{Gx}}{p^*}}{u_{GG} - \frac{u_{Gx}}{p^*L} - \frac{u_{Gx}}{p^*} + \frac{u_{xx}}{p^{*2}L}}\frac{dw^*}{dL}$$

$$= \frac{\frac{G^*}{L^2}\left(\frac{u_{xx}}{p^{*2}} - \frac{u_{Gx}}{p^*}\right)}{u_{GG} - \frac{u_{Gx}}{p^*L} - \frac{u_{Gx}}{p^*} + \frac{u_{xx}}{p^{*2}L}}$$

Observe from the proof of lemma 1, if we evaluate $\frac{dG}{dp}$ in equilibrium and substitute $x_i^* = \frac{w^* - \frac{G^*}{L}}{p^*}$, it becomes the coefficient of $\frac{dp^*}{dL}$ in the above equation. Also, it follows from lemma 2 that in equilibrium, the coefficient of $\frac{dw^*}{dL}$ is $\frac{dG}{dw}$. The expression on the R.H.S. can also be written as $\frac{G^*}{L^2}\frac{dG}{dw}$. Thus, rewriting the above equation using these facts yields the following:

$$\frac{dG^*}{dL} - \frac{dG}{dp}\frac{dp^*}{dL} - \frac{dG}{dw}\frac{dw^*}{dL} = \frac{G^*}{L^2}\frac{dG}{dw} \tag{20}$$

Now, we consider the private good market equilibrium as given in equation 12. We use equation 7 to rewrite the L.H.S. of equation 12 and get:

$$\frac{w^*L - G(w^*, p^*, L)}{p^*} = f(L_x(w^*, p^*, N))$$

We differentiate both sides of the above equation with respect to L and obtain:[1]

$$\frac{1}{p^*}\left[L\frac{dw^*}{dL} + w^* - \frac{dG}{dw}\frac{dw^*}{dL} - \frac{dG}{dp}\frac{dp^*}{dL} - \frac{\partial G}{\partial L}\right] - \frac{w^*L - G^*}{p^{*2}}\frac{dp^*}{dL} = f'\left[\frac{dL_x}{dw}\frac{dw^*}{dL} + \frac{dL_x}{dp}\frac{dp^*}{dL}\right]$$

Using equation 7, we replace $\frac{w^*L - G^*}{p^*}$ in the above equation by X^*. Also, as $\frac{\partial G}{\partial L}$ is derived from the equilibrium value of G i.e. G^*, therefore, abusing notation once again, we replace it with $\frac{dG^*}{dL}$. Now, rearranging the terms gives the following:

$$-\frac{1}{p^*}\frac{dG^*}{dL} - \left[\frac{1}{p^*}\frac{dG}{dp} + \frac{X^*}{p^*} + f'\frac{dL_x}{dp}\right]\frac{dp^*}{dL} + \left[\frac{L}{p^*} - \frac{1}{p^*}\frac{dG}{dw} - f'\frac{dL_x}{dw}\right]\frac{dw^*}{dL} = -\frac{w^*}{p^*} \tag{21}$$

Finally, we consider the labor market equilibrium as given in equation 14 and differentiate both sides of it with respect to L and obtain:

$$1 = \frac{dL_x}{dw}\frac{dw^*}{dL} + \frac{dL_x}{dp}\frac{dp^*}{dL} + \frac{dG}{dw}\frac{dw^*}{dL} + \frac{dG}{dp}\frac{dp^*}{dL} + \frac{\partial G}{\partial L}$$

Once again, replacing $\frac{\partial G}{\partial L}$ by $\frac{dG^*}{dL}$, we rewrite the above equation as:

$$\frac{dG^*}{dL} + \left[\frac{dL_x}{dp} + \frac{dG}{dp}\right]\frac{dp^*}{dL} + \left[\frac{dL_x}{dw} + \frac{dG}{dw}\right]\frac{dw^*}{dL} = 1 \tag{22}$$

From equations 20, 21 and 22 we get the following proposition.

