E C O N O M I C S B U L L E T I N

# Optimal control with switches in the objective functional

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## *Abstract*

This note offers a proof of the necessary conditions for optimal control problems that involve a finite number of discrete switches in the objective functional over the planning horizon.

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## 1 Introduction

Some optimal control problems in economics involve a finite number of discrete switches in the objective functional over the course of the planning horizon. This note offers a proof of the necessary conditions for optimal control problems of this type.[1]

A similar problem is that of switches in the *state* equation, with no switches in the objective functional. This problem has received considerable attention (see, e.g., Kamien and Schwartz 1991, pp. 246-247), and the necessary conditions are well known. Yet, there appears to be no proof in the literature of the necessary conditions for the opposite type of control problem that involves switches in the objective functional with no switches in the state equation. This paper fills this need.

## 2 Proof of necessary conditions[2]

The goal is to derive necessary conditions to a control problem with a planning interval comprised of $n$ distinct stages, with each stage corresponding to a unique objective functional.

$$\max : J = \sum_{i=1}^{n} \int_{a(i)}^{b(i)} f^{i}[t, x(t), u(t)]dt + \phi[x(b(n))] \tag{1}$$

subject to

$$\begin{aligned} &\frac{dx(t)}{dt} = g[t, x(t), u(t)] \\ &x(a(1)) = x_0 , \ given \\ &a(i), b(i) \in \Re^{+} , \ given \\ &a(i+1) = b(i) \end{aligned} \tag{2}$$

The last equation of (2) is an assumption that ensures continuity of the various $n$ stages on the control interval $[a(1), b(n)]$, i.e., stages share common endpoints.

Let $u^{*}(t), x^{*}(t)$ represent the optimal control and state variables, respectively. To obtain the necessary conditions obeyed by optimal

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[1] One example is the lifecycle control problem of maximizing household consumption while accounting for discrete switches in family size as children leave home. Such a problem typically is handled by assuming that family size is a continuously differentiable variable (see, e.g., Bütler 2001), which then allows for the application of the standard maximum principle. But, in certain instances the researcher may want the model to bear to the added realism of discrete switches in the household's objective functional. Of course, this type of control problem appears in many places and is not limited to household optimization.

[2] This proof follows the general procedure for deriving necessary conditions in Kamien and Schwartz (1991, pp. 246-247); although, the particular type of control problem analyzed here is quite different from any examined therein.

solutions, use the law of motion in (2) to append the objective functionals with $n$ continuously differentiable multipliers $\lambda^i(t)$

$$J = \sum_{i=1}^{n} \int_{a(i)}^{b(i)} \left\{ f^i[t, x(t), u(t)] + \lambda^i(t) g[t, x(t), u(t)] - \lambda^i(t) \frac{dx(t)}{dt} \right\} dt + \phi[x(b(n))] \tag{3}$$

Using integration by parts, note that

$$\int_{a(i)}^{b(i)} \lambda^i(t) \frac{dx(t)}{dt} dt = \lambda^i(b(i)) x(b(i)) - \lambda^i(a(i)) x(a(i)) - \int_{a(i)}^{b(i)} \frac{d\lambda^i(t)}{dt} x(t) dt$$

Thus, (3) can be written

$$J = \sum_{i=1}^{n} \left\{ \int_{a(i)}^{b(i)} \left\{ f^i + \lambda^i g + \frac{d\lambda^i}{dt} x \right\} dt - \lambda^i(b(i)) x(b(i)) + \lambda^i(a(i)) x(a(i)) \right\} + \phi[x(b(n))] \tag{4}$$

The first variation of $J$ is

$$\begin{aligned} J - J^* = \delta J = & \sum_{i=1}^{n} \int_{a(i)}^{b(i)} \left[ \left\{ f_x^i + \lambda^i g_x + \frac{d\lambda^i}{dt} \right\} \delta x(t) + \left\{ f_u^i + \lambda^i g_u \right\} \delta u(t) \right] dt \\ & - \sum_{i=1}^{n} \left[ \lambda^i(b(i)) \delta x(b(i)) - \lambda^i(a(i)) \delta x(a(i)) \right] \\ & + \phi_x[x(b(n))] \delta x(b(n)) \end{aligned} \tag{5}$$

where $\delta u(t)$ is the modification to the optimal program and $\delta x(t)$ is the corresponding modification to the optimal state path. The partial derivatives of $f$ and $g$ in (5) are evaluated at $u^*(t), x^*(t)$.

