Economics Bulletin

Appendix and Supplemental material not intended for publication-Round 1

Submission Number:EB-24-00446

Submitted: Oct 16 2024. **Revised:** October 22, 2024.

Supplementary Material:
Exploring Transitional and Asymptotic Impacts of Subsistence Consumption on Wealth Inequality in an AK Growth Model

October 23, 2024

Abstract

This supplementary material demonstrates some of the statements made in the paper titled "Exploring Transitional and Asymptotic Impacts of Subsistence Consumption on Wealth Inequality in an AK Growth Model." This document is for online access and not intended for publication.

Contents

Appendix A: Derivations of (7) and (8)

We begin by characterising an individual consumption path c_i. From (3) and (6), we obtain the following equation:

$$\dot{c}_i - \epsilon_i(A-\rho_i)c_i = -\epsilon_i(A-\rho_i)c_{min} \tag{A.1}$$

which is a first-order linear differential equation. Let c_i^{comp} be the complementary function, representing the general solution of the associated homogeneous equation. Let c_i^{par} be a particular solution of (A.1). The complementary function $c_i^{comp}(t)$ can be expressed as:

$$c_i^{comp}(t) = B_1 e^{\epsilon_i(A-\rho_i)t}$$

where B_1 is a constant of integration. To find the particular solution c_i^{par}, we guess:

$$c_i^{par}(t) = B_2 \Rightarrow \frac{dc_i^{par}}{dt} = 0$$

where B_2 is constant. Substituting this into equation (A.1) yields $B_2 = c_{min}$. Therefore, the general solution for $c_i(t)$ is:

$$c_i(t) = c_i^{comp}(t) + c_i^{par}(t) = B_1 e^{\epsilon_i(A-\rho_i)t} + c_{min}.$$

At $t = 0$, we have: $c_i(0) = B_1 + c_{min}$. Thus, the general solution to the differential equation (A.1) becomes:

$$c_i = \Big[c_i(0) - c_{min}\Big] e^{\epsilon_i(A-\rho_i)t} + c_{min} \tag{A.2}$$

where $c_i(0)$ is a constant of integration.

Next, we characterize the path of individual capital holdings k_i. Using equations (2), (6), and (A.2), we get:

$$\dot{k}_i = Ak_i - \Big[c_i(0) - c_{min}\Big] e^{\epsilon_i(A-\rho_i)t} - c_{min}. \tag{A.3}$$

This is another first-order linear differential equation. Let k_i^{comp} be the complementary function, representing the general solution of the associated homogeneous equation, and k_i^{par} be a particular solution. The complementary function k_i^{comp} is given by:

$$k_i^{comp} = B_3 e^{At}$$

where B_3 is a constant of integration. For the particular solution, we guess:

$$k_i^{par} = B_4 e^{B_5 t} + B_6 \Rightarrow \frac{dk_i^{par}}{dt} = B_4 B_5 e^{B_5 t}.$$

Substituting this into equation (A.3), we obtain:

$$-(AB_4 - B_4 B_5)e^{B_5 t} - AB_6 = -\Big[c_i(0) - c_{min}\Big] e^{\epsilon_i(A-\rho_i)t} - c_{min}.$$

By comparing terms, we find: $B_5 = \epsilon_i(A-\rho_i), B_6 = \frac{c_{min}}{A}$, and $B_4 = \frac{c_i(0)-c_{min}}{A-\epsilon_i(A-\rho_i)}$. Thus, the general solution for $k_i(t)$ is:

$$k_i = k_i^{comp} + k_i^{par} = B_3 e^{At} + \frac{c_i(0) - c_{min}}{A - \epsilon_i(A - \rho_i)} e^{\epsilon_i(A-\rho_i)t} + \frac{c_{min}}{A}.$$