Proposition 1. *When the public good and the private good are gross substitute, an increase in group size leads to higher amount of the public good under a unique sufficient condition that the labor demand in the private good industry is almost inelastic. But, if they are gross complement, there is neither any unique necessary nor sufficient condition which guarantees increased provision of public good due to an increase in group size. In general, the relation between group size and the level of public good provision is ambiguous, irrespective of whether the two goods are substitute or complement.*

Proof. See the Appendix.

Let us now try to understand the intuition of proposition 1. We use the concepts of

[1] With slight abuse of notation, we use $\frac{dG}{dw}$, $\frac{dG}{dp}$, $\frac{dL_x}{dw}$ and $\frac{dL_x}{dp}$ instead of $\frac{\partial G}{\partial w}$, $\frac{\partial G}{\partial p}$, $\frac{\partial L_x}{\partial w}$ and $\frac{\partial L_x}{\partial p}$ respectively. We follow this throughout the rest of the paper.

substitution effect and income effect to explain the result. Below, Figure 1 and Figure 2 respectively show how these effects interplay to yield the result when the public and the private goods are gross substitute and complement. For simplicity, we have not drawn the indifference curves. Rather, we show only the equilibrium points, i.e. the point of tangency between the indifference curve and the budget line.

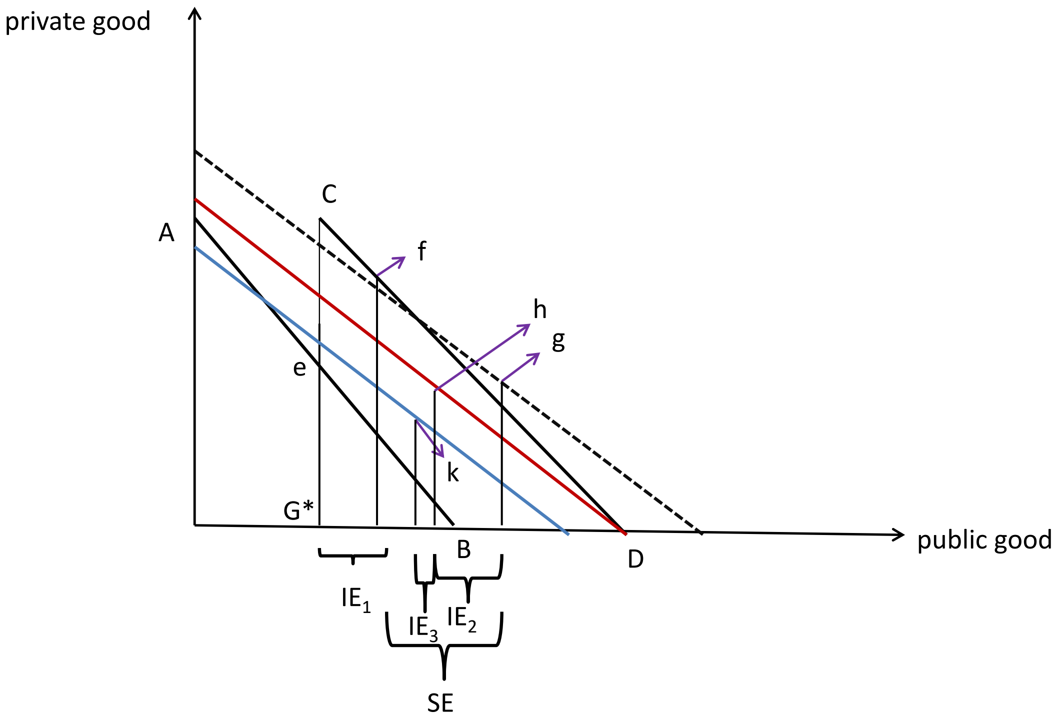

Figure 1: Public and Private Goods are Gross Substitute

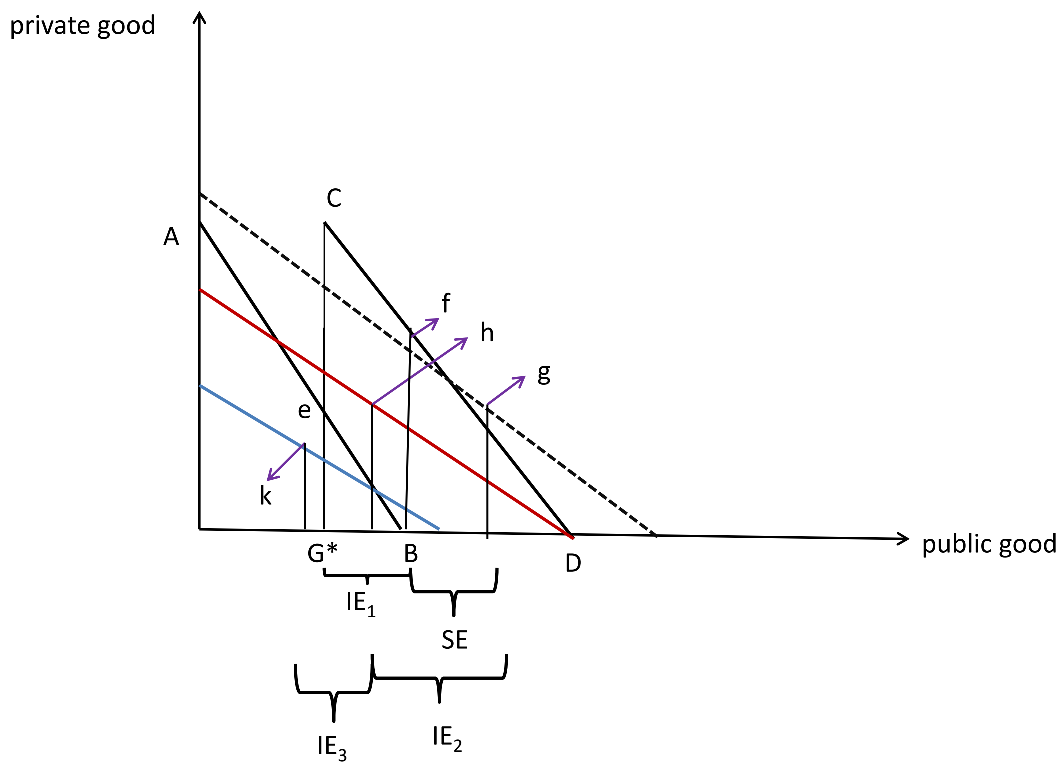

Figure 2: Public and Private Goods are Gross Complement

As the group size increases, each newly added individual consumes G^* amount of public good, even if she spends her entire income on the private consumption good.[2] This means, her budget line shifts to the right in the parallel direction, provided the consumption amount of public good is measured along the horizontal axis. In both Figure 1 and Figure 2, the initial budget line is AB. With the increase in group size, the budget line shifts to CD. As both private and public goods are normal in consumption, this increase in effective income increases the demand for both the goods. We denote this income effect by IE_1. This is represented as the movement from the point e on AB to point f on CD in both the figures. Now, increase in demand of the private good raises its price, which induces a pivotal movement of the budget line.[3] In both the figures, this is shown as the red colored line. This price rise will affect both the product and the labor markets which we discuss below.

In the product market, increase in price of the private good generates the standard substitution and income effects. We write them as SE and IE_2 respectively. In both the figures, we draw a broken line (parallel to the red colored line) to decompose the price effect into SE and IE_2. SE is shown as the movement from point f to point g, and IE_2 is shown as the migration from g to h. SE reduces the demand for the private good

[2]This is because the public good is non-excludable and non-rival in consumption.