Note that $\sum \left\{ \lambda^i(b(i)) \delta x(b(i)) - \lambda^i(a(i)) \delta x(a(i)) \right\}$ can be written

$$\left\{ \sum_{i=1}^{n-1} \lambda^i(b(i)) \delta x(b(i)) \right\} + \lambda^n(b(n)) \delta x(b(n)) - \left\{ \sum_{i=1}^{n-1} \lambda^{i+1}(a(i+1)) \delta x(a(i+1)) \right\} - \lambda^1(a(1)) \delta x(a(1))$$

Now, the continuity assumption (i.e., the last equation of (2)) allows us to rewrite this expression as

$$\left\{ \sum_{i=1}^{n-1} \lambda^i(b(i)) \delta x(b(i)) \right\} + \lambda^n(b(n)) \delta x(b(n)) - \left\{ \sum_{i=1}^{n-1} \lambda^{i+1}(b(i)) \delta x(b(i)) \right\} - \lambda^1(a(1)) \delta x(a(1))$$

We can ignore the last term (by definition the modified state path satisfies $x(a(1)) = x_0$, thus $\delta x(a(1)) = 0$) and we can substitute the above expression into (5)

$$\delta J = \sum_{i=1}^{n}\int_{a(i)}^{b(i)}\left[\left\{f_x^{\,i} + \lambda^i g_x + \frac{d\lambda^i}{dt}\right\}\delta x(t) + \left\{f_u^{\,i} + \lambda^i g_u\right\}\delta u(t)\right]dt$$

(6)
$$-\left\{\sum_{i=1}^{n-1}[\lambda^i(b(i)) - \lambda^{i+1}(b(i))]\delta x(b(i))\right\}$$

$$+\left[\phi_x[x(b(n))] - \lambda^n(b(n))\right]\delta x(b(n))$$

Define the costate variables to obey

(7a) $$\frac{d\lambda^i(t)}{dt} = -[f_x^{\,i}(t, x^*(t), u^*(t)) + \lambda^i(t) g_x(t, x^*(t), u^*(t))]$$

(7b) $$\lambda^i(b(i)) = \lambda^{i+1}(b(i))$$

(7c) $$\phi_x[x(b(n))] = \lambda^n(b(n))$$

Equation (7a) gives $n$ distinct laws of motion for the costate variable, one for each separate stage; (7b) assigns endpoint values to each of the stage specific costate variables, and, in particular, ensures the costate is piecewise continuous; and, (7c) is the traditional transversality condition. Also, choose the modification

(8) $$\delta u(t) = f_u^{\,i} + \lambda^i g_u$$

Thus, (6) becomes

(9) $$\delta J = \sum_{i=1}^{n}\int_{a(i)}^{b(i)}\left\{f_u^{\,i} + \lambda^i g_u\right\}^2 dt \geq 0$$

Thus, the chosen modification improves the objective $J$ unless

(10) $$f_u^{\,i} + \lambda^i g_u = 0 \qquad \text{for all } i$$

In sum, given (7a), (7b), and (7c), (10) is necessary for optimality, otherwise the chosen modification will improve $J$. This completes the proof of the necessary conditions for optimal control problems with stage-specific objective functionals.

**References**


[1] Bütler, Monika, 2001, Neoclassical life-cycle consumption: a textbook example. *Economic Theory* 17, 209-221.

[2] Kamien, Morton I. and Nancy L. Schwartz, 1991, *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management*, Second Edition, North-Holland Publishing Company.