The transversality condition (4) requires that $\lim_{t\to\infty} e^{-At} k_i = 0$. This implies that $\frac{\dot{k}_i}{k_i}$ must be strictly less than A. Thus, the rational choice for B_3 is zero, leading to:

$$k_i = \frac{c_i(0) - c_{min}}{A - \epsilon_i(A - \rho_i)} e^{\epsilon_i(A-\rho_i)t} + \frac{c_{min}}{A}. \tag{A.4}$$

Finally, we determine $c_i(0)$ given $k_i(0)$. From (A.4), we have:

$$k_i(0) = \frac{c_i(0) - c_{min}}{A - \epsilon_i(A - \rho_i)} + \frac{c_{min}}{A}.$$

Rearranging this equation, we obtain:

$$c_i(0) = k_i(0)[A - \epsilon_i(A - \rho_i)] - \frac{c_{min}}{A}\epsilon_i(A - \rho_i). \tag{A.5}$$

From here, equations (A.2), (A.4), and (A.5) yield the desired results.

Appendix B: Derivations of (12) and (13)

Let us begin by deriving equation (12). A suitable substitution for solving the differential equation (11) is given by:

$$s_H = \frac{1}{v} \quad \Rightarrow \quad \dot{s_H} = -\frac{1}{v^2}\dot{v}.$$

After substituting and manipulating the equation, we obtain:

$$\dot{s_H} = \Omega s_H - \Omega s_H^2 \quad \Leftrightarrow \quad \dot{v} = -\Omega v + \Omega.$$

The closed-form solution to this equation is:

$$s_H(t) = \frac{s_H(0)}{s_H(0) + (1 - s_H(0))e^{-(\eta_H - \eta_L)t}}.$$

Now, let us turn to deriving equation (13) from the differential equation (10). We define $s_H(t) \equiv \frac{1}{1+z(t)}$, which implies $\dot{s_H} = \frac{-1}{(1-z(t))^2}\dot{z}$. Using this transformation, the differential equation simplifies to the first-order linear form: $\dot{z} = -\Omega(t)z(t)$. The general solution to this equation is

$$z(t) = z(0)e^{-\int_0^t \Omega(\tau)d\tau} \tag{B.1}$$

where $z(0)$ is a constant of integration. Substituting back for $s_H(t)$ using $s_H(t) \equiv \frac{1}{1+z(t)} \Rightarrow$

$z(t) = \frac{1-s_H(t)}{s_H(t)}$, we obtain:

$$s_H(t) = \frac{s_H(0)}{s_H(0) + (1 - s_H(0))e^{-\int_0^t \Omega(\tau)d\tau}}. \tag{B.2}$$

Next, we compute $\int_0^t \Omega(\tau)d\tau$. By construction:

$$\int_0^t \Omega(\tau)d\tau = \eta_H \int_0^t \frac{1}{1 + \Phi e^{-\eta_H \tau}} d\tau - \eta_L \int_0^t \frac{1}{1 + \Phi e^{-\eta_L \tau}} d\tau. \tag{B.3}$$

To evaluate the first integral, $\int_0^t \frac{1}{1+\Phi e^{-\eta_H \tau}} d\tau$, we proceed as follows:

$$\int_0^t \frac{1}{1 + \Phi e^{-\eta_H \tau}} d\tau = \int_0^t \frac{1 + \Phi e^{-\eta_H \tau} - \Phi e^{-\eta_H \tau}}{1 + \Phi e^{-\eta_H \tau}} d\tau = \int_0^t d\tau - \int_0^t \frac{\Phi e^{-\eta_H \tau}}{1 + \Phi e^{-\eta_H \tau}} d\tau.$$

To obtain $\int_0^t d\tau - \int_0^t \frac{\Phi e^{-\eta_H \tau}}{1+\Phi e^{-\eta_H \tau}} d\tau$, we define $\chi \equiv 1 + \Phi e^{-\eta_H \tau}$. This implies

$$d\chi = -\eta_H \Phi e^{-\eta_H \tau} d\tau \text{ and } d\tau = \frac{d\chi}{-\eta_H \Phi e^{-\eta_H \tau}}.$$