[3]Observe that equation 2 can be written as $g_i + \sum_{j\in\mathcal{L}-\{i\}} g_j + px_i \leq w + \sum_{j\in\mathcal{L}-\{i\}} g_j$. This, further, means $G + px_i \leq w + \sum_{j\in\mathcal{L}-\{i\}} g_j$. Hence, with the amount of public good on the horizontal axis and the amount of private good on the vertical axis, the absolute slope of the budget line becomes $\frac{1}{p}$. Thus, an increase in p will make the budget line flatter with reduced vertical intercept and the horizontal intercept remaining unchanged.

and, in turn, increases the demand for the public good. As real income falls and both the goods are normal in consumption, IE_2 reduces the demand for both the goods. Note that both the effects reduce the demand for the private good but, while SE increases the demand for public good, IE_2 reduces it. Now, if the private good and public good are gross substitute, SE must dominate IE_2 and hence, the demand for public good increases (see Figure 1). But, if they are gross complement, IE_2 dominates SE and the demand for public good decreases (see Figure 2).

In the labor market, increase in price of the private good reduces the real wage rate offered by the firms and hence, they increase labor demand. If the labor demand is *almost* inelastic, the private good industry will not be able to absorb all the newly added individuals, leading to a temporary unemployment situation. This will reduce the wage rate and hence wage income, causing the demand for public good to fall. We denote this income effect by IE_3. Due to fall in the wage income, the budget line shifts parallel in the downward direction which is shown by the blue colored line in both the figures. IE_3 is depicted as the migration from point h to point k.

Observe from Figure 1, if the public and private goods are gross substitute, IE_1 and SE together dominate the combined effect of IE_2 and IE_3. This leads to an increase in the provision of public good. Moreover, due to this, the excess labor in the private good sector migrates to produce the increased amount of the public good, and the full employment is restored.

On the other hand, if the public and private goods are gross complement, although IE_1 increases the demand for the public good, IE_2 dominates SE to reduce it. This makes the impact of increase in group size on the amount of public good provision ambiguous, irrespective of the magnitude of IE_3. Figure 2 shows a situation where the magnitude of IE_2 is slightly greater than that of SE. Consequently, the point h lies to the north-east of the point e, denoting a higher amount of public good at h relative to e. But, if the magnitude of IE_2 is sufficiently higher than that of SE, then h can be to the south-west of e, representing a lower amount of the public good. Thus, the size of IE_2 relative to that of SE is sufficient to generate ambiguity in the amount of public good, even without taking IE_3 into consideration.

Bag and Mondal (2014) assume multiple varieties of a single private good. Hence, the private good producing industry is monopolistic competitive in nature. In this set up, they show that if the private good and public good are gross complement, an increase in group size leads to higher amount of provision of the public good. Whereas, if they are gross substitute, the relation between group size and the level of public good provision is inverted U shaped. Clearly, this result is different from the one stated in our proposition 1. Hence, it is important to understand the reason behind this divergence of result.

In Bag and Mondal (2014), an increase in group size leads to an income effect which is identical to our IE_1. This increases the demand for the public good. Moreover, as group size increases, more varieties are produced resulting in a fall in the price of the composite private good. The standard substitution and income effects raise the demand for private good. In this situation, if the public good and private good are gross complement, the demand for public good also increases. Thus, in terms of our model, the combined effects of IE_1, SE and IE_2 increase the demand for the public good. This leads to a positive monotonic relation between group size and level of provision of the public good. In contrast, if the public good and private good are gross substitute, although IE_1 increases the demand for the public good, the combined effects of SE and IE_2 reduce it, leading to a non-monotonic relation.

From the above discussion it is clear that the difference in result arises from the fact that while in Bag and Mondal (2014), price of the private good reduces, it increases in our model. This stems from the difference in nature of private good industry. For this, while they obtained a positive monotonic relation when the two goods are gross complement, we obtained it in the situation where the two goods are gross substitute but, of course, under a sufficient condition. While they found non-monotonic relation in case of gross substitute, we find ambiguity in case of gross complement. Nevertheless, it is clear that the result of Bag and Mondal (2014) is not robust, at least in the context of market structure producing the private good.