As a result,

$$\int_0^t d\tau - \int_0^t \frac{\Phi e^{-\eta_H \tau}}{1 + \Phi e^{-\eta_H \tau}} d\tau = t + \frac{1}{\eta_H} \int_{\tau=0}^{\tau=t} dln \mid \chi \mid = t + \frac{1}{\eta_H} \Big[ln(1 + \Phi e^{-\eta_H t}) - ln(1 + \Phi) \Big]$$

, where we can ignore absolute notation from the last expression since $\chi(\tau) \geq 0$ for all τ. Therefore, the result is:

$$\int_0^t \frac{1}{1 + \Phi e^{-\eta_H \tau}} d\tau = t - \frac{1}{\eta_H} \Big[ln(1 + \Phi) - ln(1 + \Phi e^{-\eta_H t}) \Big]. \tag{B.4}$$

Similarly, for the second integral, i.e., $\int_0^t \frac{1}{1+\Phi e^{-\eta_L \tau}} d\tau$, we can show that,

$$\int_0^t \frac{1}{1 + \Phi e^{-\eta_L \tau}} d\tau = t - \frac{1}{\eta_L} \Big[ln(1 + \Phi) - ln(1 + \Phi e^{-\eta_L t}) \Big]. \tag{B.5}$$

Substituting (B.4) and (B.5) into (B.3) yields:

$$\int_0^t \Omega(\tau)d\tau = (\eta_H - \eta_L)t + ln(1 + \Phi e^{-\eta_H t}) - ln(1 + \Phi e^{-\eta_L t}). \tag{B.6}$$

From this point, the remaining steps follow straightforwardly.

Appendix C: Proof of Lemma 2

This appendix is devoted to analyze the time path of $V(t) \equiv \left[\frac{1+\Phi e^{-\eta_L t}}{1+\Phi e^{-\eta_H t}}\right]$. Depending the sign of $\eta_H - \eta_L$, three cases are considered.
Case 1: $\eta_H - \eta_L = 0$
In this case, it is obvious that

$$V(t) = 1 \text{ for all } t \geq 0. \tag{C.1}$$

Case 2: $\eta_H - \eta_L > 0$
In this case, $V(t)$ is a continuously differentiable function defined on $[0, \infty)$. To simplify the analysis, we transform the independent variable by letting $z = e^t \geq 1$. Then, we define the function as

$$W(z) \equiv \frac{1 + \Phi z^{-\eta_L}}{1 + \Phi z^{-\eta_L}}. \tag{C.2}$$

This transformation helps in characterizing the graph of the function more easily.

The function $W(z)$ is continuously differentiable on $[1, \infty)$ and is bounded below by one. Particularly, $W(1) = W(\infty) < W(z)$ for all finite $z > 1$. Also, this function is bounded above by a finite number and has a unique global maximum at $z = z_{Max} > 1$. To illustrate this, consider

$$W'(z) = (1 + \Phi z^{-\eta_H})^{-2} \cdot \frac{\Phi}{z} \cdot \left[\eta_H(z^{-\eta_H} + z^{-\eta_H - \eta_L}) - \eta_L(z^{-\eta_L} + z^{-\eta_H - \eta_L})\right]. \tag{C.3}$$

Since $(1 + \Phi z^{-\eta_H})^{-2} \cdot \frac{\Phi}{z} > 0$, the sign of $W'(z)$ is determined by the sign of the term

$$\left[\eta_H(z^{-\eta_H} + z^{-\eta_H - \eta_L}) - \eta_L(z^{-\eta_L} + z^{-\eta_H - \eta_L})\right].$$

Rewriting, we get:

$$W'(z) \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0 \Leftrightarrow \underbrace{\eta_H(1 + \Phi z^{-\eta_L})}_{\equiv LHS(z)} \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} \underbrace{\eta_L(z^{\eta_H - \eta_L} + \Phi z^{-\eta_L})}_{\equiv RHS(z)}. \tag{C.4}$$

Given $0 < \eta_H < \eta_L$, at $z = 1$ (or $t = 0$), $W'(1) > 0$ because $\eta_H(1 + \Phi) > \eta_L(1 + \Phi)$. By continuity, $W(z)$ is increasing when z slightly increases from 1. However, for sufficiently large z, $z^{\eta_H - \eta_L}$ becomes large enough so that $\eta_L(z^{\eta_H - \eta_L} + \Phi z^{-\eta_L})$ exceeds $\eta_H(1 + \Phi z^{-\eta_L})$, leading to $W'(z) < 0$. This implies that $W(z)$ must have a unique global maximum $z_{Max} > 1$ such that

$$W(z_{Max}) > W(z) \text{ for all } z \geq 1 \text{ and } z \neq z_{Max}.$$

To demonstrate the existence and uniqueness of z_{Max}, we compare the graphs of $LHS(z)$ and $RHS(z)$ in the Euclidean plane. Let us first focus on $LHS(z)$. We observe that $LHS(z = 1) = \eta_H(1 + \Phi) > \eta_H$. Also, $LHS(z)$ strictly decreasing for all $z > 1$, with

$\lim_{z\to\infty} LHS(z) = \eta_H > 0$. Next, we turn our attention to $RHS(z)$. At $z = 1$, $RHS(z = 1) = \eta_L(1+\Phi) < \eta_H(1+\Phi)$. It is important to note that the difference $\eta_L(1+\Phi) - \eta_H$ can be positive, negative or zero. Additionally, the derivative of $RHS(z)$ is given by:

$$RHS'(z) = \eta_L z^{-\eta_L - 1}\Big[(\eta_H - \eta_L)(z)^{\eta_H} - \eta_L \Phi\Big].$$

This derivative indicates that $RHS(z)$ behaves as follows:

(i) $RHS(z)$ is strictly increasing without bound if $(\eta_H - \eta_L)(1)^{\eta_H} - \eta_L\Phi \geq 0$, or

(ii) $RHS(z)$ is strictly decreasing until it reaches $z = \left(\frac{\eta_L \Phi}{\eta_H - \eta_L}\right)^{\frac{1}{\eta_H}}$, after which it strictly increases without bound if $(\eta_H - \eta_L)(1)^{\eta_H} - \eta_L\Phi < 0$.

In both cases, there is a unique $z_{Max} > 1$ solving $LHS(z_{Max}) = RHS(z_{Max})$.

In summary, we can conclude that $W(z)$ is a bounded function defined on $[1, \infty)$. The function has the following properties:

(i) $W(1) = 1$.

(ii) It is strictly increasing for $z \in (1, z_{Max})$.

(iii) It reaches its maximum at $z = z_{Max} > 1$ where z_{Max} solves the following equation:

$$\eta_H(1 + \Phi(z_{Max})^{-\eta_L}) - \eta_L\Big[(z_{Max})^{\eta_H - \eta_L} + \Phi(z_{Max})^{-\eta_L}\Big] = 0. \tag{C.5}$$

(iv) It is strictly decreasing for $z \in (z_{Max}, \infty)$.

(v) As z increases without bound, $W(z)$ converges to 1.

In addition, z_{Max} solves the following non-linear equation:

$$\eta_H(1 + \Phi(z_{Max})^{-\eta_L}) - \eta_L\Big[(z_{Max})^{\eta_H - \eta_L} + \Phi(z_{Max})^{-\eta_L}\Big] = 0. \tag{C.6}$$

Since $t = lnz$ is a monotonic transformation, the properties of $W(z)$ apply similarly to $V(t)$. Specifically, $V(t)$ is a bounded function defined on $[0, \infty)$ with the following characteristics:

(i) $V(0) = 1$.

(ii) It is strictly increasing for $t \in (0, t_{Max})$.

(iii) It reaches its maximum at $t = t_{Max} > 1$ where $t_{max} = lnz_{max}$.

(iv) It is strictly decreasing for $t \in (t_{Max}, \infty)$.

(v) As t increases without bound, $V(z)$ converges to 1.