Moreover, there are some issues in Bag and Mondal (2014) which are worth mentioning. *First*, the general equilibrium model which they develop is for short run period. They assume that in equilibrium, each firm producing a particular variety of the private good is earning normal profit. This, in fact, allows them to determine the optimal number of varieties in the market. Now, the standard microeconomic theory shows that in short run equilibrium, a monopolistic competitive firm can earn supernormal profit. It earns normal profit only in the long run equilibrium. Thus, their assumption can only be a special case. *Second*, their modeling technique allows the determination of equilibrium wage rate as unity. Consequently, it leaves out an additional income effect which is IE_3 in our model.

In the following section, we consider some possible extensions of the basic model and use the intuition of proposition 1 to discuss the relation between the group size and the amount of the public good(s).

4 Some Discussions

4.1 Homogeneous Preference and Multiple Public Goods

Let there be two distinct pure public goods which are labeled as G_1 and G_2. First, assume that each public good is gross substitute with the private good. Then, it is easy to see that an increase in group size generates the same substitution and income effects i.e. SE and IE_1, IE_2 and IE_3. Hence, under the same sufficient condition as mentioned in proposition 1 i.e. labor demand in the private good industry is *almost* inelastic, the amount of both the public goods increases.

Now, without loss of generality, suppose G_1 is gross substitute and G_2 is gross complement, both in relation to the private good. Also, we retain the condition that labor demand in private good industry is almost inelastic. In this case, as group size increases, IE_1 increases the demand for both G_1, G_2 and the private good. Consequently, price of the private good increases. Then, SE and IE_2 reduce the demand for it. Due to the relationship with the private good, the demand for G_1 increases while that of G_2 decreases. As labor demand in private good industry is marginally elastic, it fails to absorb all the new group members, thus creating an unemployment. Meanwhile, due to the fall in demand for G_2, some labor are also released from its production. This adds to the pool of unemployment. As a result, the wage rate falls which generates IE_3. But, this income effect is relatively stronger than the corresponding IE_3 of section 3. There is a possibility that the fall in demand for G_1 due to IE_3 is more than the increase in its demand due to the combined effects of SE, IE_1 and IE_2. In such case, the amount of G_1

falls. Thus, the *sufficient condition* mentioned in proposition 1 may not be sufficient to establish a positive monotonic relation between the group size and the level of provision of G_1 (despite being gross substitute with the private good), if there is at least one more public good which is gross complement with the private good.

Finally, let us assume that both G_1 and G_2 are gross complement with the private good. It can be easily verified that here IE_1 will increase the demand for both the public goods, but SE, IE_2 and IE_3 will combine to reduce it. Thus, the relation between the group size and the amount of each public good becomes ambiguous. Moreover, due to the facts that the labor demand in private good industry is almost inelastic and labor is released from the production of both the public goods, it is possible to have unemployment in equilibrium.

4.2 Heterogeneous Preference and Single Public Good

We divide the given set of individuals into two non-empty partitions and label them as $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L}_1, \mathcal{L}_2 \neq \Phi$, $\mathcal{L}_1 \cup \mathcal{L}_2 = \mathcal{L}$ and $\mathcal{L}_1 \cap \mathcal{L}_2 = \Phi$. Without loss of generality, assume that for individuals in $\mathcal{L}_1$, the public good and the private good are gross substitute, while they are gross complement for individuals in $\mathcal{L}_2$. Suppose the size of $\mathcal{L}_1$ increases. As usual, IE_1 increases the demand for both the goods within $\mathcal{L}_1$. The price of private good rises. This generates SE and IE_2 which not only affects the consumption pattern of the individuals of $\mathcal{L}_1$, but also of $\mathcal{L}_2$. As the demand of private good falls due to these effects, the demand for the public good increases in $\mathcal{L}_1$ while it decreases in $\mathcal{L}_2$. Thus, the net effect on the demand for public good is ambiguous. Clearly, it will remain so even if we incorporate IE_3. Hence, the *sufficient condition* stated in proposition 1 is not sufficient to establish a direct relation between the amount of public good and the group size even if the group considers the public and private goods as gross substitute, provided there is at least one group which considers the same two goods as gross complement.