In essence, we can state:

$$V(0) = 1, V(t) > 1 \text{ for all finite } t > 0 \text{ and, } \lim_{t\to\infty} V(t) = 1. \tag{C.7}$$

Case 3: $\eta_H - \eta_L < 0$

We can apply the same terminology as in the previous cases to address this scenario. First, let us consider equation (C.2). Under the assumption $\eta_L > \eta_H > 0$, we observe that $W(z = 1) = \lim_{t\to\infty} W(z) = 1$. To characterize the behaviour of the curve in the interval $(1, \infty)$, we examine the first derivative, given by equation (C.4).

Focusing on (C.4), we note that $W'(z = 1) < 1$. By continuity, $W(z)$ is decreasing as z

slightly increases from 1. However, if z becomes sufficiently large such that $z^{\eta_H-\eta_L}$ diminishes enough, it could cause $\eta_L(z^{\eta_H-\eta_L}+\Phi z^{-\eta_L})$ to become smaller than $\eta_H(1+\Phi z^{-\eta_L})$. In this situation, $W'(z)>0$. If this inequality holds for some $z_0>1$, it will continue to hold for $z_0+\epsilon$, where ϵ is any positive constant. This suggests the existence of a unique global minimum at $z_{min}>1$ such that:

$$W(z_{min})<W(z) \text{ for all } z\geq 1 \text{ and } z\neq z_{min}.$$

Thus, $W(z)$ reaches its minimum at z_{min}, after which it starts to increase as z becomes larger.

To establish the existence and uniqueness of z_{min}, we examine the behaviour of the graphs of $LHS(z)$ and $RHS(z)$ on the Euclidean plane. To begin with, consider $LHS(z)$. We can see that $LHS(z=1)=\eta_H(1+\Phi)>\eta_H$. Also, $LHS(z)$ is strictly decreasing for all $z>1$ and $\lim_{z\to\infty} LHS(z)=\eta_H>0$.

Next, consider $RHS(z)$. We can see that $RHS(z=1)=\eta_L(1+\Phi)>\eta_H(1+\Phi)$. In addition, $\lim_{t\to\infty} RHS(z)=0$. Not only that, the derivative of $RHS(z)$ is given by

$$RHS'(z)=\eta_L z^{-\eta_L-1}\Big[(\eta_H-\eta_L)(z)^{\eta_H}-\eta_L\Phi\Big]<0\,\forall z\geq 1.$$

This means that $RHS(z)$ is strictly decreasing for all $z\geq 1$. As a result of these properties, there exists a unique $z_{min}>1$ such that $LHS(z_{min})=RHS(z_{min})$.

In summary, $W(z)$ is a bounded function defined on $[1,\infty)$. It takes the value of one when $z=1$, strictly decreasing for all $z\in(1,z_{min})$, reaches its minimum when $z=z_{min}>1$, strictly increasing for all $z\in(z_{min},\infty)$. Additionally, as z increases without bound, $W(z)$ converges to 1. In addition, z_{min} solves the following non-linear equation:

$$\eta_H(1+\Phi(z_{min})^{-\eta_L})-\eta_L\Big[(z_{min})^{\eta_H-\eta_L}+\Phi(z_{min})^{-\eta_L}\Big]=0. \tag{C.8}$$

As in the previous case, the key characteristics of $V(t)$ are straightforward to establish. Specifically, $V(t)$ is a bounded function defined on $[0,\infty)$ with the following characteristics:

(i) $V(0)=1$.

(ii) It is strictly decreasing for $t\in(0,t_{min})$.

(iii) It reaches its minimum at $t=t_{min}>1$ where $t_{min}=lnz_{min}$.

(iv) It is strictly increasing for $t\in(t_{min},\infty)$.

(v) As t increases without bound, $V(z)$ converges to 1.

In essence, we can state:

$$V(0)=1, V(t)<1 \text{ for all finite } t>0 \text{ and } \lim_{t\to\infty} V(t)=1. \tag{C.9}$$

The proof is complete. **Q.E.D.**