4.3 Heterogeneous Preference and Multiple Public Goods

We borrow the notations from the sections 4.1 and 4.2 and assume that G_1 and G_2 are the two public goods and $\mathcal{L}_1$ and $\mathcal{L}_2$ are the two non-empty partitions of the set of individuals. Further, let the individuals of $\mathcal{L}_1$ consider G_1 (G_2) and the private good as gross substitute (complement). The converse is assumed for the individuals of $\mathcal{L}_2$. Irrespective of the group whose size increases, IE_1 will increase the demand for both the public goods. But, SE and IE_2 will combine to increase the demand for each public good in one group (where the public and private goods are gross substitute) while reducing it in another group (where the two goods are gross complement). Hence, it is easy to see that the impact of increase in group size on the amount of each public good becomes ambiguous.

The purpose of section 4 is to see whether the result obtained in proposition 1 holds in situations which are extensions of the basic model. From the above discussions we can conclude that, in general, the relation between the group size and the amount of public good is ambiguous, irrespective of whether the public good is gross substitute or complement with the private good.

5 Conclusion

In this paper we assume a finite and non-empty set of identical individuals, each consuming a private good and a public good, which is provided by the voluntary contribution of all the individuals. The private good industry is perfectly competitive. Also, each individual has a general quasi-concave utility function. We explore the relation between the group size and the amount of public good provided using a general equilibrium framework.

We find that if the public and private goods are gross substitute and the labor demand in private good industry is almost inelastic, there is a positive monotonic relation between the two. Hence, the group size paradox does not exist. But, if the two goods are gross complement, the relation becomes ambiguous. This result is different from what Bag and Mondal (2014) obtained assuming a CES utility function and multiple varieties of a single public good. In the later part, we consider some possible extensions of the basic model and, after intuitive discussion, we find that in general, the relation between the group size and the level of public good is ambiguous.

References

Andreoni, J.R. (1988). Privately provided public goods in a large economy: The limits of altruism, J. Public Econ., 35, 57-73.

Bag, P.K. and D. Mondal (2014). Group size paradox and public goods, Economics Letters, 125(2), 215-218.

Bergstrom, T., L. Blume and H. Varian (1986). On the private provision of public goods, J. Public Econ., 29, 25-49.

Chamberlin, J. (1974). Provision of collective goods as a function of group size, Amer. Polit. Science Rev., 68, 707-716.

Esteban, J. and D. Ray (2001). Collective action and the group size paradox, Amer. Polit. Science Rev., 95(3), 663-672.

McGuire, M.C. (1974). Group size, group homogeneity, and the aggregate provision of a pure public good under Cournot behavior, Public Choice, 18, 107-126.

Mondal, D. (2015). Private provision of public good and immiserizing growth, Soc. Choice Welf., 45, 29-49.

Olson, M. (1965). The logic of collective action, Cambridge MA, Harvard University Press.

Pecorino, P. and A. Temimi (2008). The group size paradox revisited, J. Public Econ. Th., 10(5), 785-799.

Pecorino, P. (2009). Monopolistic competition, growth and public good provision, Econ. Journal, 119, 298-307.

Appendix

Proof of Lemma 1. We totally differentiate both sides of equation 5 and assume $dw = dL = 0$ to obtain:

$$u_{GG}dG + u_{Gx}\left(-\frac{1}{pL}dG - \frac{w-\frac{G}{L}}{p^2}dp\right) + \frac{u_x}{p^2}dp - \frac{1}{p}\left[u_{Gx}dG + u_{xx}\left(-\frac{1}{pL}dG - \frac{w-\frac{G}{L}}{p^2}dp\right)\right] = 0$$

Substituting $\frac{w-\frac{G}{L}}{p} = \frac{w-g_i}{p} = x_i$ in the above equation and after rearranging the terms we get:

$$\frac{dG}{dp} = \frac{u_{Gx}\frac{x_i}{p} - \frac{u_x}{p^2} - u_{xx}\frac{x_i}{p^2}}{u_{GG} - \frac{u_{Gx}}{pL} - \frac{u_{Gx}}{p} + \frac{u_{xx}}{p^2 L}}$$

Given the assumptions on the utility function, observe that the denominator of the above equation is negative. But, the sign of the numerator is indefinite. Hence, we have the following two possibilities:

i If $\frac{x_i}{p}\left(u_{Gx} - \frac{u_{xx}}{p}\right) < \frac{u_x}{p^2}$ or $px_i\left(u_{Gx} - \frac{u_{xx}}{p}\right) < u_x$, then the numerator becomes negative and we have $\frac{dG}{dp} > 0$. This implies that the private good and the public good are gross substitute.

ii If $px_i\left(u_{Gx} - \frac{u_{xx}}{p}\right) > u_x$, then the numerator becomes positive and we have $\frac{dG}{dp} < 0$. This implies that the private good and the public good are gross complement.

This completes the proof of the lemma.

Proof of Lemma 2. We totally differentiate both sides of equation 5 and assume $dL = dp = 0$ to obtain:

$$u_{GG}dG + u_{Gx}\left(-\frac{1}{pL}dG + \frac{1}{p}dw\right) - \frac{1}{p}\left[u_{Gx}dG + u_{xx}\left(-\frac{1}{pL}dG + \frac{1}{p}dw\right)\right] = 0$$

After rearranging the terms we get:

$$\frac{dG}{dw} = \frac{\frac{1}{p}\left(\frac{u_{xx}}{p} - u_{Gx}\right)}{u_{GG} - \frac{u_{Gx}}{pL} - \frac{u_{Gx}}{p} + \frac{u_{xx}}{p^2 L}}$$

Given the assumptions on the utility function, it is clear that both the numerator and the denominator of the above equation are negative. Hence, we have $\frac{dG}{dw} > 0$.
This completes the proof of the lemma.

Proof of Lemma 3. *Part (i).* On total differentiation of both sides of equation 8 and assuming $dN = 0$ we obtain:

$$f'dp + \frac{pf''}{N}dL_x - dw = 0$$

Assuming $dp = 0$ we have:

$$\frac{dL_x}{dw} = \frac{N}{pf''}$$

Since $p, N > 0$ and $f'' < 0$, it is clear that $\frac{dL_x}{dw} < 0$.
Part (ii). Assuming $dw = 0$ we get:

$$\frac{dL_x}{dp} = -\frac{f'N}{pf''}$$

Since $p, N, f' > 0$ and $f'' < 0$, we have $\frac{dL_x}{dp} > 0$.
This completes the proof of the lemma.
Proof of Proposition 1. We solve for $\frac{dG^*}{dL}$ from equations 20, 21 and 22 using the Cramer's rule. For this, we define the following two determinants.

$$D = \begin{vmatrix} 1 & -\frac{dG}{dp} & -\frac{dG}{dw} \\ -\frac{1}{p^*} & -\frac{1}{p^*}\frac{dG}{dp} - \frac{X^*}{p^*} - f'\frac{dL_x}{dp} & \frac{L}{p^*} - \frac{1}{p^*}\frac{dG}{dw} - f'\frac{dL_x}{dw} \\ 1 & \frac{dL_x}{dp} + \frac{dG}{dp} & \frac{dL_x}{dw} + \frac{dG}{dw} \end{vmatrix}$$

Evaluating the above determinant gives:

$$D = 2\left(f' - \frac{1}{p^*}\right)\left(\frac{dG}{dp}\frac{dL_x}{dw} - \frac{dG}{dw}\frac{dL_x}{dp}\right) - \frac{1}{p^*}\left(X^*\frac{dL_x}{dw} + 2X^*\frac{dG}{dw} + L\frac{dL_x}{dp} + 2L\frac{dG}{dp}\right)$$

From equation 8 we have $f' = \frac{w^*}{p^*}$ in equilibrium. Substituting this in the above equation we get:

$$D = 2\left(\frac{w^* - 1}{p^*}\right)\left(\frac{dG}{dp}\frac{dL_x}{dw} - \frac{dG}{dw}\frac{dL_x}{dp}\right) - \frac{1}{p^*}\left(X^*\frac{dL_x}{dw} + 2X^*\frac{dG}{dw} + L\frac{dL_x}{dp} + 2L\frac{dG}{dp}\right)$$

We define another determinant D_L by replacing the first column of the coefficient matrix with the constant vector and get:

$$D_L = \begin{vmatrix} \frac{G^*}{L^2}\frac{dG}{dw} & -\frac{dG}{dp} & -\frac{dG}{dw} \\ -\frac{w^*}{p^*} & -\frac{1}{p^*}\frac{dG}{dp} - \frac{X^*}{p^*} - f'\frac{dL_x}{dp} & \frac{L}{p^*} - \frac{1}{p^*}\frac{dG}{dw} - f'\frac{dL_x}{dw} \\ 1 & \frac{dL_x}{dp} + \frac{dG}{dp} & \frac{dL_x}{dw} + \frac{dG}{dw} \end{vmatrix}$$

Expanding the above determinant and after substituting $\frac{w^*}{p^*} = f'$ from equation 8 we obtain the following:

$$D_L = \frac{G^*}{L^2}\frac{dG}{dw}\left(\frac{dL_x}{dw}\frac{dG}{dp} - \frac{dL_x}{dp}\frac{dG}{dw}\right)\left(f' - \frac{1}{p^*}\right)$$
$$-\frac{1}{p^*}\left[\frac{G^*X^*}{L^2}\frac{dL_x}{dw}\frac{dG}{dw} + \frac{G^*X^*}{L^2}\left(\frac{dG}{dw}\right)^2 + \frac{G^*}{L}\frac{dG}{dw}\frac{dL_x}{dp} + \frac{G^*}{L}\frac{dG}{dw}\frac{dG}{dp} + L\frac{dG}{dp} + X^*\frac{dG}{dw}\right]$$

Replacing f' by $\frac{w^*}{p^*}$ from equation 8 we finally obtain:

$$D_L = \frac{G^*}{L^2}\frac{dG}{dw}\left(\frac{dL_x}{dw}\frac{dG}{dp} - \frac{dL_x}{dp}\frac{dG}{dw}\right)\left(\frac{w^* - 1}{p^*}\right)$$
$$-\frac{1}{p^*}\left[\frac{G^*X^*}{L^2}\frac{dL_x}{dw}\frac{dG}{dw} + \frac{G^*X^*}{L^2}\left(\frac{dG}{dw}\right)^2 + \frac{G^*}{L}\frac{dG}{dw}\frac{dL_x}{dp} + \frac{G^*}{L}\frac{dG}{dw}\frac{dG}{dp} + L\frac{dG}{dp} + X^*\frac{dG}{dw}\right]$$

Therefore, we have:

$$\frac{dG^*}{dL} = \frac{D_L}{D}$$

From assumption 2 we have $w^* \geq 1$. Also, lemma 2 shows that $\frac{dG}{dw} > 0$ and lemma 3 shows that $\frac{dL_x}{dw} < 0$ and $\frac{dL_x}{dp} > 0$. Using these facts, the sign of both D_L and D becomes indefinite, irrespective of whether the public good and the private good are gross substitute or complement. Hence, the sign of $\frac{dG^*}{dL}$ is also indefinite.
Now, define the elasticity of labor demand in private good industry as $\epsilon_{Lx} = \frac{dL_x}{dw}\frac{w}{L_x}$, then if $\frac{dL_x}{dw} \approx 0$, it is sufficient to ensure that $\epsilon_{Lx} \approx 0$. Clearly, this condition alone sufficiently makes both $D_L < 0$ and $D < 0$, provided $\frac{dG}{dp} > 0$ i.e. the public and private goods are gross substitute. In such case we have $\frac{dG^*}{dL} > 0$.
On the other hand, it is easy to see that if the two goods are gross complement i.e. $\frac{dG}{dp} < 0$, there is neither any unique necessary nor sufficient condition that can produce $\frac{dG^*}{dL} > 0$.
This completes the proof of the proposition.